Discrete Mathematics
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Lecture - 21
Mathematical Induction (Part 8)

Welcome to the fourth video lecture in week 5. So we have been looking at induction formulas, two weeks.

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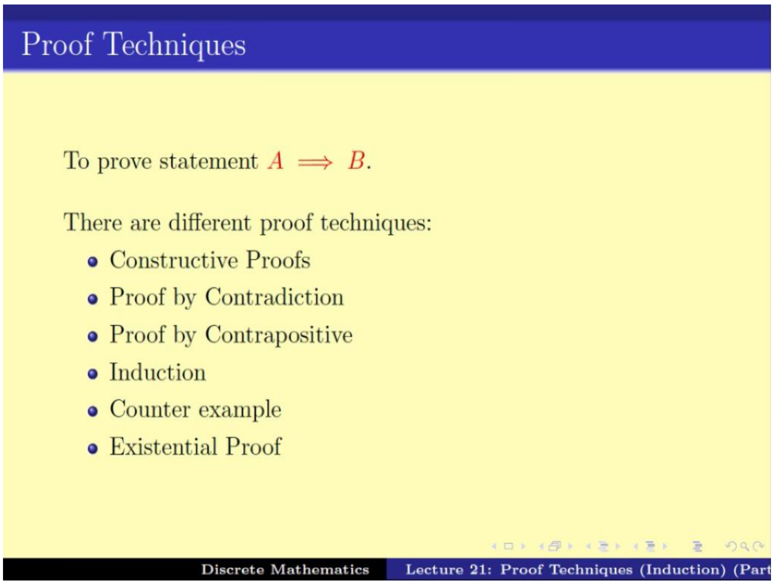

So before that we looked at various other proof techniques, how to prove a statement like A implies B using constructive proof or prediction, contrapositive and compared level. But for certain problems in discrete mathematics proving it by induction is a bit useful at any approach.

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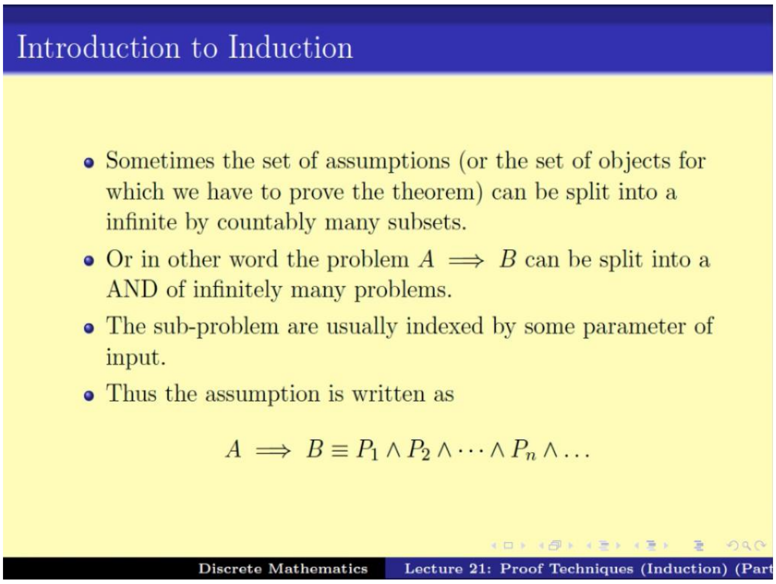

So the main idea is that, one can stitch up the assumptions into possibly infinite number of subsets. This implies that, the problem A implies B can split into AND of infinitely many problems. They are usually parameterized by some parameter of the input.

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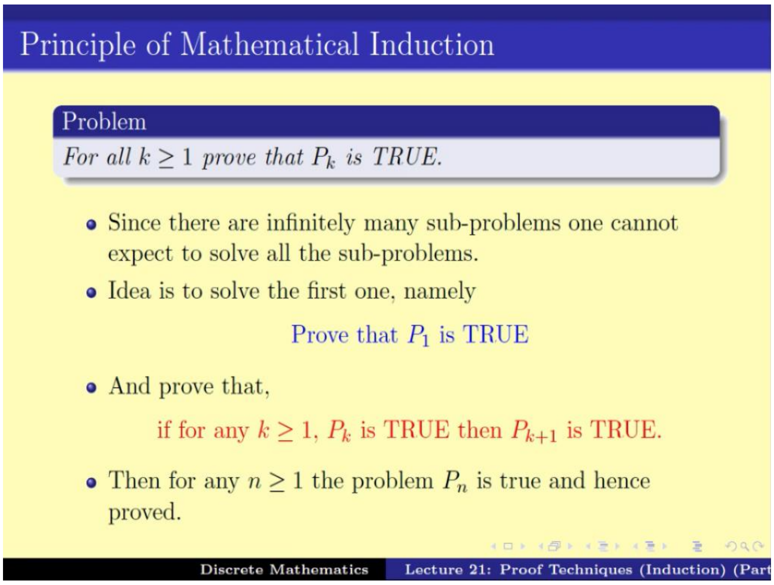

