Discrete Mathematics
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Lecture - 22
Introduction to Graph Theory

Welcome everybody sixth week of discrete mathematics. So we will start this week with introduction to graph theory, which is one of the very important ways of modelling problems. So we start with this.

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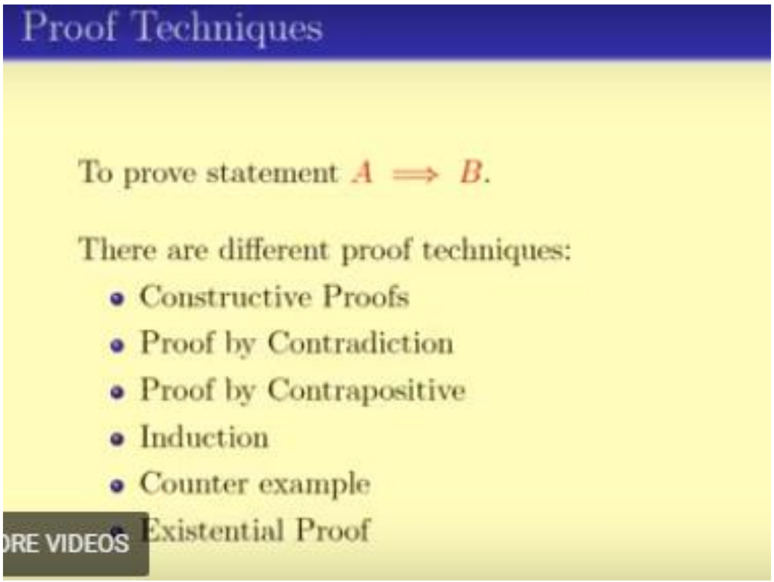

Till now, we have new technique we have looked at how to solve a problem like a implies B and we have gone through a number of proof techniques right. We have not yet gone to this technique of existential proof. We will possibly take a look at them later on, but we have till now gone through constructive proofs, proof by Contradiction, proof by Contrapositive, Induction, Counter example and so on.

Now let us ask very important question how to solve the problem?

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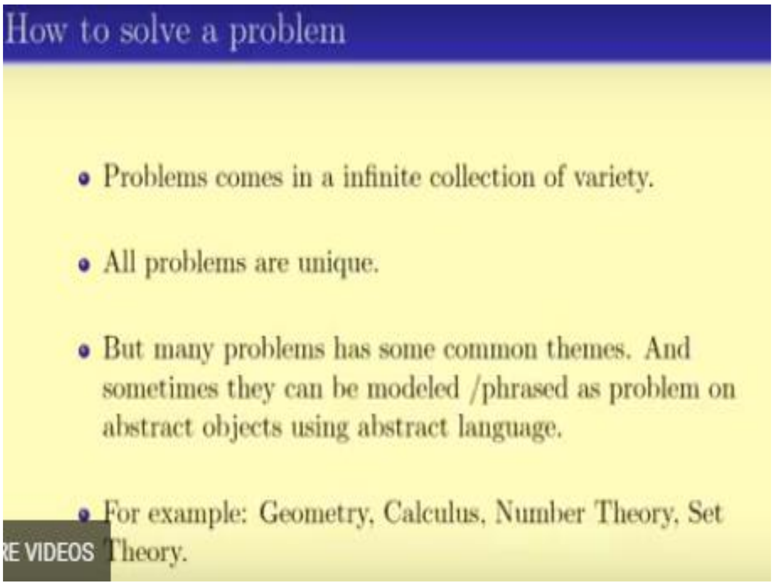

First of all, problems come in an infinite collection of varieties. So it is not very clear whether the problems, whether the two problems are same or not, or have they any similarity, all of them are unique. So there is as such no particular thumb rule about how to solve a problem so it so happens that many problems have some common themes and sometimes they can be modelled or rather the problem can be phrased.

And the problem of some abstract object using some abstract language. Now the advantage of doing so is that one can try to attempt the problems or using the abstract language. For example, we have seen that in your high school how we can solve various problems in geometry or how they are various problems can be phrased in trigonometry or calculus or number theory or set theory and so on. So different problems can be phrased

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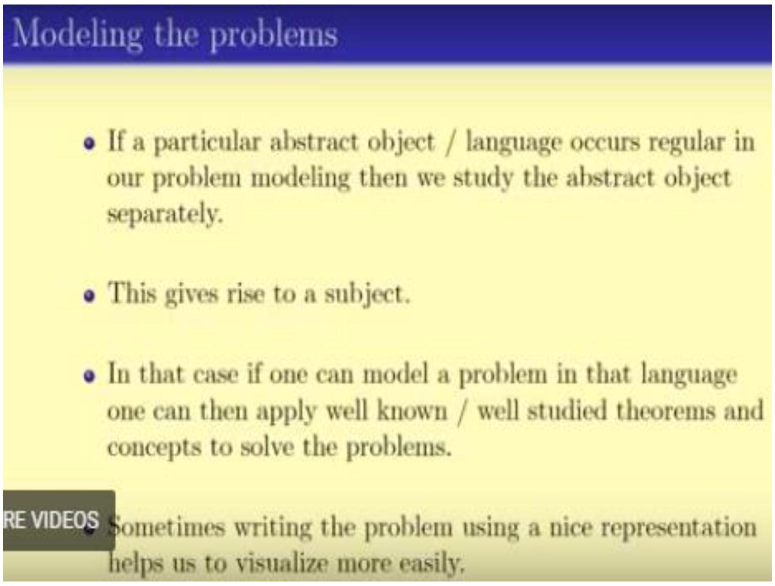

As a problem in this abstract objects. Now if a particular abstract object occurs regularly to our problem regularly, then we would like to study this object separately for example if you remember in the case of geometry, we started studying things about circles and lines and angles because they were arising in real life quite a lot of time. Similarly, we realize that many problems in geometry or otherwise can be written as.

The ratio of the hypotenuse versus the sides of the triangle, which in turn gives us sin, tan, and cos, and then we start describing trigonometry. So these are abstract language, trigonometry, geometry and so on and by studying them we possibly come up with a unifying theory of attacking the problems. So once we have a particular abstract object appearing quite regularly.

We start studying it separately this gives rise to a new subject and in that case once you get a problem, you would like to phrase it or convert it to the language of the abstract object or the abstract language, so that we can apply the variable terms, right. So if you can convert a problem in a problem in geometry, we can apply theorems by Pythagoras theorem. We do not have to prove it again because we know those theorems.

So these are the advantage of understanding or converting of these problems or problems in to abstract language. Sometimes also writing the problems in a different 23rd language can help in understanding or visualizing the problem better.

