Discrete Mathematics
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Lecture - 19
Mathematical Induction (Part 6)

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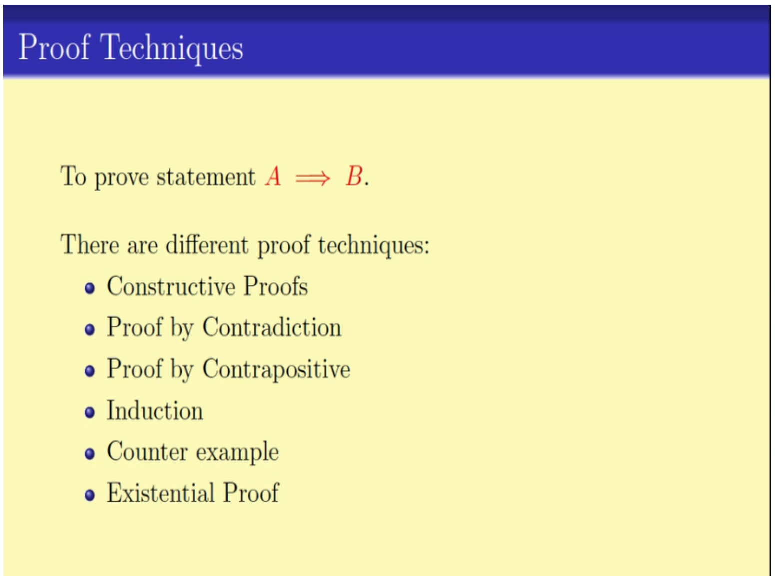

Welcome to the second video lecture in week 5 of Discrete Mathematics. We have been looking at the proof technique called induction. Till now, we have seen a quite a number of different proof techniques namely, constructive proofs, proof by contradiction, proof by contra positive, counter example and induction.

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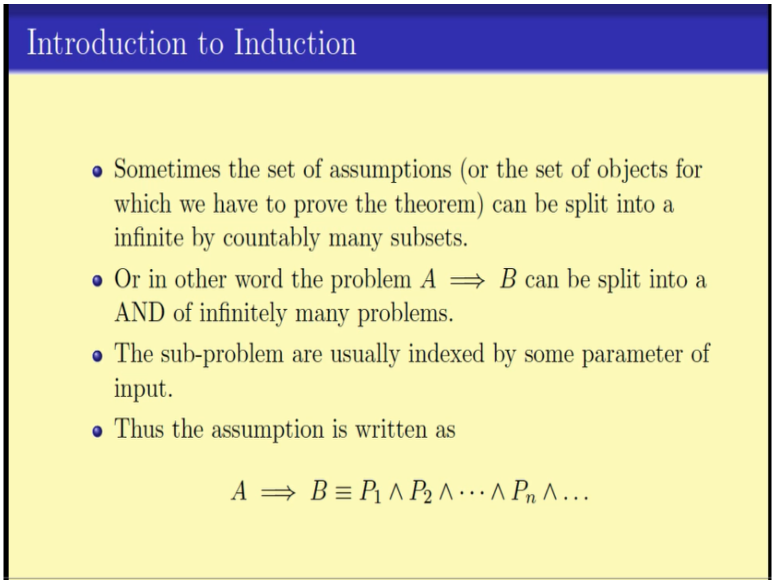

Last week and this week, we have been looking at this induction. Now what is induction? The main idea is that sometimes the set of assumptions can be split up into infinite by countably many subsets. So, that in turn is that the problem, which is A implies B can be split up into and of infinitely many problems. Each of this sub-problem are usually indexed by some parameter of input and we like the problem A implies B as P1 and P2 and so on till Pn (()) (01:19).

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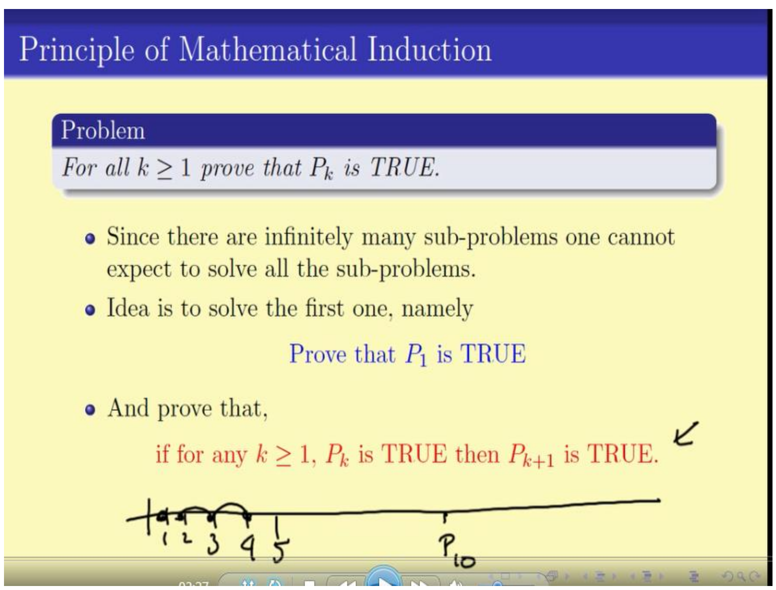

Now this problem, A implies B becomes this problem, for all k greater than or equal to 1 prove that P k is true. Then of course one obvious way of solving them, let me first prove P 1 is true, then prove P 2 is true, then prove P 2 is true, and so on and so forth. Unfortunately, since there are infinitely many sub-problems, one cannot expect to solve all the sub-problems. So, we have to come up with some cleverer way of solving the infinite cases in one (()) (02:08).