So the problem A implies B is same as P1 AND P2 AND so on till infinity. So this problem A implies B thus becomes for all k prove Pk is TRUE. One cannot go about proving all the problems individually because there are infinitely many of them. So one have to come up with a clearer way of solving it. The idea is that first solve the first case namely P1 and then solve that for all k if you can solve Pk is TRUE then solve Pk plus one is TRUE.

By doing so, you will be able to hopefully prove for all n Pn is TRUE because you have proved P1 is TRUE and one is TRUE therefore two is TRUE, the two is TRUE therefore three is TRUE and so on So if you go on like that you will end up proving Pn is TRUE for all n greater than or equal to one.

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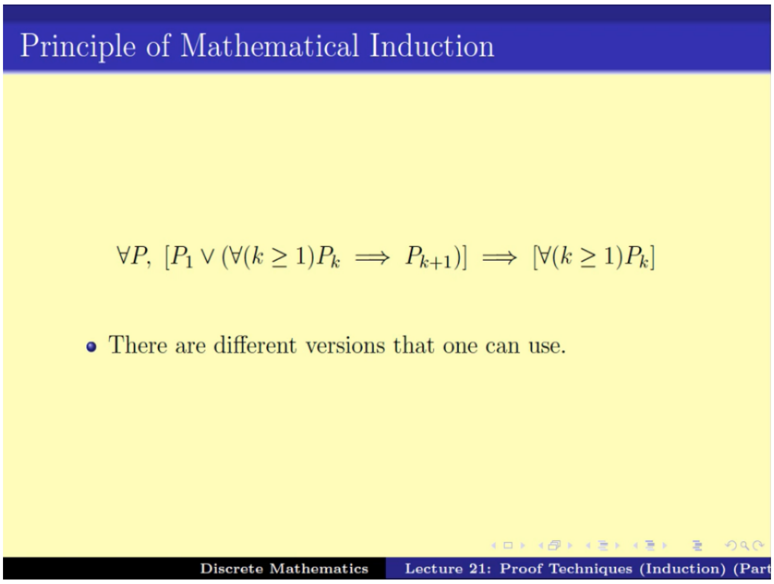

So the fact that by doing so we will be able to proof for all n is guaranteed by what is known as principle of mathematical induction which basically states that proving for all k greater than or equal to one Pk it is sufficient to prove P1 and then to prove that all k greater than or equal to one, Pk implies Pk plus one. Now this is the most basic form of principle of mathematical induction.

You can get quite number of different versions of this. And depending on the problem we might have to use the right version of it. So we have gone over few of the versions. Let me go over them again.

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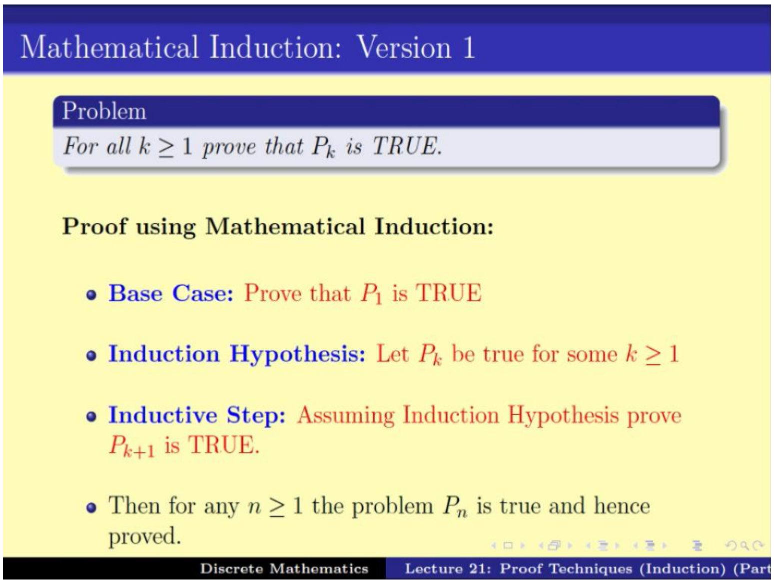

