Discrete Mathematics
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Lecture -02
Sets, Relations and Functions

Welcome everybody to the second video lecture on discrete maths. So in this particular video, we will be looking at the definitions and notations of sets, relations and functions. These definitions and notations will be essential for setting up the course and we will be used throughout the course. So it is very important for us to ensure that we understand these concepts very well.

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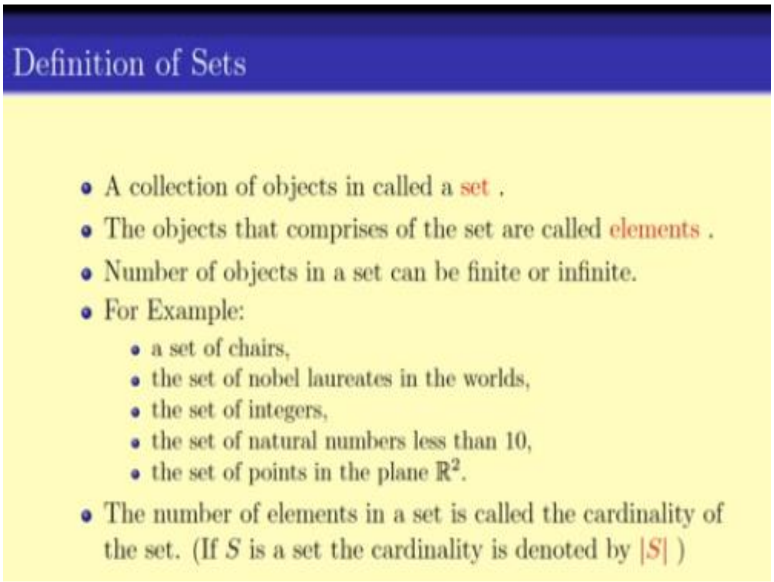

Let us start with the definition of sets, so what is a set? Any collection of objects is called a set. The objects that comprises of the set are called the elements. The number of objects in the set can be finite or infinite. For example, say, I can say a set of chairs or the set of Nobel laureates in the world or the set of integers or the set of natural numbers less than 10. So this comprises of 1, 2, 3, 4, 5, 6, 7, 8, 9 or the set of points in the plane R2.

Note that the set of integers and set of points in the plane R2 are actually infinite sets meaning the number of elements in this set are infinite. So we can have set which are finite or we can set, have sets which are infinite. The number of elements in a set is called the cardinality of set, it is usually denoted by this notation. so if S this the set then S between two vertical lines denotes the cardinality of the set.

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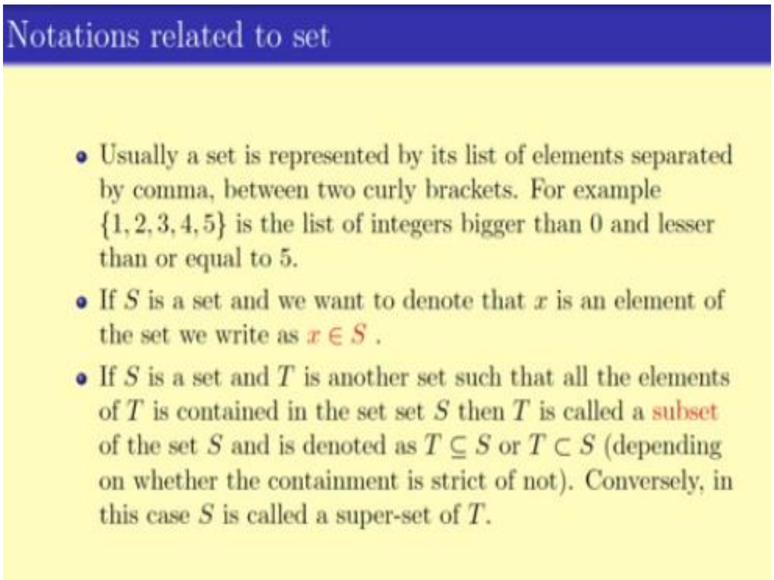

Moving on to some more notations about set, usually when I have to describe a set or represent a set, it is done so by putting the elements separated by commas and between 2 curly brackets. For example, here between this 2 curly brackets of this and this I have the elements 1, 2, 3, 4 and 5. They are all separated by the comma, that means this represents the set 1, 2, 3, 4 and 5. Okay?

Now if I have to pick an element out of a set, we usually use the notation of S contain in is set S. So this particular notation will be used again and again in this notes, in this course and is very common notation to be used is x contained in S. If S is a set, I can take a subset of this S by taking some elements out of this set S. So if T is such a set that means every element of T is contained inside S.

Also then we say that T is a subset of S and it is usually denoted by this term, this notation T contained in S or sometimes it is used just notation without the equality sign the below which means that T is strictly containing S. This means that there is at least one element in S that is not contained in T. Conversely in this case, S is called a superset of T.