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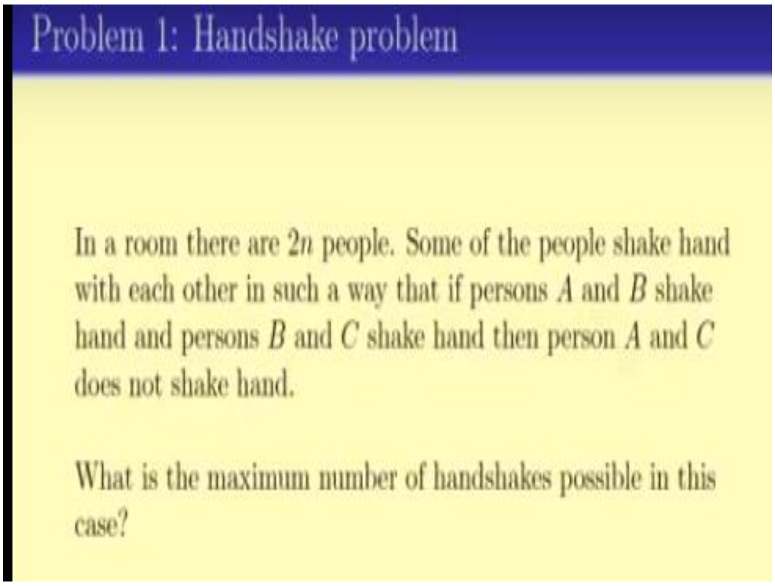

So let us start with three problems and that means motivate us to study a particular subject. The first problem is in a room there are 2n people, some of the people shake hand with each other, but in such a way that if person a and b shake hands and person b and c shake hands, then person a and c does not shake hands. Question is that what is the maximum number of handshakes possible in this case, right?

Let me quickly ask how many handshakes can be possible when there are 2n people, when I do not have the condition of this condition. So I have 2n people, how many possible handshakes can there. I we leave that as a problem and move on to the next problem that comes up.

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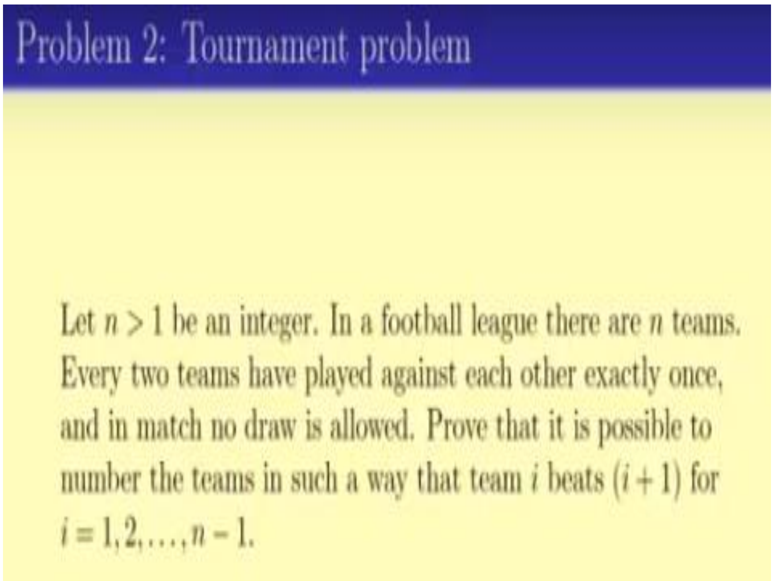

The next problem is known as the tournament problem. So in a football league, there are n teams. Every two team has played against each other and there is no draw, so either when team A plays with team B, either A has defeated B or B has defeated A. Prove that it is possible to number the teams in such a way that the first team will defeat the second team, the second team will defeat the third team, third team defeats fourth team and so on.

The n minus one team defeats the n team, right. So this is one more problem. It is a problem like the other problem on a real-life problem, but how do you solve this problem.

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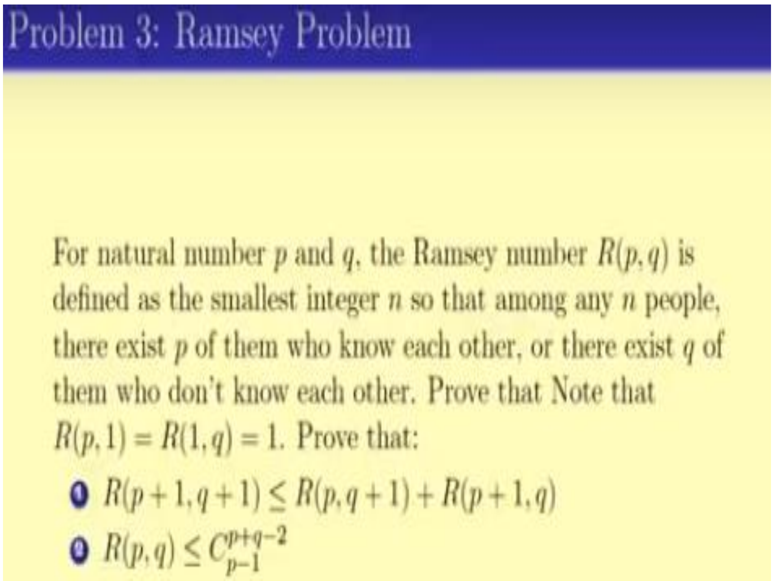

Let us look at one more problem. This is called the Ramsey number, so we say that for a number p and q natural number, which the Ramsey number R pq is the smallest integer n, so that among any n people there exist pn who know each other or there exist qn who do not know each other, so what does the statement say? It states that if R pq is equals 100, it means that give me any hundred people.

There will be either p of them who know each other or q of the people who do not know each other. Now, I would like to understand how big this number R pq can be? Can it be 100 or 4? And it says the following thing, prove that R p plus 1, q plus 1 is less than Rp, q plus 1 plus Rp plus 1, q and Rpq is less than p plus q minus 2 p minus 1. This is basically p plus q minus 2 p minus 1.

If you recall this particular problem was done as part of the induction problem, but how do we get this recurrence is the main question. So in induction problem, we have told that if we have this recurrence and we have this base cases, can we prove this statement? But I am asking the question here is that how do you prove this particular statement or this particular recurrence, right?

So here are three problems that we have seen just now that we could solve these three problems once again.

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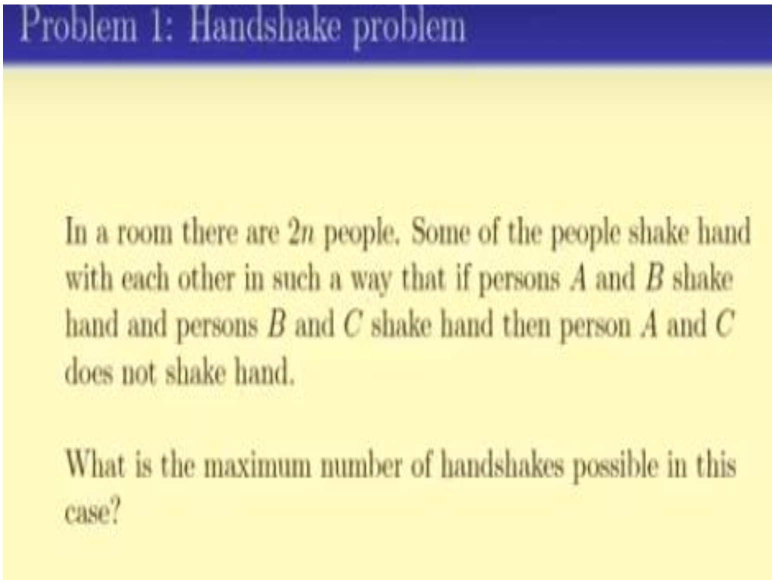

We have this handshake problem where we say that there are 2 n people and there were no three people have shaken hand with each other, then what is the maximum number of handshakes possible?