To start with and the first version, the basic version, which says that prove K is greater than equal to one Pk is TRUE. We have the base case which is proving that P1 is TRUE. Induction Hypothesis says that Pk is TRUE for some k greater than or equal to one and Inductive Step says that assuming Induction Hypothesis prove Pk plus one is TRUE. In version two, basically says that we can change the base case to some other number r and that would imply that for all k greater than or equal to r, Pk is TRUE.

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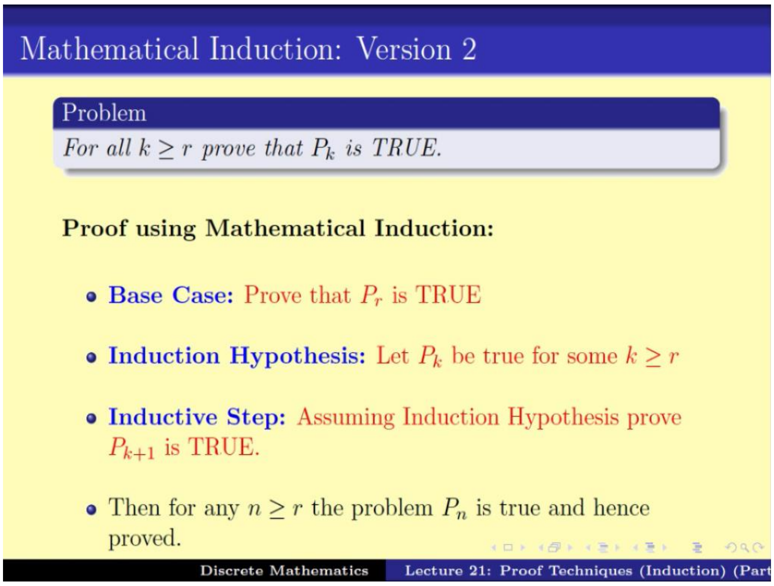

So for proving that k is greater than or equal to r, Pk is TRUE, we have to start with the base case with Pr and then the remaining same will rule. But sometimes proving the inductive step that is proving Pk is TRUE implies Pk plus one is TRUE can be copied here. One might have to prove Pk is TRUE therefore Pk plus two is TRUE or something like that.

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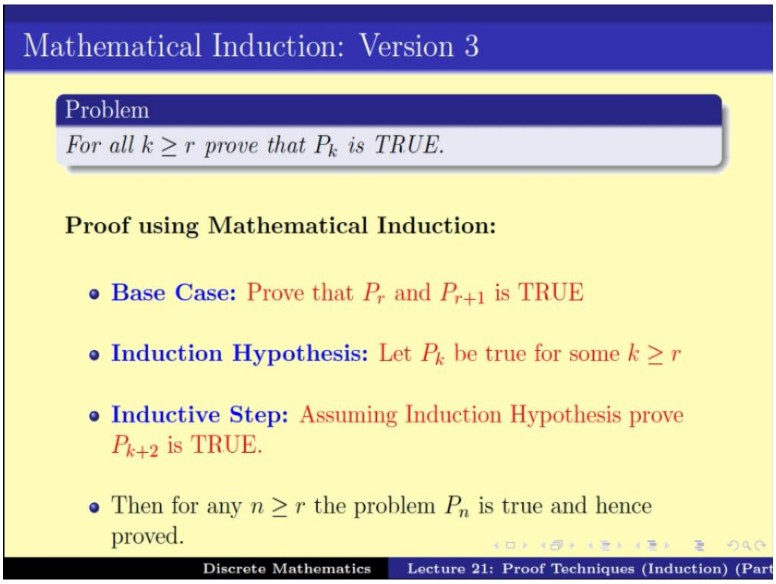

So in that k we have to come up with the new version of that. So we are having one of them that prove Pr and Pr plus one is TRUE and then assuming Pk is TRUE prove Pk plus two is TRUE. Now for all them, and they were repeating many times in the last few lectures, you should ensure that all the cases are satisfying. As long as you can mention that all these cases are covered, it is fine.