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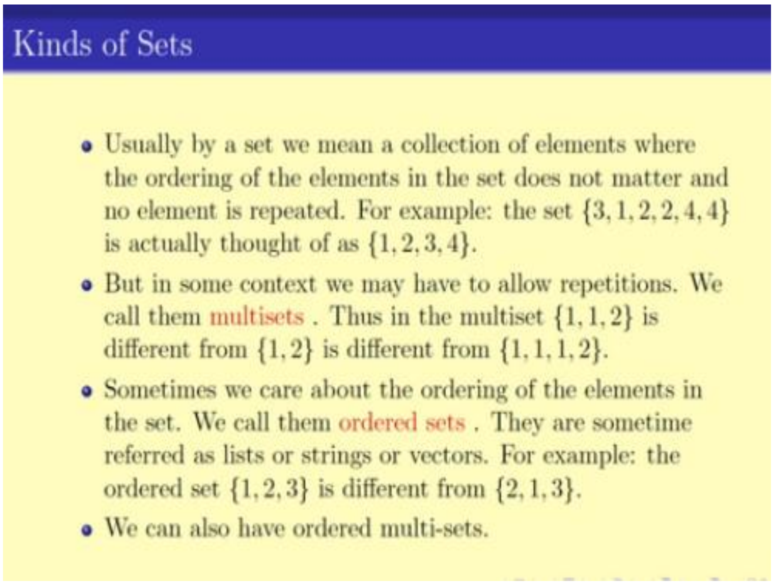

Now, there can be different kinds of sets. So here, I have defined 4 different sets, styles and this is very important for a setting up the notational framework. Firstly, when people talk about set, they usually mean a collection of objects where repetition is not allowed and the ordering does not matter. So in this example, in the first example, if I tell you the set comprises of the element 3, 1, 2, 2, 4 and 4.

It is exactly same as the set containing 1, 2, 3 and 4. The repeated elements of 2 does not matter and similarly, the ordering does not matter. But in certain context, we might have to use these repetitions. These are called multisets, so multiset is a set where repetitions matter that means they set of 1, 1 and 2 is different from the set of 1 and 2. The reason being in the first case, in set of 1, 1 and 2. there are 2 ones whereas in the case 1, 2. There is only one, one.

And similarly, it is also different from three 1's and 2. So when we want to refer to such a set where repetitions are allowed or repetitions should be taken into consideration, we call it a multiset. The third kind of set is what we call as ordered sets. So these are the sets where the ordering matters. So that means that if I tell you the set comprises of 1, 2 and 3, this is different from if I tell you that the set comprises of 3, 2 and 1.

Or I say that the set comprises of 3, 1 and 2, because the ordering of the element in the set matters. Now when we consider ordered sets, sometimes it has other names people used in different context, sometimes it is called a list. A list meaning, it is a list of objects or list of elements where the ordering does matter, similarly we sometimes call it as string or vectors.

Now, we can also have sets which are ordered multisets meaning sets where both repetitions as well as ordering matters.

We will be using this terminology again and again in this course, so it is very important to get used to this particular terminology.

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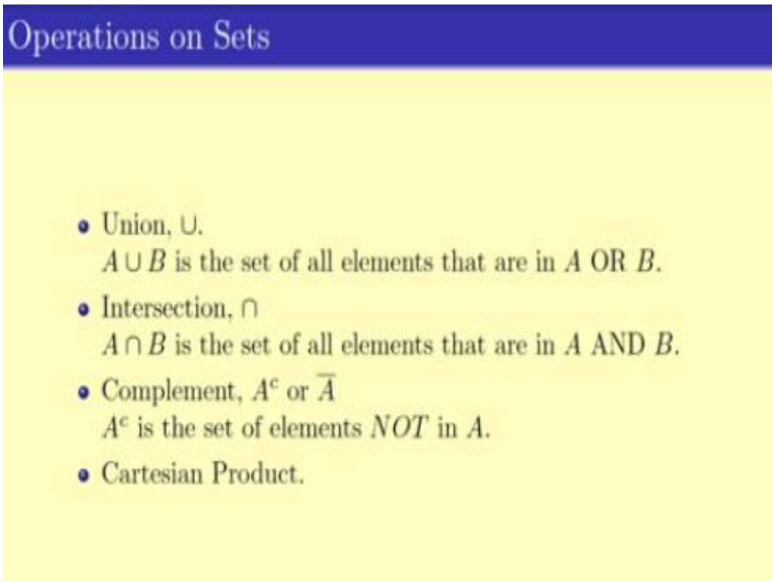

Now, given a set or given sets, we can have some set of operations on the set. There are 3 main operations that one can do and they pretty well known. First of all is the union, so A Union B, but A and B are 2 sets is the set of all elements that is contained in either A or B. So it comprises of everything together. Similarly, the other one which is intersection, so A Intersection B is the set of all elements that are contained in both A and B.

The third is the complement, this is the sets of elements that are not contained in A. So this definition of compliment, assume that there is a universal set. We sometimes referred it to U or omega and the compliment is all the elements in the universal set that are not contained in A. It is usually denoted as A Power C or A with a straight line on top. There is one more operation on set that is done and that is called the Cartesian product.