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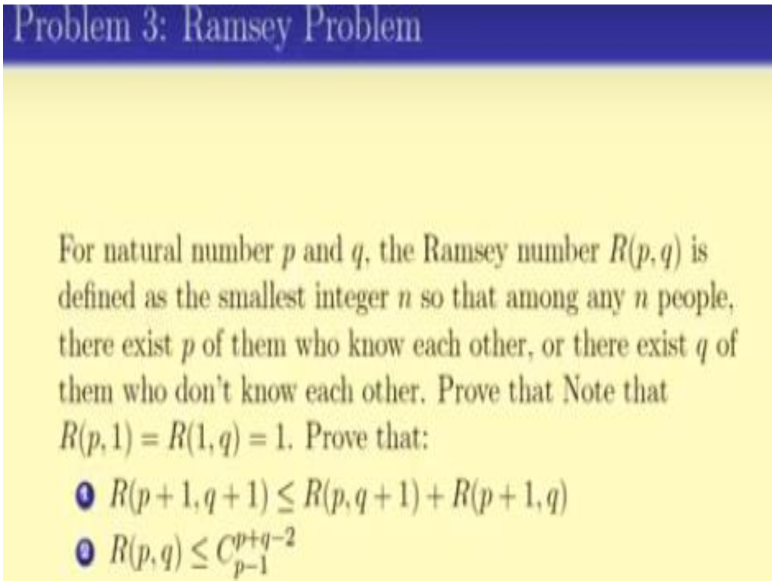

Second problem in a football tournament, we have to prove that I can order P team so that team one defeats team 2, team 2 defeats team 3 and so on. And the Ramsey problem, which says that what is the smallest number n such that there must be q of the people in any room of people with n size, there must be q of them among each other or there exists q of them who do not know each other.

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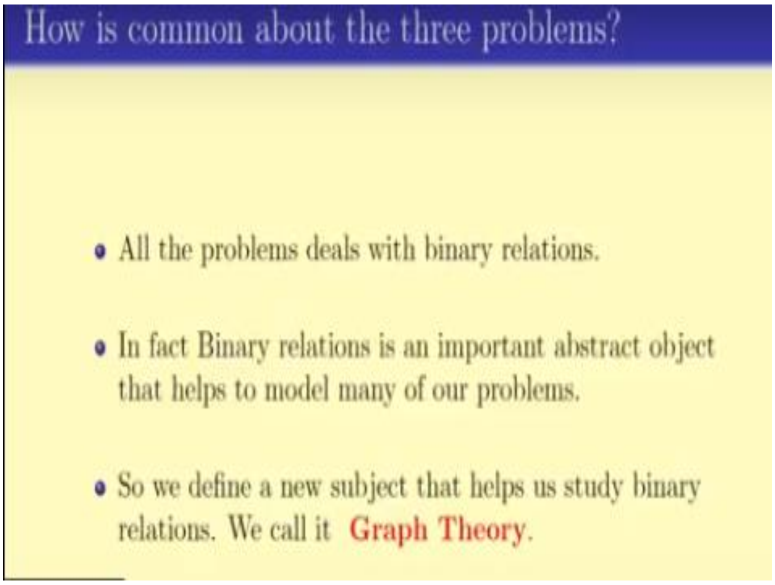

Now what is common among these problems? First thing you notice that all the problems deals with binary relations. The first problem deals with two people shaking hand with each other. The second problem deals with team a defeating team b or team b defeating team a. And the third problem deals with whether two people know each other or not. So these are the binary relations right.

Given two people what is the relation between them and they can be ordered for that matter. So binary relations are a very important abstract model that helps us to model many of our problems. We will see many examples of this in the next one or two weeks and the subject that helps us study tips binary relation is what we call as graph theory. This is what we will be studying. Now graph theory to start with we have to define what a graph is.

A graph is a particular representation of the binary relation okay.

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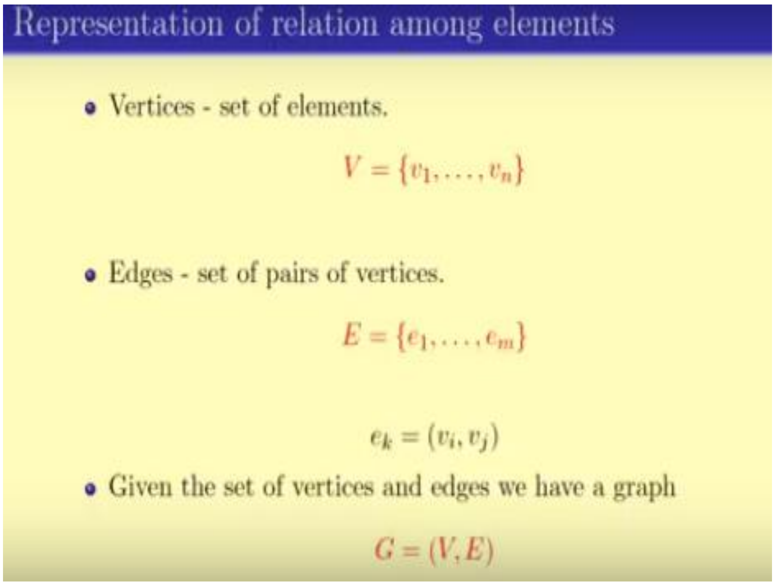

So to define relations among this element, we really need all the set of vertices which are the set of elements. So let v1 to vn be the set of elements. The relations that we are talking about are relations on the pair of vertices. So these are called edges. So the sets of pairs of vertices. So in other words if I usually denote it as e, e1 to en, but ek is of the form some vi, vj, which means that so this means that vi m vj are related right.

So if someone gives you the set of vertices and the set of edges then that leaves us the graph, which is called G equals to V, E. So a graph is nothing but a set of vertices and a set of binary relations. So E is a subset of V cross V. Now this set of relations we can use this relation to define lots and lots of things. This relation can be either reflexive or not right or in other words vi, vj can be different from vj, vi, right

So depending on restrictions or what properties you put on them, on this relation, we would get different kind of graphs. So these are all notation stuff the way you visualize this set graphs is by drawing the vertices as plots.