You can convince yourself that all the all the various versions ensures that. So if you can assure that all the cases are covered then we had the proof of the whole problem.

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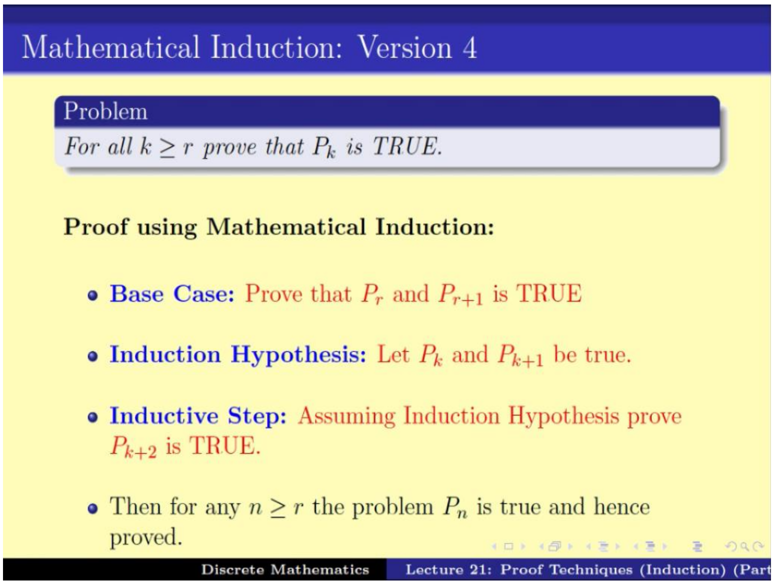

So we also have another version which states that if we cannot prove that Pk implies Pk plus two, but maybe we can prove it that if Pk and Pk plus one is TRUE then Pk plus two is TRUE then also we get the same result. So these are defined versions and depending on the problem or depending on what can be proved within the Inductive step we have to choose our defined parts here.

The problem can more than one parameter and in that case we might have to induct on multiple parameters and again there are quite number of front versions that one can apply. Here also since we are dealing with multiple parameters, the goal is to ensure that all the possible points are offered within this case in the two dimensional grid. So in other words, if you have to prove that for all p, q, the problem parameters like p, q is TRUE.

There is various way of doing it, so one of them is say prove that one or q is TRUE and as you will probably will prove for the parameter p, q prove that, prove it for p plus one or q. Note that this will ensure that all the points is equidimensional grid is covered and hence we have the full answer, full proof. We can also define versions here for example we can start with the different base here, that means, one, q, and p, one are both sort of base cases.

And then if p, q is TRUE then p plus and q plus one is TRUE. It will also be a sufficient condition for it, or as we saw in another example. We can also induct on p plus q, that is p plus q less than equal to k is TRUE then they can prove that for all p prime, q prime but p prime plus q prime is k plus one is TRUE then we have the whole problem again.

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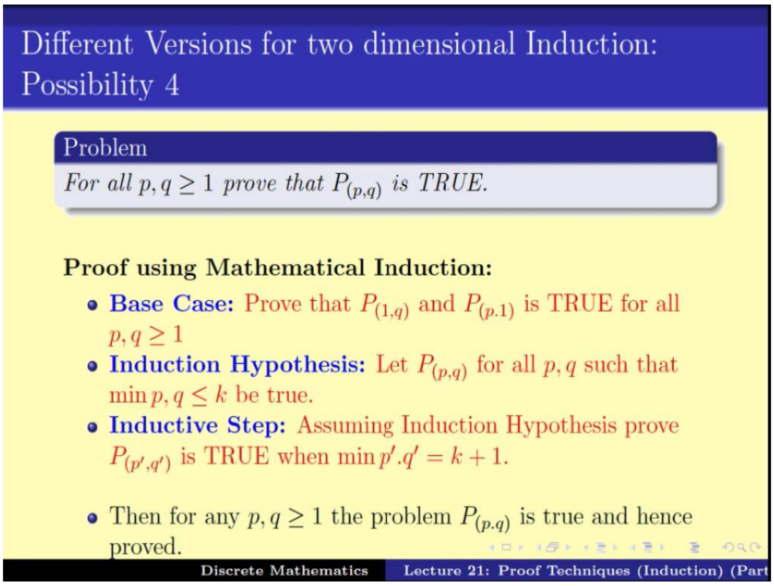

One more example that may have been going on, what they are telling is the, you feel that minimum of p comma q and so on. So it is clear that there are quite number of versions and the versions can be, you can yourselves change the version, you can also choose the different version.