The idea is first prove, P 1 is true and you mean that you can prove P k is true, prove that P k plus 1 true. So, it is like, if I write down all the integers or natural numbers 1, 2, 3, 4, 5 and so on in this number line. This says the first prove P 1 is true, so I proved that P 1 is true. Now, the second set, the first line says that, if I can, if I know P k is true then I know P k plus 1 is true. Since I know P 1 is true, therefore P 1 plus 1 is P 2 is true.

Since I know P2 is true, therefore I know P 3 is true.Since I know P 3 is true, so I know P 4 is true. So, this way, the idea is that I will be able to prove anything further if you tell me can I prove P 10 is true. Yes, I keep on continuing like this.P 1 is true, therefore P 2 is true, therefore P 3 is true, therefore P 4 is true, therefore P 5 is true, therefore P 6 is true, therefore P 7 is true, therefore P 8 is true, therefore P 9 is true, and therefore P 10 is true.

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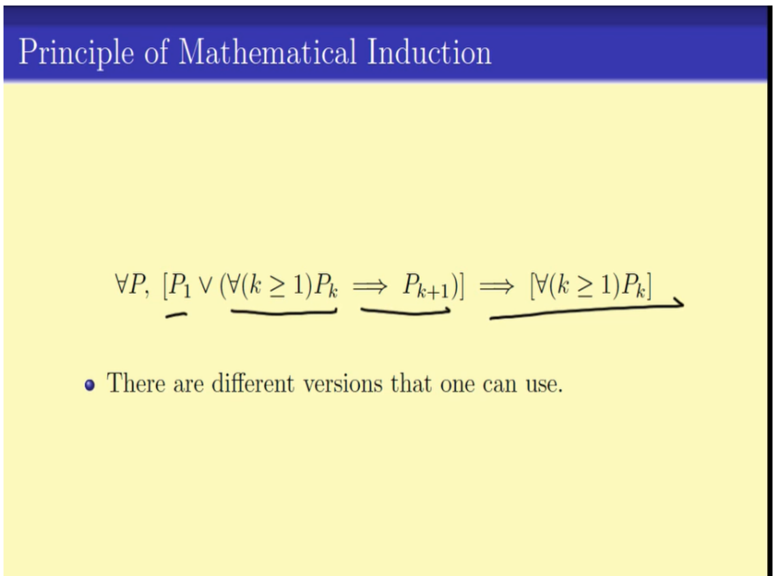

So this way, I will be able prove that for all n greater than or equal to 1, the problem P n is true and hence proved. Unfortunately, this does not follow, this proof technique does not follow from the usual rules of propositional logic. We need a particular action for it, which says that this proof technique is actually correct.

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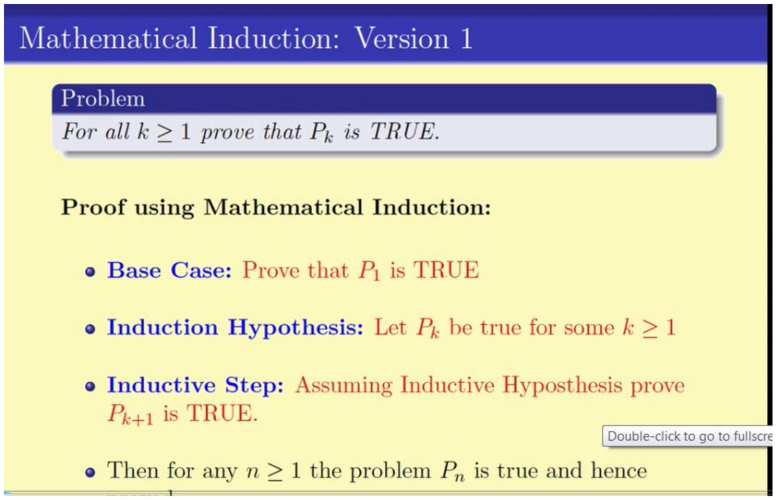

We call this one the principle of mathematical induction and this is what it says that if you solve it accodingly then if you solve P 1 is true and then for all case you can solve, P k is true implies P k plus 1 is true then you end up proving that P k is true for all k greater than equal to 1. So, this is very useful proof technique to have. We have already seen quite a number of applications of this proof technique.

And we will keep on seeing more and more interesting proof techniques using, using interesting problem solving using this proof technique.

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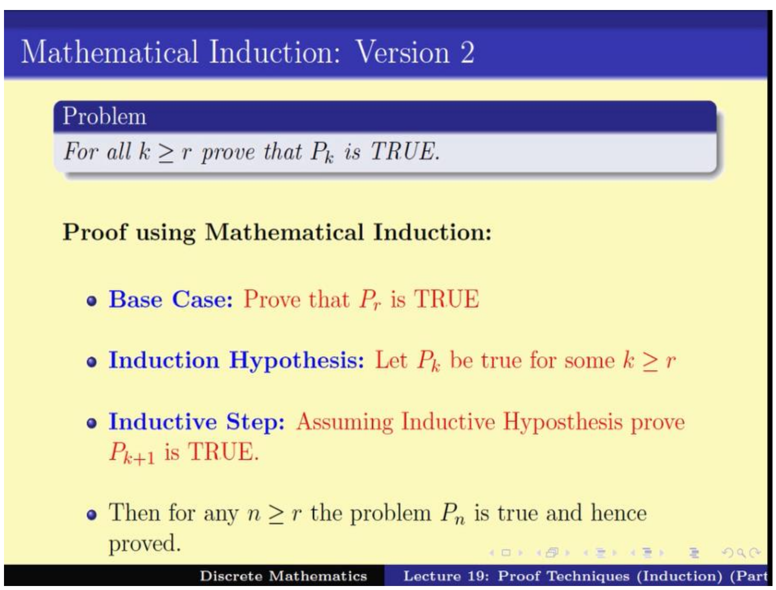

So this mathematical induction has different versions of it and we might be interested in applying some one of the other one. The version one is of course what we just now solved, which is the, if

I have to prove that for all k greater than equal to 1, P k is true. The idea is prove P 1 is true then assuming P k is true, prove P k plus 1 is true and if you can solve then then you will solve, solve the whole problem.