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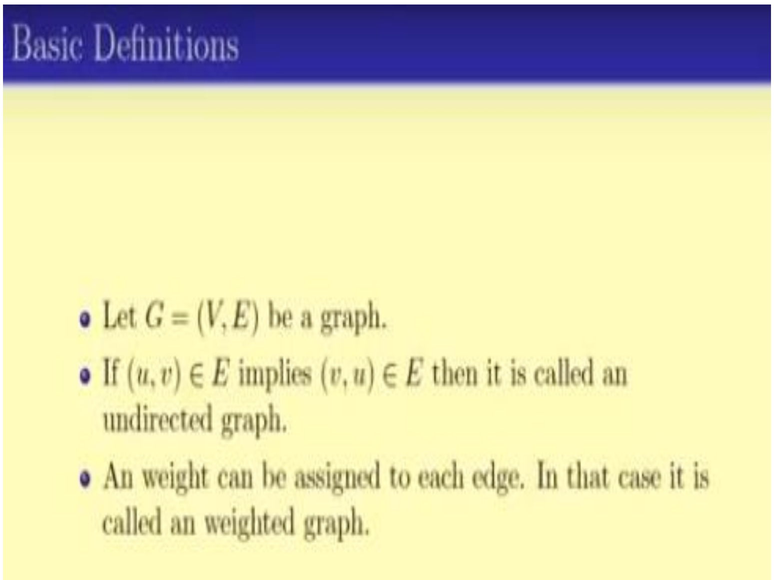

We will come to that. First of all in the basic definition, let G be a graph, if I have a uv and if it is a reflexive relation, that is u, v is an edge implies v, u is the edge, we call it as an undirected graph else we call it as directed graph and sometimes for some purpose we can convert this whole problem in to various different kinds of, we can make this abstract data structure much more complicated and in particular we can put a weight can be assigned to each edge and in that case we can have a weighted graph.

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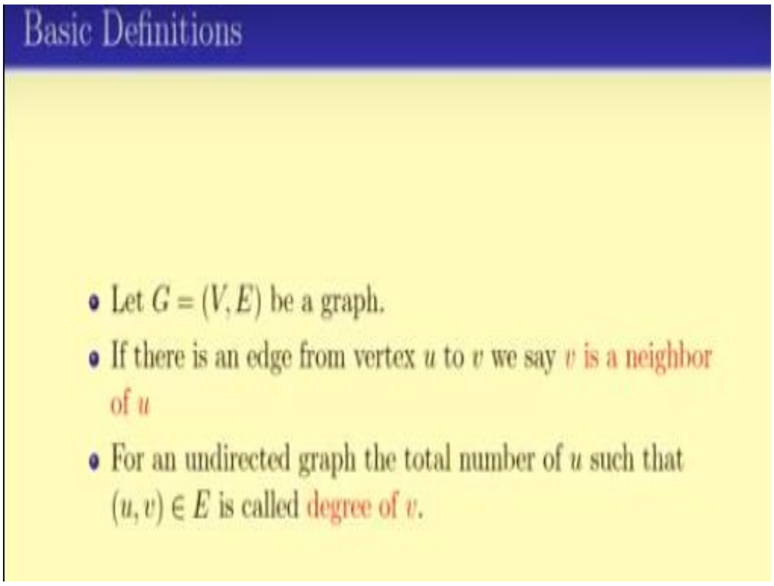

So G is a graph and if there is an edge between u to v, then we say that v is a neighbor of u and that is another definition that is there and for any undirected graph the total number of total degree of u such that a number of pairs u v that are there, is known as the degree of v, all right. So this is the number of element or the number of vertices that has an edge to v. So as I told you, we were going to represent this graph as blobs, so like this a, b, c, d, e, f, g are vertices in a graph now the edges are usually drawn by lines.

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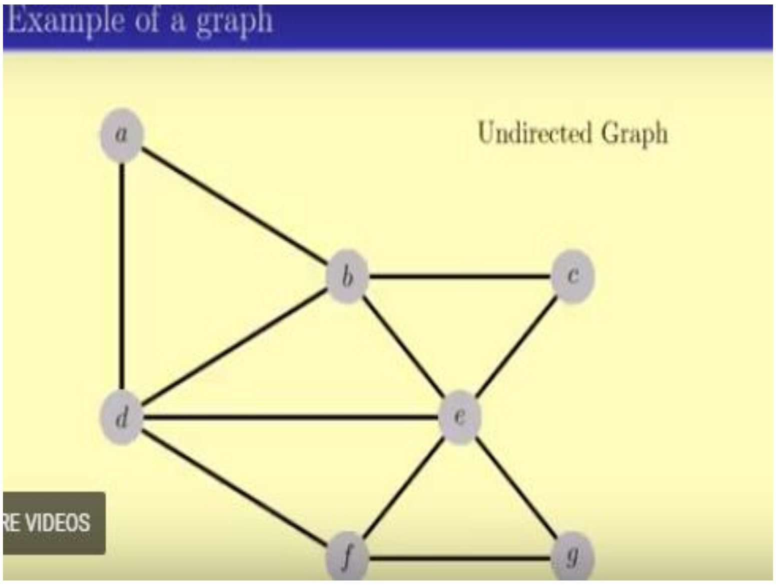

If I draw lines like this basically says that a, b is an edge. Also this one basically says that a, b is an edge and so on and so forth. The edges are usually represented using the line drawn between the blobs or vertices. Now as you can see here d is called a neighbour of a because there is an edge from a to b. Also the other thing is that since a has two vertices the degree of a is which we call as is two. So degree of a is two because it has two lines, right.

So this is what we call us undirected graph. When we have undirected graph that means I do not care whether a, b or b, a is in the edge. In that case there is no ordering between a and b. So we usually in that case draw it with this kind of stuff just a straight line between the two vertices that related. Now, if the two vertices are not, if the relationship is not reflexive, that is a and b need not, a is related to b does not imply b is related to a.

In that case, we have to design in a different way of writing. We will come up with different representation. Also sometimes this relationship can have a weight associated with it. For example, think of a, b, c, d here as some the cities and then a to d the line will be either they are flight from a to d and I want to represent the time taken by the flight. In that case we would like to put some value on the edges, we denote the flight time.

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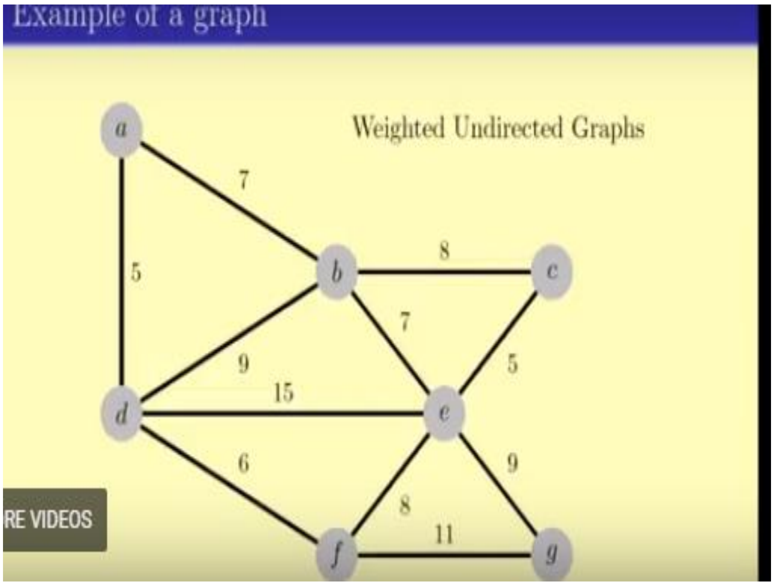

So in that we will get something like this. We have the weight. We will put weights on the edges and then you get a weighted undirected graph. This weights can represent whatever we want to represent. We will see various examples of this. Also if we do not want to have a undirected graph or another says we do not, if binary relation is not reflexive, we usually denote the non-reflexive edges using the arrows.