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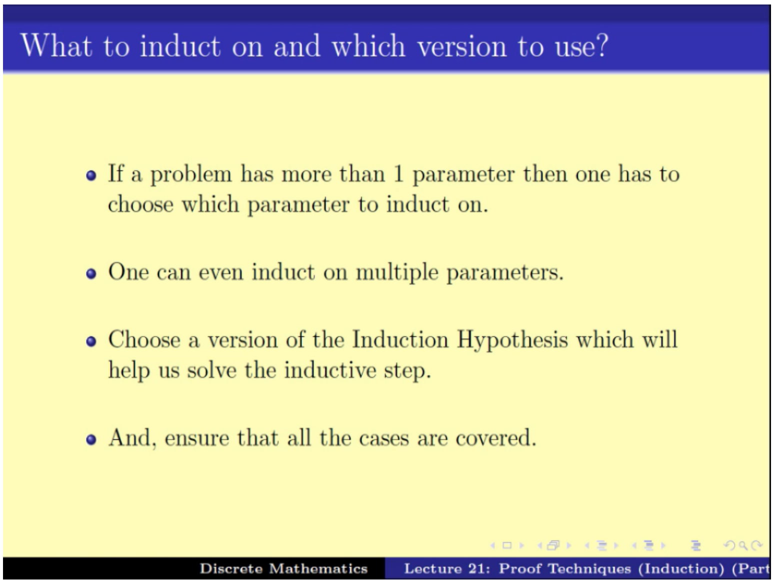

All that you have to do is that, you have to choose the Inductive Hypothesis which will help us prove the Inductive Step and ensure that all the cases are covered. In the last video we saw a very interesting application of this induction where we could solve the AM-GM Inequality which was clearly a bit complicated problem and solved it using applying an induction version. If we deal we can have at that time, we call it the backward induction.

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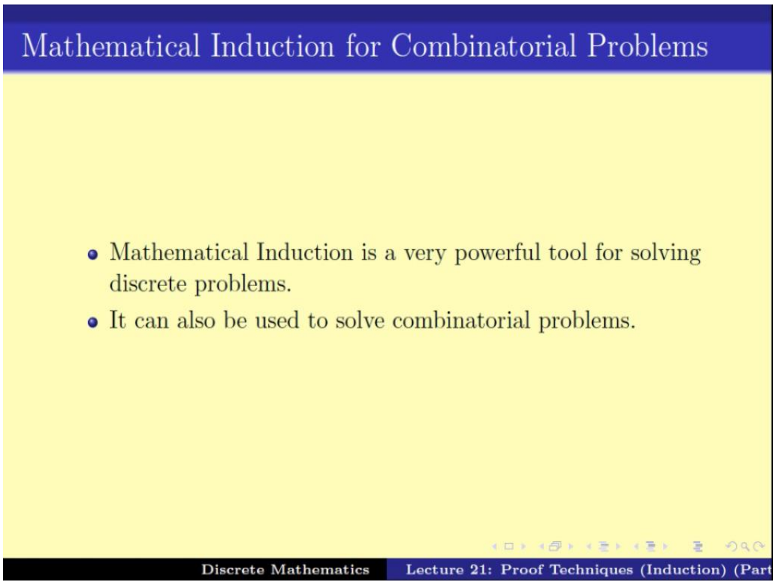

Now this mathematical induction would also be useful for solving, not only the discrete problems like number three problems and so on, I can also be used to solve combinatorial problems. What were really combinatorial problems? Problems which deals with combinatorial objects, not just numbers.