The version 2 is a similar version except that the change shift the base case.So, if you have to prove that the statement is true for k greater than equal to r. We start with the base case of P r is true then assuming the induction hypothesis, we prove that P k plus 1 is true and that will help us to solve all problem.

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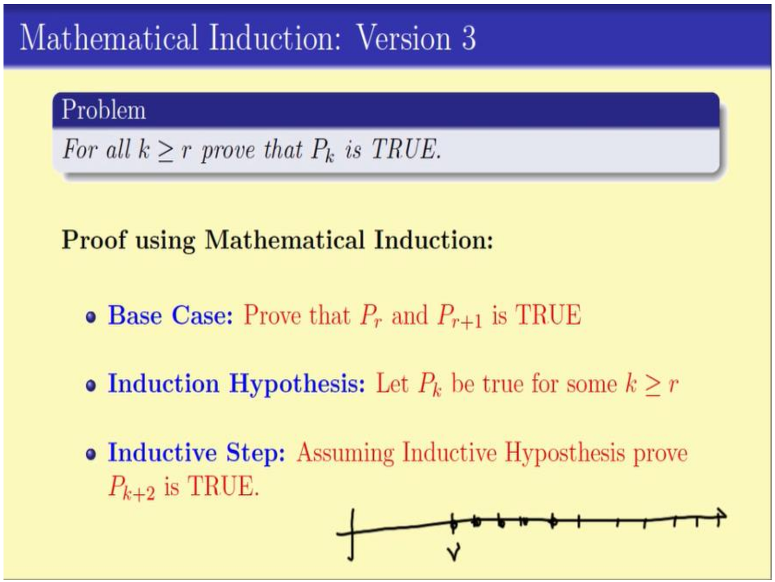

In version 3, which also solve the same problem, but we have a different inductive step and hence a different base case.So here, we have to again prove that for all k greater than equal to r, P k is true. The way to prove it is, first prove P r and P r plus 1 is true and then assuming that P k is to true, prove that P k plus 2 is true. Again I repeat, all the versions just ensure that if n r, all the integrals greater than r are covered by this step.

If I follow the step, so this says that okay, let me start with P r, it is true and P r plus 1 is true this base case. In that base case, it says that if k is true then k plus 2 is true. So, r is true therefore r plus 2 is true. If r plus 1 is true therefore r plus 3 is true. If now r plus 2 is true therefore r plus four is true and so on and so forth.We can again even see ensure that all the points greater than r are covered by this step.

Now one might of course ask, why to use different versions, it all depends on the problem. For some problem, proving P k plus 1 is true assuming P k is true as easy whereas for some problem proving P k plus 2 is true assuming P k is true (()) (07:33). So different problems give us different easiness of approving a particular step and depending on the inductive step that we can solve easily or base case changes or version that we apply of the induction mathematical induction changes.

So, this was in version 3 and the version 3 of 4 says if you can prove that k is true and k+2 is true with the base scale P r and P r plus 1, we have the problem is true for all n greater than or equal to r.

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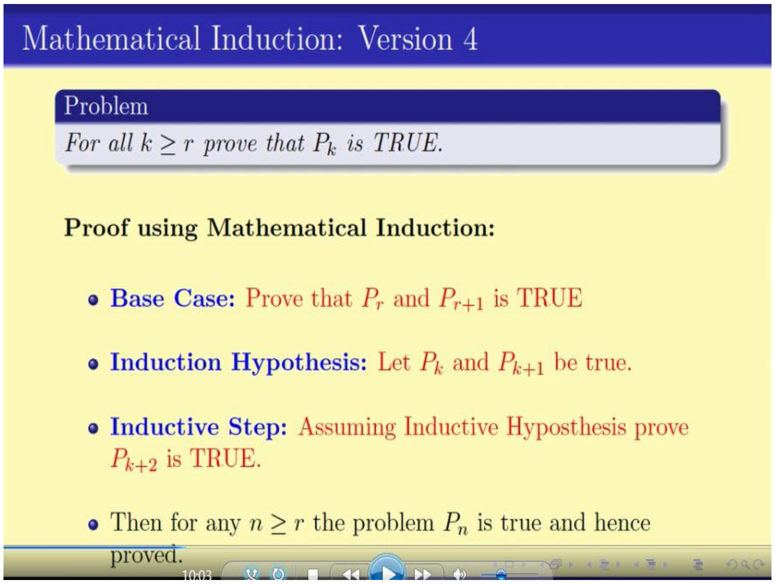

In the last video lecture, we saw another version, this problem is again same, the base case is also same. Except in the induction hypothesis, we have to assume both P k and P k plus 1 is true and assuming that we have to prove P k plus 2 is true. Again the reason, if that there are problems as we saw in last class or last video lecture that proving that P k is true requires us to assume something of P k and P k plus 1 is true.

So, there are problems, but this particular induction step is easy to prove and then accordingly you choose the correct version of mathematical induction. Now can be many other versions that you can think of problems, for every problem there can be a different version required and as long as you ensure that the basic concept that all the points are covered, you can apply that induction hypothesis or in the mathematical induction logic.

Now, in all of them you assume that we induct on k in the same fluffs k was increasing, so we kind of kept on solving for induct on k that means we prove for k or k plus 1 then we prove k plus 2 or k then P plus k plus 1 and so on.