So this arrow we use at d is related to a because the edge is d to a, but since is there is no edge from a to d it means that a is not related to d. You can think of it as a one-way path. So if these are cities, so this says that there is a flight from d to a, but there is no flight from a to d. So that is represented using this kind of graph and we call it as a weighted directed graphs. Of course if you do not have the weights, we just call it as directed graphs. Now graphs are, your must have seen graphs in various places.

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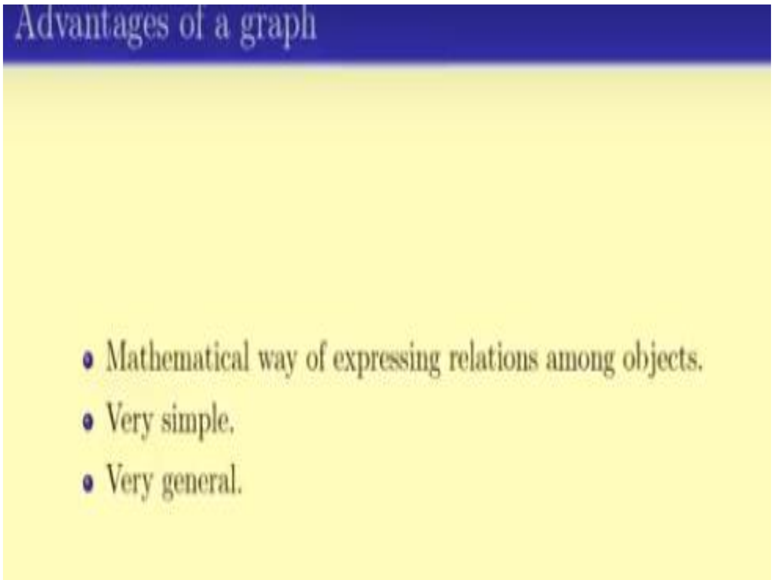

And they are very useful way of representing data and it also helps us to formulate problems in a nice way.

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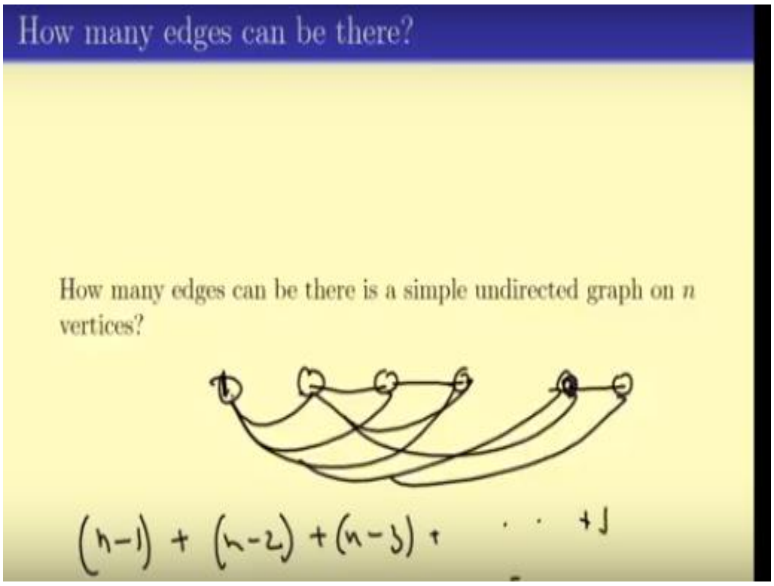

Before we move, let us ask some quick questions. How many edges can there be a simple undirected graph on n vertices. Okay so let us see. So the first vertex so if I have say n vertices, first vertex can have edges to all of them right, which is n minus 1 plus second vertex, now already have edge to the first one, can have edge to second and all of them, which is n minus 2.

Third one can have edge to everything on this side, which is n minus 3 and we keep on adding till this vertex, which has edge to 1. So this is actually equals to n into n minus 1 by 2. The other way of looking at it is that between any two vertices I can draw an edge, so total number of edges that can be there is number of ways I can choose two vertices from this end vertex, which is of course equal to this also.

So the total number of edges that can be there in an undirected graph of n vertices is n into two or n to n minus 1 by 2.

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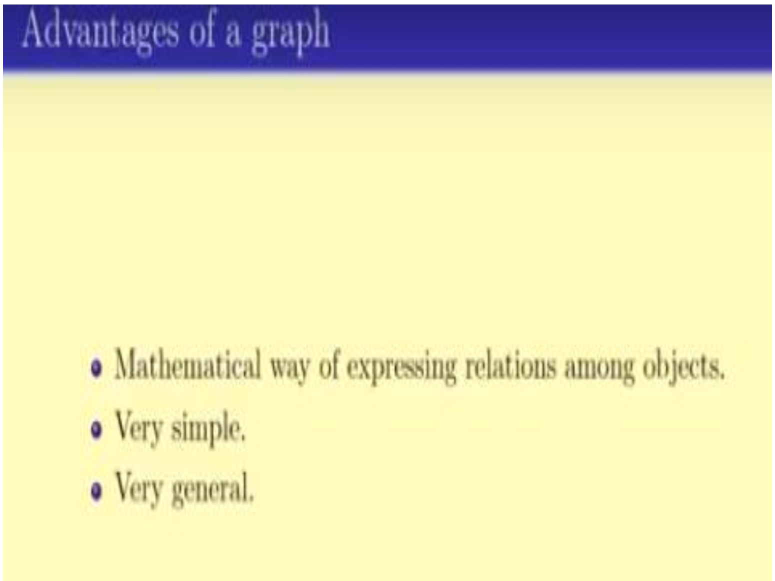

Now graphs are a nice way of expressing relationship between objects particularly binary relations. They are very simple yet they are very general in the sense that they are simple to understand what they are and yet they are used to represent quite a number of complicated relationship and problems. There are various examples of graphs class that can be there, but I have pulled out a few of them. Here is one, so this is called the friendship graph.

So every person is a vertex and we say that if two persons are friends, then there is edge between the respective vertices. This is used a lot for understanding things like social networks like Facebook. So all the users are vertices. Between any two vertex, I draw an edge if they are friends in Facebook. The question is that how well connected they are and so on.