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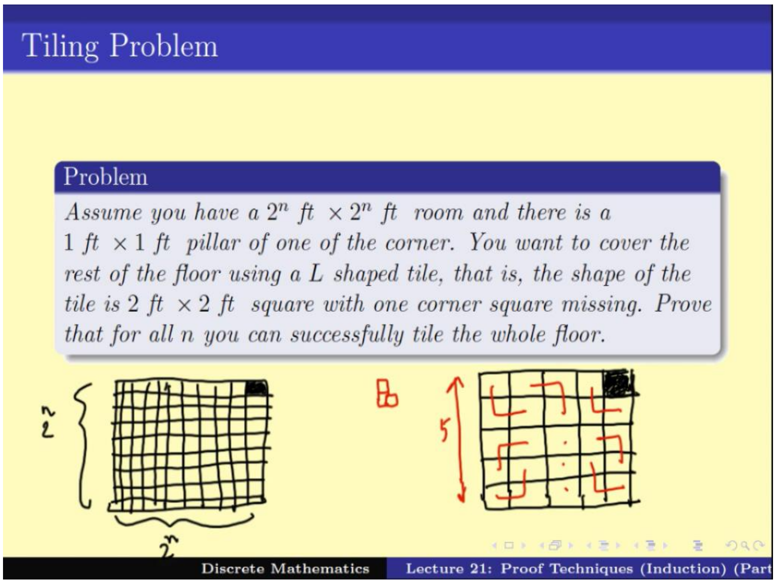

So here is that simple problem. So you have two power n plus two power n room with a pillar in one corner. So let me draw the room which is something like this. The dimension of this one is, this is two power n, dimension of this one is also same two power n and of course this is broken up into various kind of tiles or grids kind of, right? Something like this. is floor and is drawing an oblique structure over here and there is a pillar at one of the corners.

That means this corner is covered with a pillar. So you cannot move anything with the corners. So your room looks like this. It is almost a square, except that one corner, one square foot is not. Now you have a tile. A tile is basically something of this form. We have one cross, one cross, one cross tile. An L shaped tile that you have and the question is that, can you use this particular tile to cover the whole floor.

You are not allowed to break the tile; you are not allowed to leave any space of the floor uncovered. So you have to use exactly that many number of tiles. Above that for all n you can successful tile before this one. Okay, so let us do some quick checks now. First of all a tile covers three squares, right? And how many squares are there in the whole room? It is two power n times two power n minus one which is four power n minus one.

Now if I have to cover the whole room using this L shaped tile, I have to ensure that, this must happen that four must divide, 4n, sheet must divide 4n minus one. Now is this true?

Now I will leave it you guys as an exercise to prove that, so prove this statement that three does divide 4n minus one, divided does divide by 4n minus one. So it is possible that maybe we can use this grey colored L shaped tile to cover the whole floor.

But just because three divides 4n minus one, it is not necessary that we would be able to cover the whole floor, right? Say for example, let us take a quick example here. If instead say I have, instead of this two power n, two power n, I have a five plus five, so, one…If this is my room that they have and the corner room, corner through this corner is not there, this is a five plus five floor, five feet multiplied by five feet and this one plus one tile.

And it is even checked that there are twenty-four available squares, three divides twenty-four, but if I want to tile this whole space using three cross one tile just (()) (13:52) we are going to put one here and we put one here, put one here, maybe I can put maybe one here, put one here. I can put one here but then see this three here, I cannot because the size of the shape L I will not be able to cover this three things, right?

Even I have to cover these three spaces using this L shaped tile, I have to break this tile and that is not allowed. So it is obvious that, in fact it is clear from the example that, just if the number of space allowed, available is divisible by three, you can divide by, you can tile it using the extra tile.

The problem says that if the, instead of five which is not two power n, it is not a power of n it is five**,** instead of that they have two power n, plus two power n size room, then we will be able to solve tile using this particular time, right? So how do we do it? So we have to of course give a tiling for that. We have to show that a particular tiling exists. So we have to prove this particular problem in the industry.