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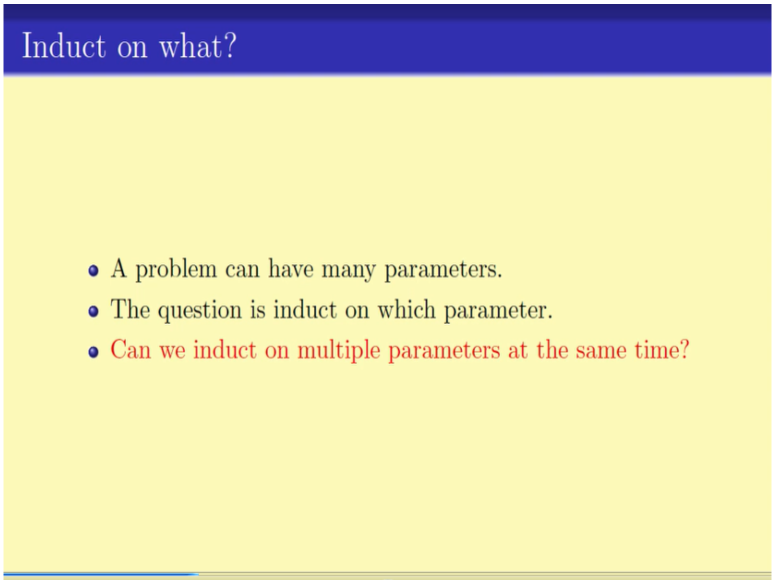

But sometimes, we might have problems where there are multiple videos, so there can be multiple parameters in that case what parameter, which parameter to induct on? In fact, can you induct on multiple parameters at the same time?

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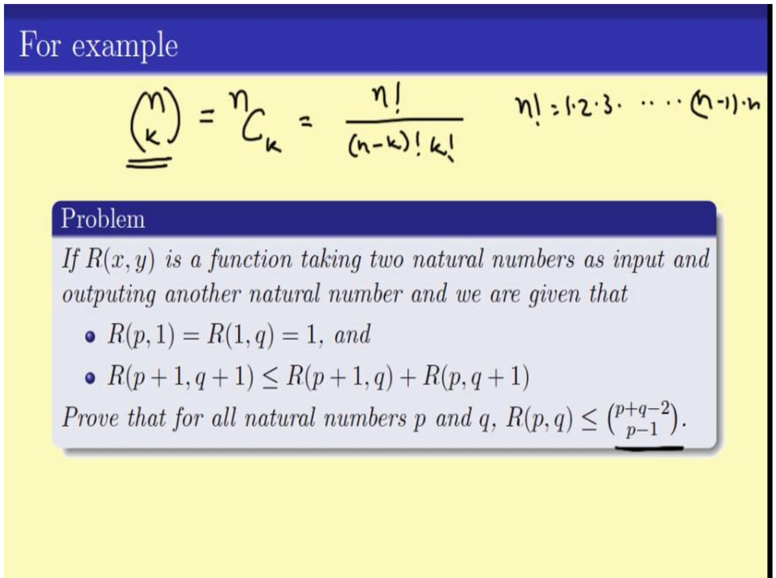

So let us look at an example. So here is an example, so let R x y be a function, which takes to natural number as input and outputs are natural number. The rule is that, R p comma 1 is equal to R 1 comma p equals to 1 for R p and q in natural numbers and r p plus 1 or q plus 1 is less than or equal to R p plus 1 q plus R p q plus 1. If this is the case, then can you prove that for all natural numbers p and q, R p q is less than p plus q minus 2 to choose p minus 1.

So this is the binomial term, so let me just quickly write down here what the binomial term is. In case if I have n and k this is n k, do we call this on n through k sometimes it is denoted as n c k as we always use the first notation, this notation. So, this is n factorial divided by n minus k factorial times k factorial.And what is in this number factorial? So n factorial is of course 1 times, 2 times, 3 till n, n minus 1 times n.

So that is n minus 1 factorial, similarly k factorial and minus 2 factorial. So, the problem of R is that, can you prove that R p q is less than p plus q minus 2 choose p minus 1? Now as you can see here there are two different parameters here, p and q. Now which one do you induct on? On the other words, which one do you use to split up the problem in two cases?

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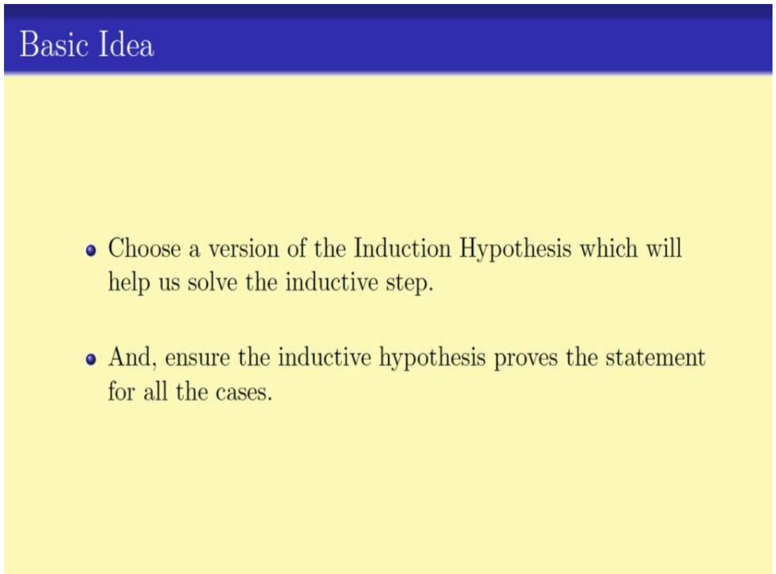

Now for solving any problem in induction, the basic idea is choose a version of the induction hypothesis or induction version, which will help us to solve the inductive step.Number one and we also will ensure that inductive step proves all the cases, right.