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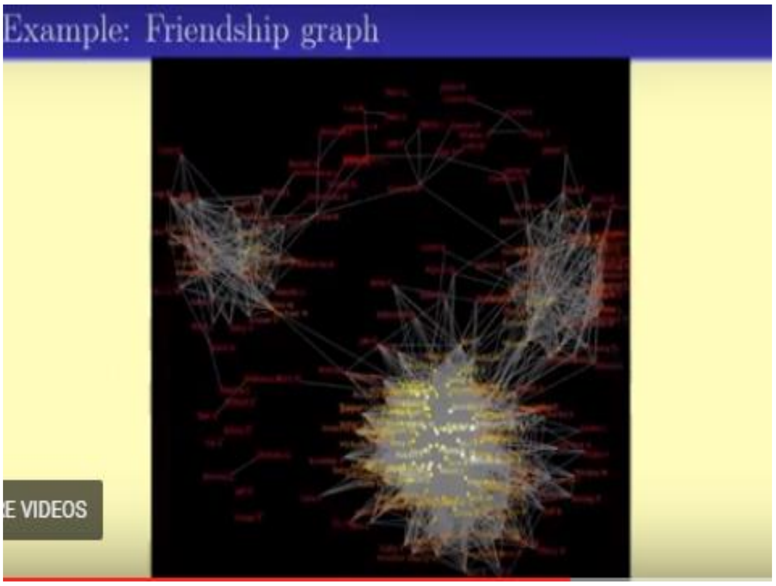

So here is an example of a friendship graph, typical friendship graph from one of these. So I have this point, every vertices. So this is one vertex, this is one vertex, this is one vertex, and so on. These all vertex represents and there is an edge if they are friends with each other in social network. This helps us to understand whether there are various structures in the friendship graph.

Further as you can see in this particular friendship graph, one can see this one and say okay this set, may be this set of people are the ones who are in a particular country, these set of people are in another country and so this set of people in another country. So this is called clustering problem or understanding the friendship graph to understand some structure on the graphs or some convenient one.

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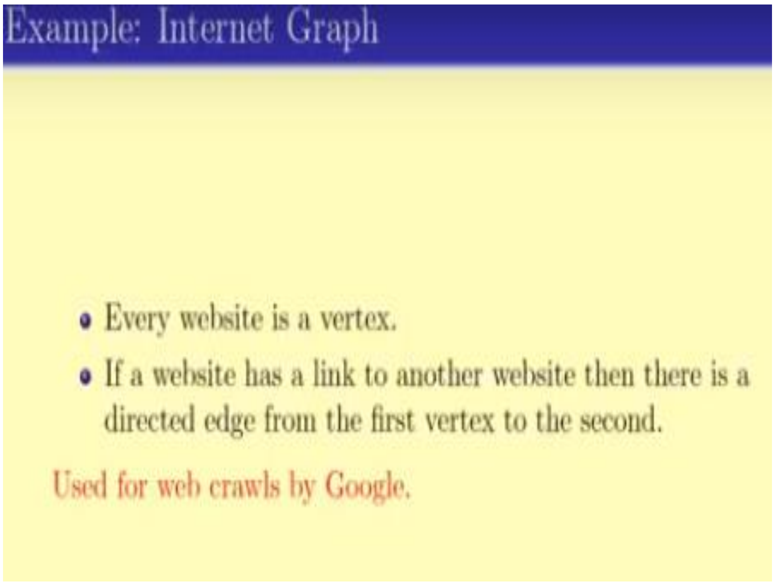

Another example is the internet graph. This is used a lot for Google crawl, but every website is a vertex and you know website has a link to another website, then there is a direct edge from the first vertex to the second vertex. This of course gives us another thing a directed graph and this is used for web crawls by Google. When you search something on Google, Google uses this one.

So these are examples of real life problems or real life data that are represented in graphs and used a lot in our modern day technologies.

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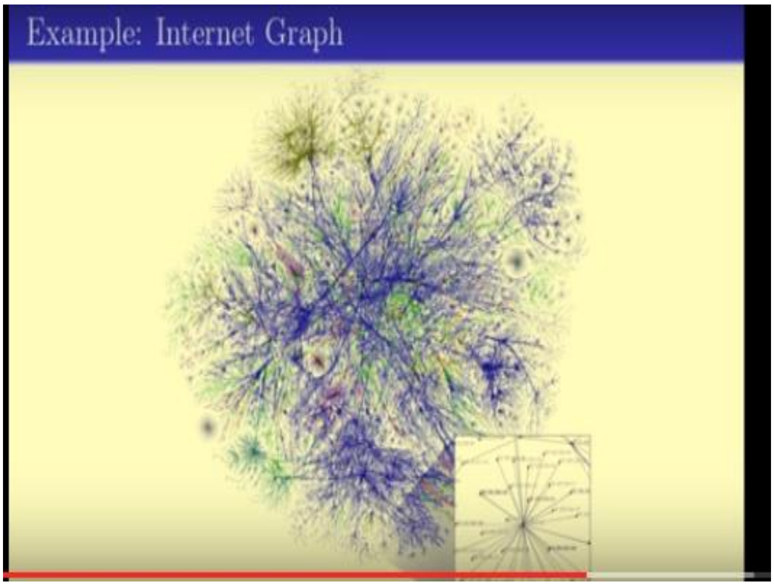

So this is the internet browser. I think this was a few years ago. This is the representation of the internet graph.

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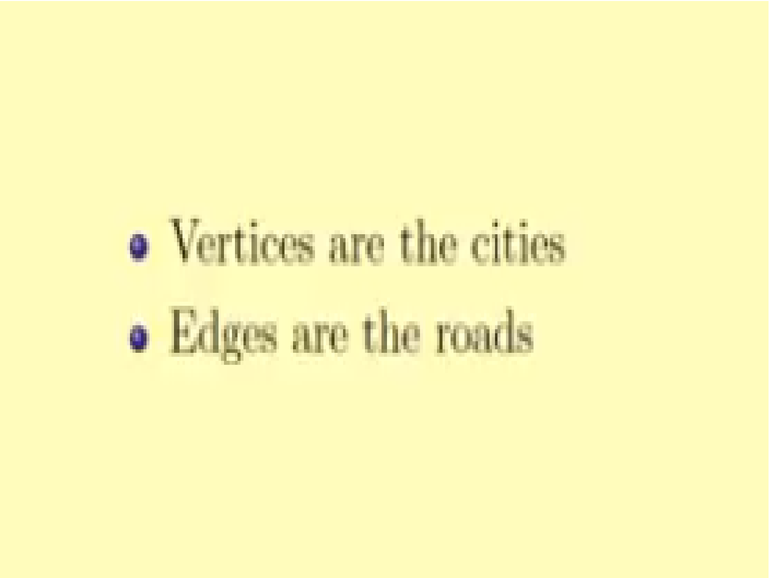

Now another way of one other values of this graph are of course road network where we have vertices are cities and edges are roads.