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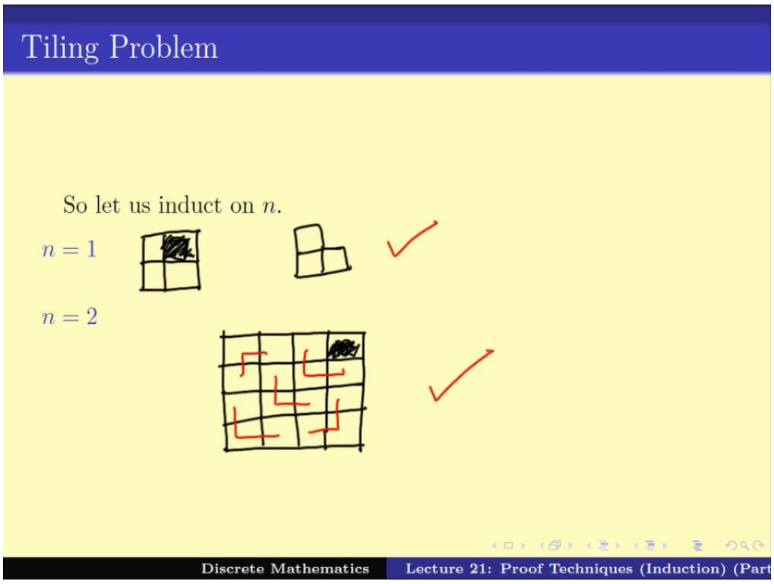

Of course to start with, let us look at through some simple cases, n equals to one. Now n equals to one that means two power n is two, that means the floor looks like a two plus two feet floor with one of the corner not available, like this is not there and I have tile which is actually, exactly the size of the floor. So if n equals to one then the tile is itself this shape and size of the floor.

So I can just take this tile and put it there and therefore n equals to one is good, right? How about this n equals to two, meaning two power two is four. So I have this four by four shaped flow with this corner not there and how do I use this one tile? Can I tile it? Now there can be quite a number of ways of tiling it possibly. For example, maybe I might be able to plan it like this, no, it will not work. It is kind of… Clear, then I will not be able to tile this one.

So another example, let us see, if that make another I can prove, tile it. Like this one you got there. So maybe what I can do is that, again tile this like this. Okay, that looks interesting, so I am able to tile it like this. So the n equals to two case also can be tiled, right? I tile it using five of this type, no two of them are what have been and I have covered the full space. Now this is for the case of n equals to two.

Can I do anything about anything about anything else, n equals to for general length?

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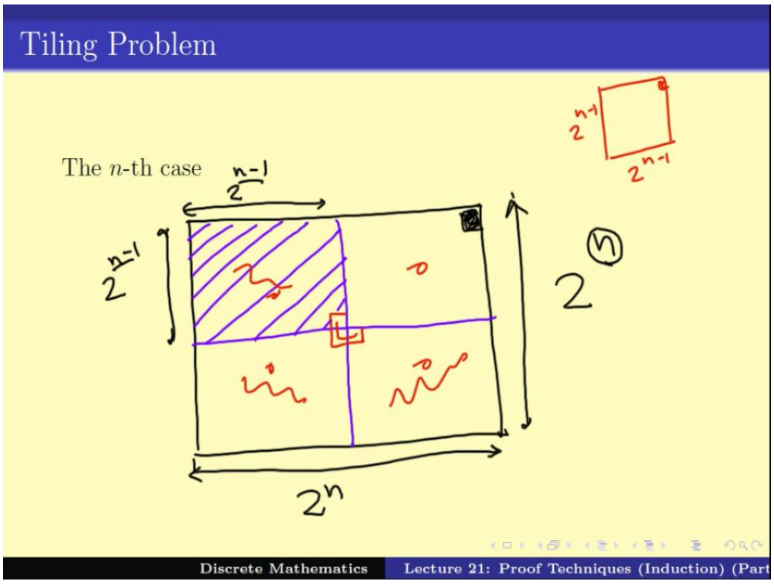

See, I am getting of course, since n is the only integer here that we have, we can induct on n. And if this induct on n, how does it look like. So here is the big floor that I have. This shape is two power n and this shape is again two power n and the corner square is not there. How do I use as different case. Now, okay, one technique to is to let us split up this whole floor into two parts or at the most to four parts. Split one in the middle and this way.

Now what do I have? Now each of them is of the size two power n minus one, because I have split into two parts into two power n minus one. Thus I have broken the bigger instance into a smaller instance. I have taken the floor when the floor is of size two power n when broken up into two; the case for the floor is of size two power n minus one. And now, if I use induction, further if I assume that the floor of size 2n minus one times 2n minus one with one of the corner is missing can be tiled.

Then can I view the tile? This particular example, or the nth case. Now here what we can do is that, we can put a tile, simple tile at the corner here. Now if I put it at the corner like this one, knows that the remaining, that means this space or the shaded space that I am drawing is a 2n minus one plus 2n minus one square with the corner missing. And this gives us the idea for the induction.