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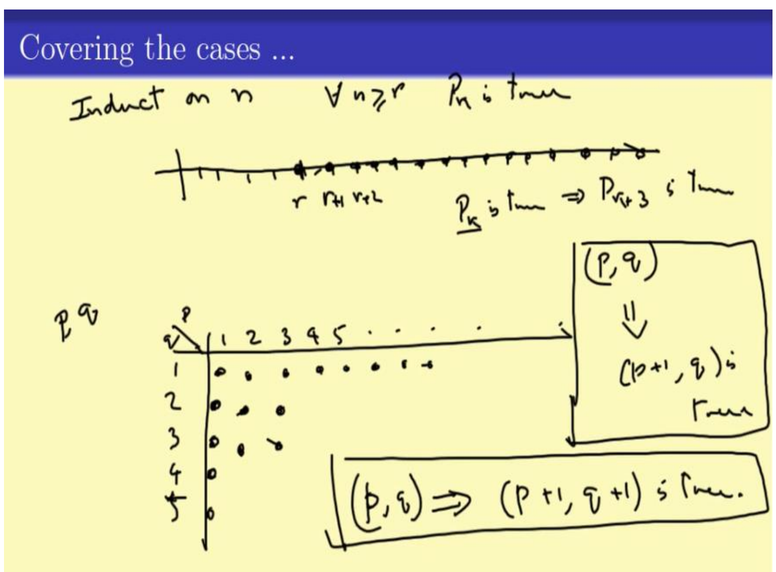

So, what do I mean by covering all the cases. Till now we have only induction on a particular number, say n right. So in that case, as I told you again and again, it is basically ensuring that for all the integer values greater than say induct on r and I have to prove that for all n greater than equal to r, P r is true, your P n is true. How do you do it? You start with the rth step, you somehow ensure that r is true.

And then somehow ensure that r plus 1 is true and somehow ensure that r plus 2 is true and so on. So, one technique is of course r is true then r plus 1 is true and in other case 2 and k plus 1 is true, then r is true, then r plus is true, then r plus 2 is true, then r plus 3 is true and so on. Or I can also have things of this much completed thing that P r is true implying P r plus 3 is true.

And if I have the base case that r is true, r plus 1 is true, and r plus 2 is true then this will say that okay, r is true, therefore r plus 3 is true, r plus 1 is true, therefore, so this is the k actually.So P r plus 1 is true, therefore r plus 4 is true, and so on and you kind of convince yourself that you will be able to fill up all the dots on the right side of r. But now, when I have two parameters p and q, it is like instead of the particular in I have agreed for p and q, right?

So p can be 1, 2, 3, 4, 5 and so on, q can be 1, 2, 3, 4, 5 and so on. Now I have to somehow come up with a version so that there is a case to solve. Let us look at some of the examples, maybe I can of course I would prove that p equals to 1 and q be anything is true. So p equals 1, q be anything is true, so I managed to prove all this point, right. And then I will end up proving that if p comma q is true then p plus 1 q is true and you think end up proving this statement and that means that okay.

This is true 1, 1 is true therefore 2 1 is true.2 1 is true therefore 3 1 is true, 3 1 is true therefore 4 1 is true and so on or similarly 1 2 is true therefore 2 2 is true, 2 2 is true therefore 3 2 is true and 3 2 is true therefore 4 2 is true and 4. So this can be one particular way of ensuring that everything gets filled up. Another way we can do that, we start with the assumption that all the p 1 for 1 comma p is true for all p and q comma 1 is true for q.

And may be then something form p plus, sorry p comma q is true implies p plus 1 comma q plus 1 is true. This can be another of these versions and note that if I have these dots filling up on the all boundary cases, then okay. 1 plus 1 is true therefore 2 plus, 2 comma 2 is true, 2 comma 2 is true therefore 3 comma 3 is true and so on. Now 1 comma 2 is true, therefore, 1 comma 2, 2 comma 3 is true, 3 comma 1 is, sorry, 2 comma 1 is true therefore 3 comma is 2 is true and so on.

So in other words, there are various case at the end of filling the two dimensional grid. We have to ensure them thicker version, which fills up the two-dimensional grid. In the case of induction 1 of k, you fill up the real line, the integers line, the natural lines. In case of two dimensional induction where we induct on two things you fill up in two dimensional grid.

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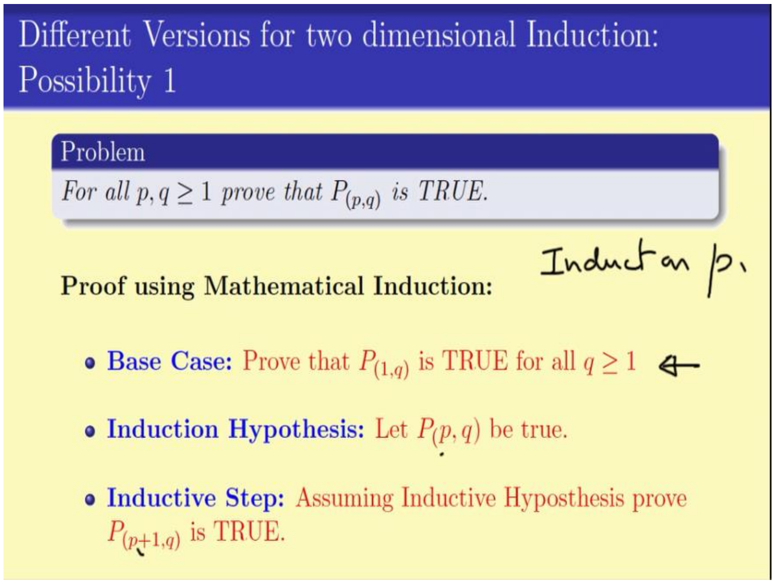

So there are different versions that are available and you will be going through a few of them, so say the version, possibility 1 can be that if we have to prove for all of p q. First prove that 1 comma q is true and then as you induct p q is true prove that p plus 1 q is also true. Now when this happens what is the term that we are going up or going down. In this case, assuming p we are going to p plus 1. So in this case we say that we induct on P.