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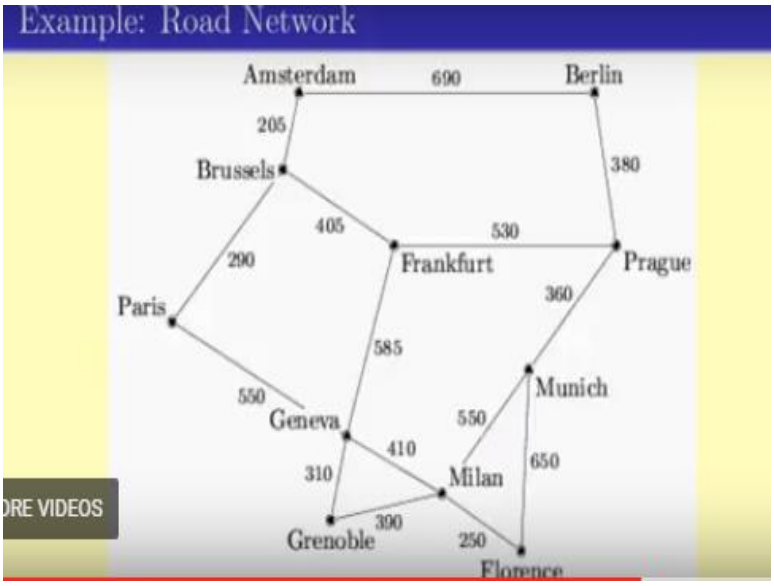

In fact we can also have numbers assigned to them, which basically indicates the number of the distance between them. We say Brussels to Paris is distance 290 kilometres, so maybe this is a road network that is there if you ever open a map you see such structure like this. So graphs are also used for designing road network, railway network and so on.

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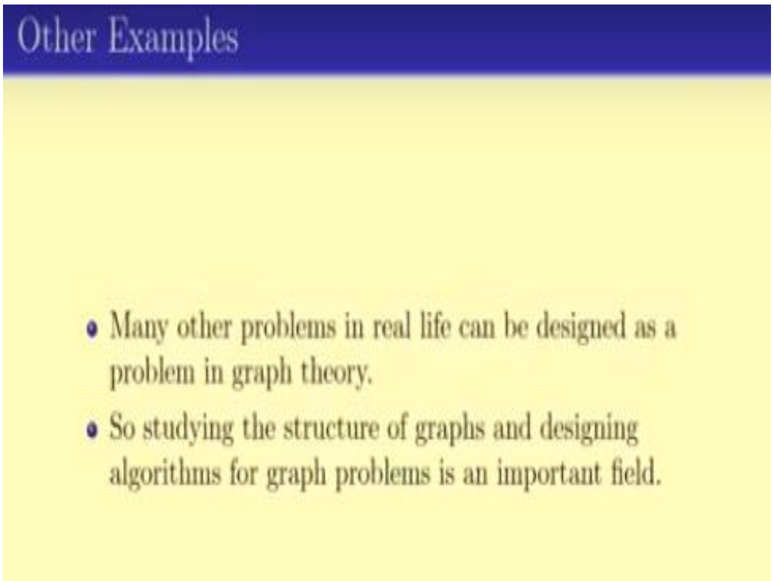

There are many other problems in real life that can be designed as problems in graph theory. Thus studying the structures of graphs and designing algorithms for graph problems is an important field. So studying the structure is what we will be focusing on this particular subject, in this particular class. Now this was all introduction to graphs, how that it helps us to solve problems.

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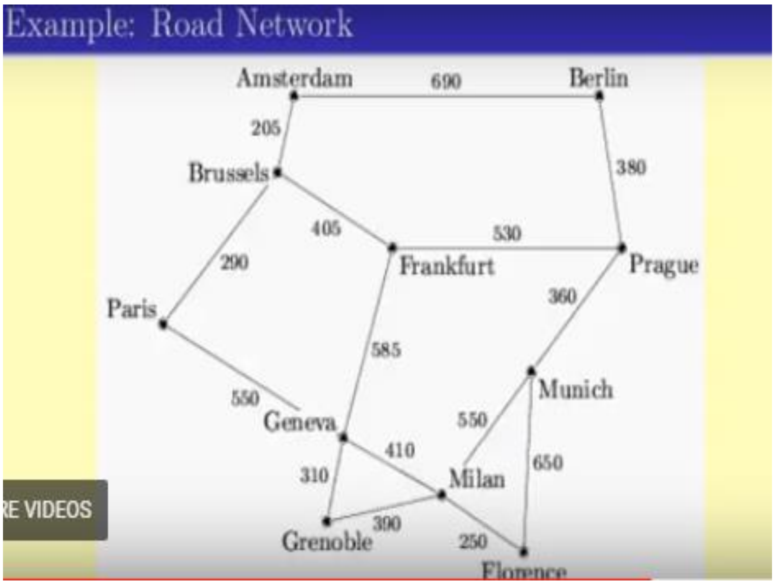

So let us see here is the problem that we have right. The first problem in a room there are 2n people, some of the people shake hand with each other in such a way that if a and b shake hand and b and c shake hand, and a and c does not shake hand, what is the maximum number of handshakes possible. Can you model this problem using graphs? It is not very hard to see what is going on.

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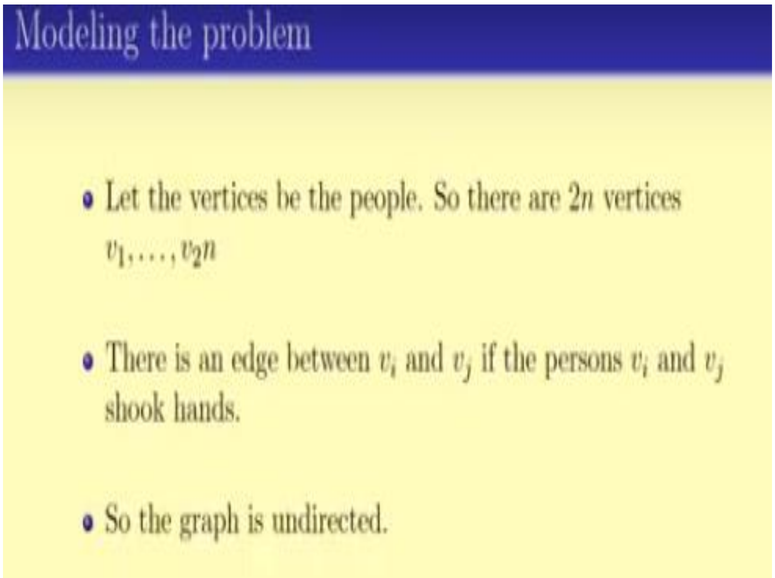

First of all, let there be 2n vertices, every vertex represents a person, so there are 2n vertices, v1, v2n and if two persons shake hand with each other, you plot an edge. So there is an edge between vi and vj if vi and vj shook hand. Now if vi shakes hand with vj, then vj must shake hand with vi. In other words, this relationship is reflexive and which means that the graph is undirected.