Now if using induction, we can say, or the induction hypothesis says that, for case n minus one I can tile using some way, so there is some of tile, this particular thing, there is some way of tiling this particular thing and there is some way of tiling this particular thing. Note that all the three squares and this square are all basically the same structure, meaning 2 power n minus one, two power n minus one with one square missing.

All of them are exactly of this structure, just rotating. So if I know how to tile with 2n minus one plus 2n minus one floor, then I will be able to use it to tile two power n floor. So in other words using the n minus one case I can solve the nth case. So the idea is that I take the instance, the problem instance of the n and break it down into smaller instance which is n minus one instances and I can use the induction hypothesis now to solve it, right?

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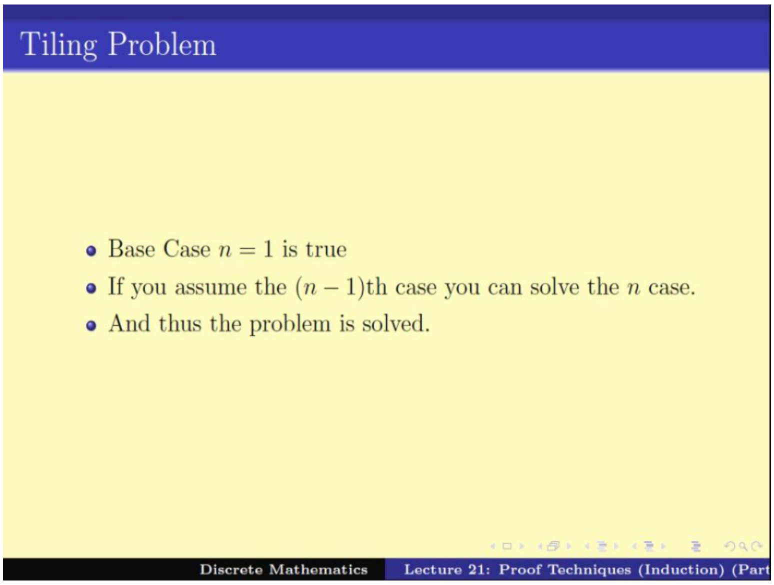

So what will we get? So it has that the base case n equals to one, we can solve it trivially and if we assume that n minus one case, we can solve the nth case, recall this is exactly the version one. We are applying the version one to the first one of the induction through this particular combinatorial problem and using that we have solved the whole problem. So this is a typical application of induction on combinatorial objects.

The idea is take the combinatorial objects with all the n and somehow break it down or reduce it to smaller instances.

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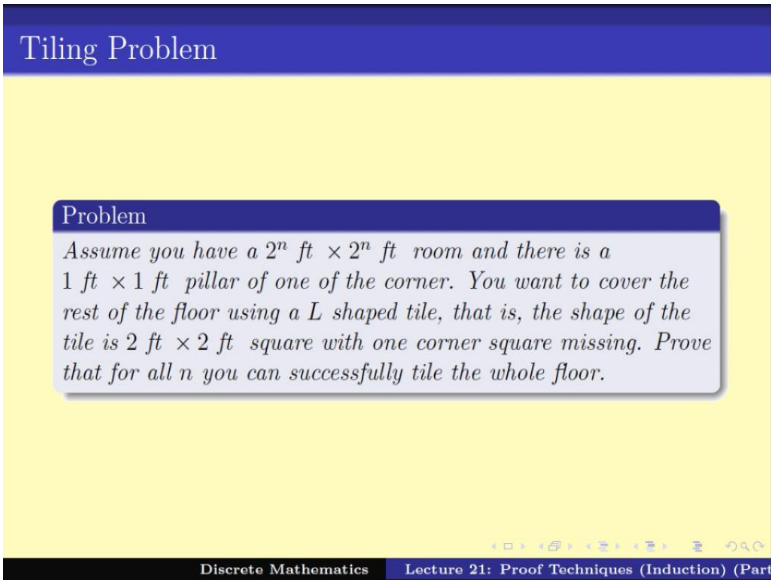

So here we have to use the induction hypothesis to prove the tiling problem. In the coming weeks we will be seeing lot more of the combinatorial objects and how to solve many of the problems using induction. So for that we will need to come up with a different language of talking about combinatorial object and gives that is what we will be calling graphs. So next week we will be introducing graph theory and them for lot of problems in graph theory. Thank you.