Verify for yourself that this particular induction step will indeed prove that of course along with this particular base case. We will indeed prove that for all p q this p of p q is true. So in other words the whole of two dimension is covered by this particular induction step.

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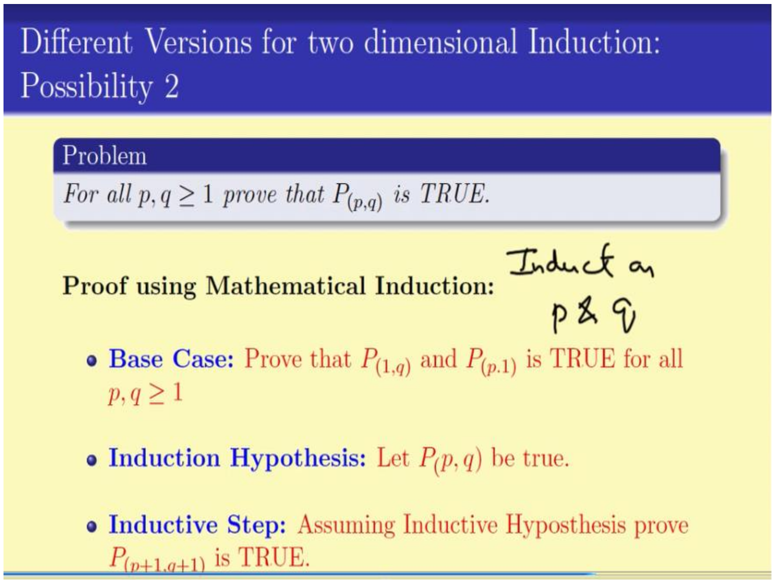

Another possibility is first prove that 1 comma q and p comma 1 is true and then assuming that p comma q is true prove that p plus 1 q plus 1 is true. We just looked at this particular case and in this case what we induct on? We are inducting on actually both p plus 1 and q plus 1 or p and q, right. So, simultaneously we induct on p and q, we are you introducing both p and q. So in this case we induct on p and q, so we push both p and q.

Again convince yourself that this will help us or ensure that this particular version ensures that we cover all these cases or all the two dimensional case.

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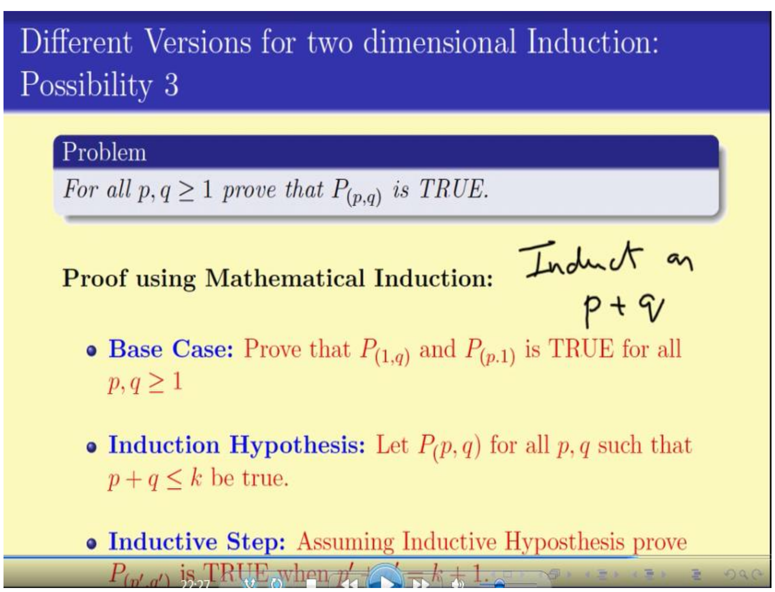

Now version 3 again is the same thing as the base case are same, but induction hypothesis now says that prove that this P p q is true for all p and q such that p plus q is less than k. And then prove that you can prove it for p prime for q prime is true, when p prime comma q prime is k +1. So, in other words now, here we are inducting on p plus q, we are saying that if p plus q is k that I can use it to prove the case when p plus q is k plus 1.

Again, check that this indeed helps us to cover the all the cases or all the two dimensional points.

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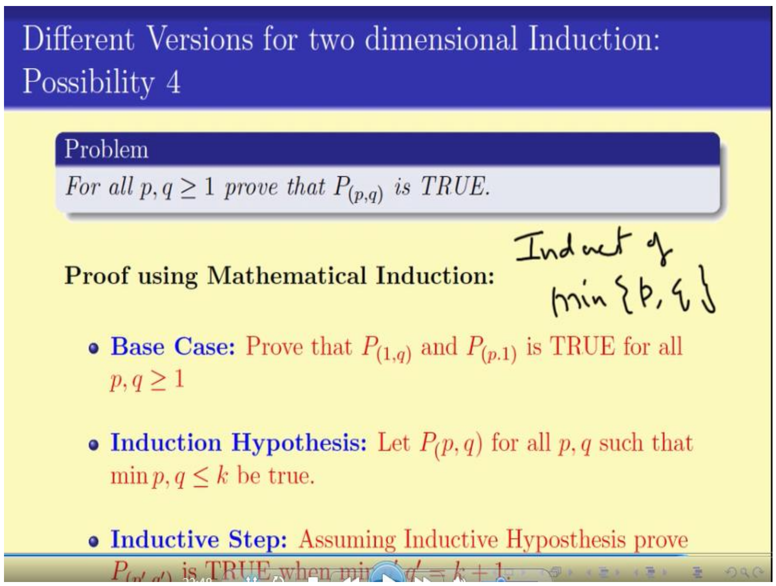

Then more versions that can be there, for example we can have this fourth version where we can induct on minimum of p comma q, so minimum p comma q is k then we can share that okay. I can prove it by minimum of p prime comma q prime is k plus 1.So, here we induct on minimum of p comma q. So, if there are a quite a number of versions that are available as long as you can come up with a version where you can ensure that all the points on which we have to prove.