Now what is it we are saying about this condition that if a b shakes hand and bc shakes hand, and ac cannot shake hands. So if I see something like that, there is a vertex, they shake hand and they shake hands, and they cannot have a shaking hand here, which in other words means that there is no triangle. We call this one a triangle, right. So there is no triangle and now we are asking how many handshakes can there be?

So in other words, we are asking that how many edges can there be? So the problem reduces to a nice problem in graph theory, which says that a graph on 2n vertices without a triangle how many edges can there be. So we have converted this problem into a nice problem in graph theory.

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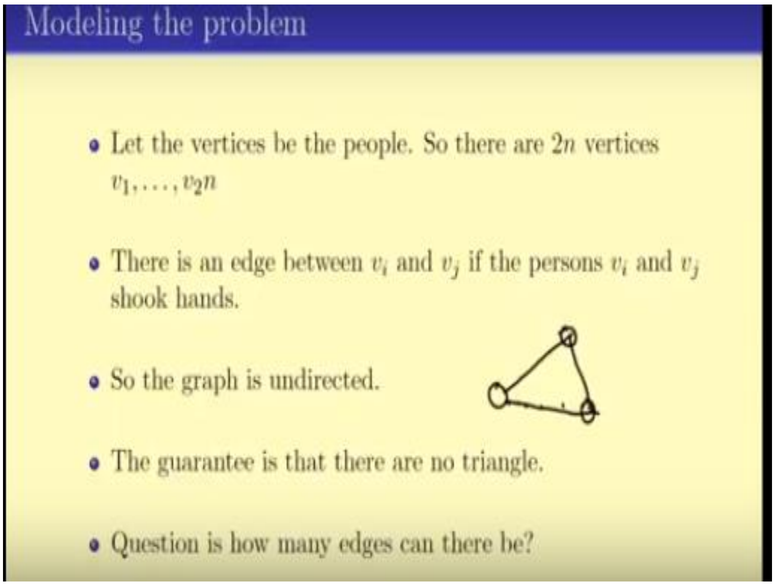

With observation, if the condition was that there was no triangles were not there, then how many edges there be. It will have all the possible edges that can there be in a graph right, which is of course n into 2, we just now saw some time ago, this n chose 2. But now we have been given the condition that there cannot be any triangle. So can we guess now how many edges can there be where there is no triangle.

Can you give me a bump, of course n chose 2 is an upper bound, but whatever something less than that. Can you come up with some example, can you come with a logarithm or other words can we create a graph, which does not have a triangle and yet has quite a number of edges. So here is one thing if I have n vertices in this vertex in this part, n vertices here and the edges are all going from left to right, okay.

These are the edges. This is something called a bipartite graph meaning. It is a graph where I have split the vertices into two parts, you name them. They are usually named as left and right. Each of them has n vertices each and the edge set can be the complete set or it can be a subset of the complete set, between any two vertex I can draw an edge, any two vertex one from left, one from right.

Note that this particular graph cannot have a triangle, will not have a triangle right. So if we had a triangle, if there is a triangle or something of this form right, something going like this, something going like this and right ankle like this, but then I have not drawn a single edge between 2 vertices L, so that cannot be possible right. So this L cannot have a triangle and how many edges are there in this graph?

There are n vertices here, all the pairs can be there, so the size of the edge can be n square. So here is a graph, which does not have a triangle, but has n square number of edges. So the answer of the question that was asked, how many edges can there be in a graph on 2n vertices without any triangle, we have got a lower bound, which is the n square. We have taken the n square.

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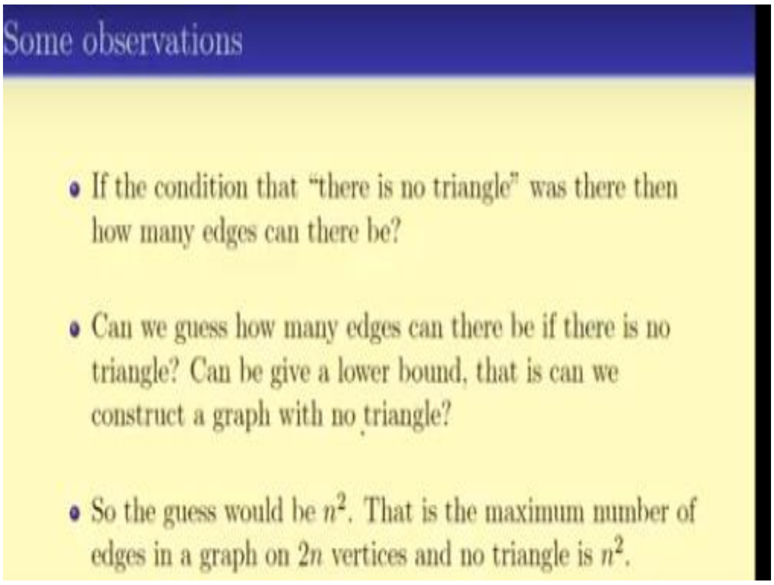

Now, it is told the n square is the right answer, but can we prove that? That is the question, how do we prove that a graph on 2n vertices without a triangle can have at most n square edges and that it is tried. So the guess would be n square that is the maximum number of edges in a graph of 2n vertices with no triangle is n square. Can you prove that and that is what we will be doing in the next class.

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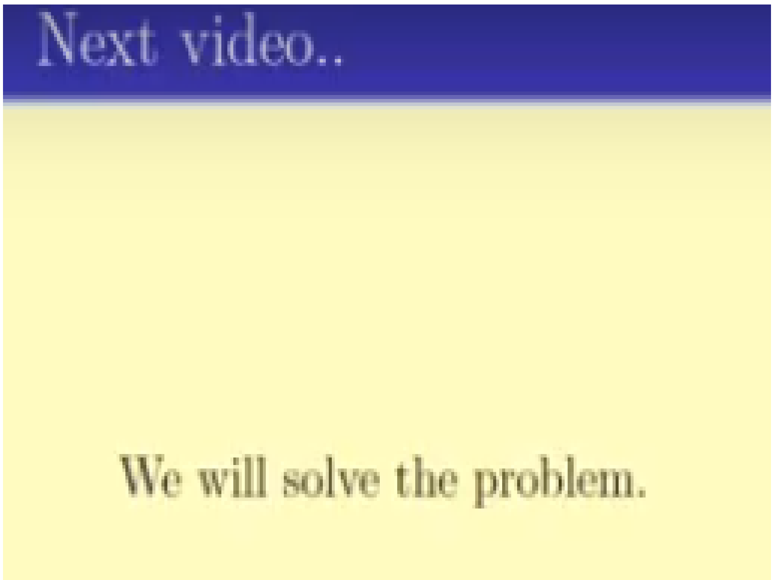

In the next video lecture, we will solve this problem and also we will see how to represent other problems using graph theory. Thank you.