So in this case a two dimensional grid or sometimes one dimensional grid or sometimes three dimensional grid or something then will be true, right. So as long as they cover all of them, all the necessary cases by this idea, by this inductive step which should be true. So that to our problem, which version should be apply?

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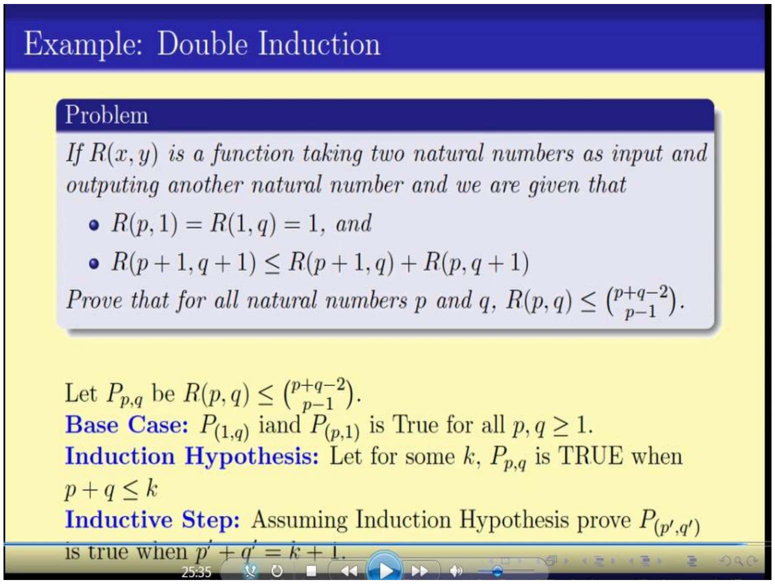

So, I will be applying the version 3, but if base case is that prove for p 1 comma q and p of p comma 1 is true and we induct on p plus q that means if I assume the p plus q equals to k is true then I can solve the case p plus q equals to k plus 1, right? So in this problem, clearly p q equals to the R p q, p q is basically in statement that R p q is less than or equal to p plus q minus 2 choose p minus 1.

So, you will prove that base case is P 1 q and P q 1 is true for all p and q. We have the induction hypothesis then you have to assume that p q case is true as long as p plus k is q because there is k and assuming the induction hypothesis prove that for any p prime, q prime is true, when p prime, plus q prime equals to k plus 1.

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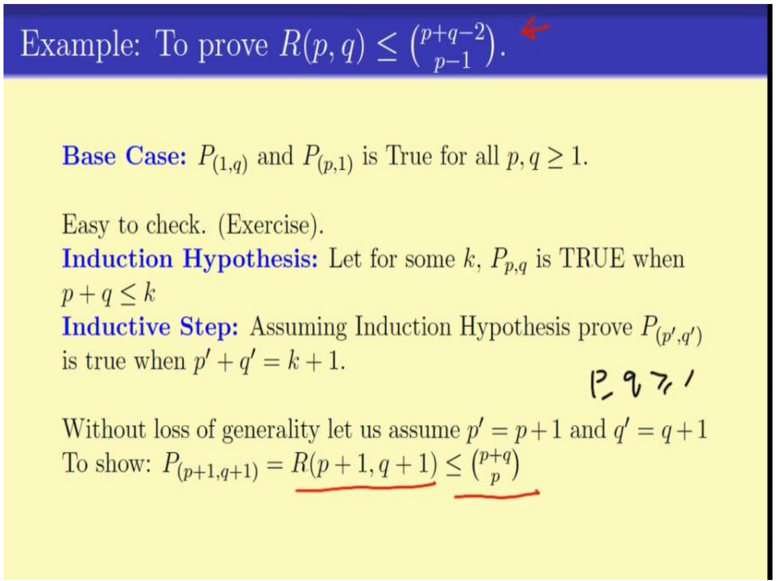

So, here are three cases to prove and again once you get the version, right version to apply it should not be a complicated thing to check, okay. Let us see, so base case I leave you to you guys to check that base case indeed hold in this case. Induction hypothesis, we know what we have to assume and inductive step is that we have to prove that if p prime plus q prime equals to k plus 1, prove that p prime plus comma q prime k is true.

So, here without loss of (()) (26:10) let us assume that p prime equals to p plus 1 and q prime equals to q plus 1 for some p and q, positive, right p and q greater than zero, sorry comma q is greater than 1 and both p and q is greater than 1. Now to show that the p plus 1, q plus 1 holds that it means that I have to prove this thing that r of p plus 1 q plus 1 is less than, so this is what we have to prove. So, if you put p plus 1 and q plus 1, we get p plus q choose p.

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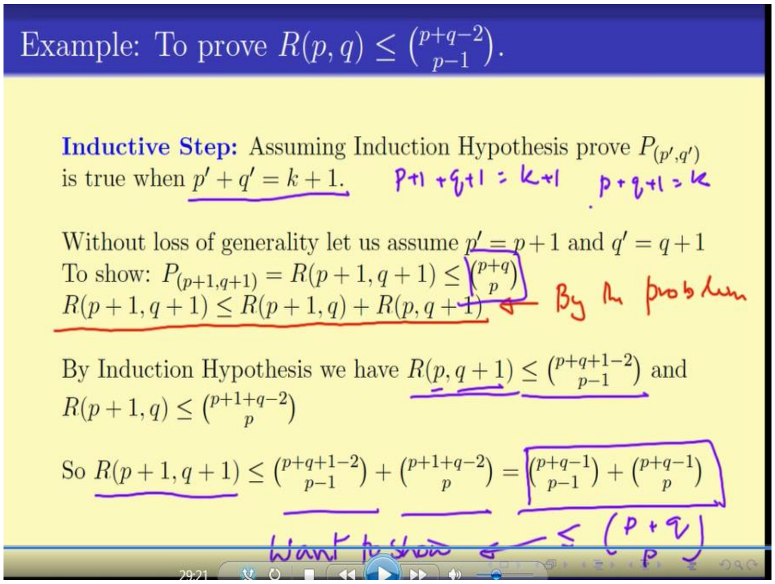

So this is what you have to show, right. Now, we know that by the recurrence we know that R p plus 1 q plus 1 is less than or equal to R p plus 1 q and R p q plus 1. This by the recurrence, by the problem definition, is in the problem definition. And by induction hypothesis, we know that this is less than this why? Because we assume that p prime comma q prime, p prime plus q prime equals to k plus 1 or in other words p plus 1 plus q plus 1 is k plus 1.

So p plus q plus 1 is equals to k and we know that by induction hypothesis as long as this plus this is less than or equal to k we have the problem or we know how to solve it or that by induction hypothesis, right. So, this plus this, all of this comma this is less than whatever that is and similarly R of p plus 1 comma q. So plugging these two values up there, we have R p plus 1 q plus 1 is less than this plus this, which simplifies to this number. So what we have to prove now.

We all, we have to prove is that this quantity is less than this quantity, which is p plus q choose p that is all we have to show. So, this is what we want to show. And if you can show that then we will be having the full proof.

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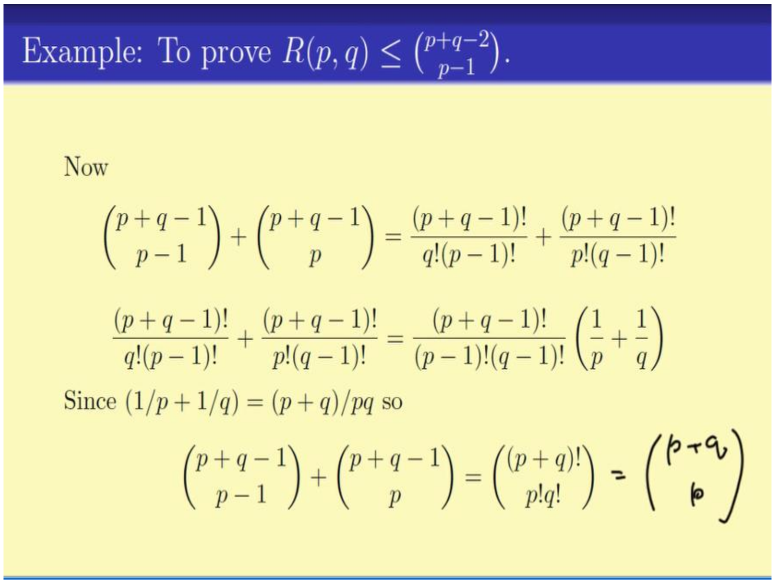

Now just to complete the algebra here, now all that we are doing is some algebra that this plus this which is what was there in the last slide, the last one and this one. Now just to apply some algebra, this one is equal to I just unfolded the definition, if p plus 1 minus 1 factorial by q factorial time t (()) (29:55) factorial and so on. And you can just take out p plus q minus 1 factorial by q p minus 1 factorial time 2 minus actual out.

And then it gets 1 by t plus 1 by q and since 1 by p plus 1 by q is p plus q by p q. So p plus q times p plus q minus 1 factorial is in indeed p plus q factorial. And similarly p times p minus 1 factorial is p factorial and q times q minus one factorial q factorial thus we have that this one is less than this, which is by the way recall that this is actually was p plus q choose q or choose p whatever both of them are right and which is what we wanted to prove.

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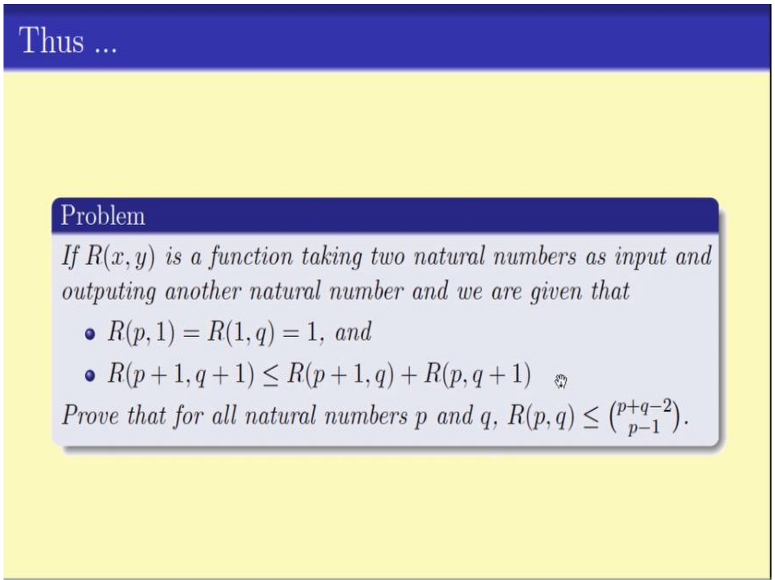

Thus we have been able to prove this particular problem using the inductives, a mathematical induction on two variables. We have seen a number of different versions of induction hypothesis that can be applied depending on the problem one might have to choose the right version to apply. We will continue with our more studies on induction, proof to the induction in the next couple of days. Thank you.