We will be talking about this Cartesian product after a couple of slides.

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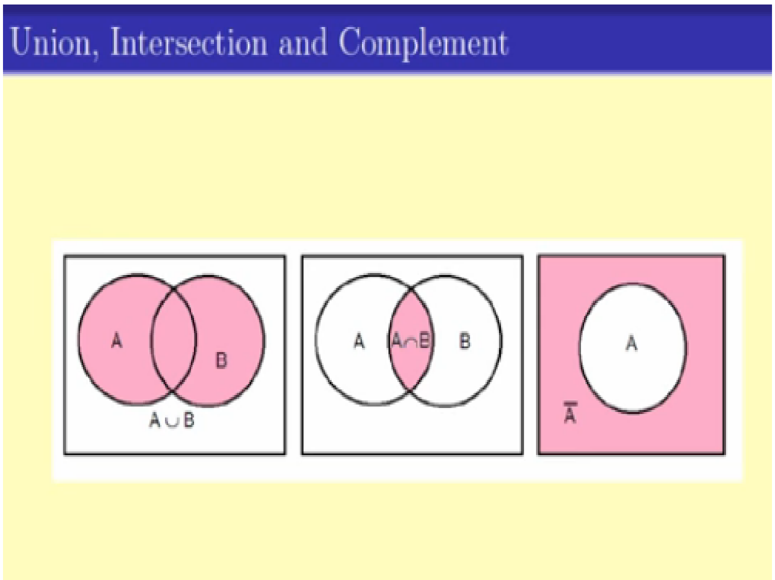

So here is a pictorial description of Union, Intersection and Complement. So for example, look at the first picture, if A and B are these two round objects to this round circle here represent the set B. Similarly, this round object here represents the set A and the outer boundary wall which is the rectangle represents the universal set same for all of them. Now A Union B is the set of all elements that are there in either A or B or both.

So it is all the elements that are there in the pink area in the first diagram. Similarly, A intersection B is the set of all elements that are in the pink area in the second picture and A compliment is the pink area in the third picture. So this is what we call as Venn diagram and we use this particular form of diagram a lot since it gives a pictorial view. It helps us to think in a slightly more easier way.

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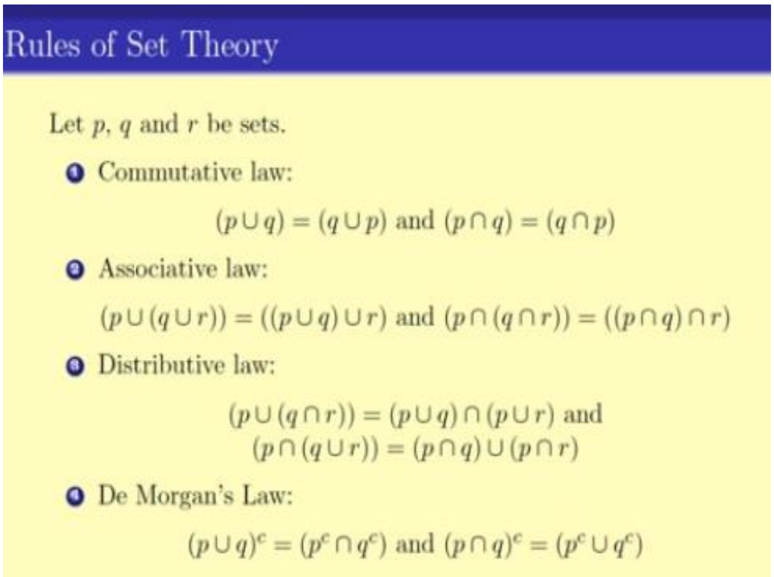

Now, the set of these 3 operations that are there that is the union intersection and compliment satisfies some set of laws. I have written down some obvious set of laws. These are the most natural ones and our most important ones. Number 1, that is commutative law which means that P Union Q is same as Q Union P and P Intersection Q is same as Q Intersection P. The other law is the associative law.

We say that P Union Q Union R is same as you first do the union of Q, P and Q and then you do a union with R. Similarly, for the set of intersection, this is called the associative law. The third law which is an interesting law is the distributive law. This says that P Union Q Intersection R is same as you first do the union of P with Q and P with R and then intersection with it between these 2 sets.

Similarly, P intersection Q Union R is same as P Intersection Q Union P Intersection R and the fourth law is the De Morgan's law which says that P Union Q compliment is same as P complement intersection Q compliment and similarly, P intersection Q compliment is P complement union Q compliment. Now, these laws are provable laws that means you can prove these laws from the definition of intersection and union and complement.

I leave it to you guys to check that these laws are indeed correct. In the 2 video lectures that I will spend on solving problems. I will prove that these laws are actually true but before that I encourage you guys to prove it for yourself or try to prove it for yourself that these laws are indeed correct. They should follow from the definition of union intersection and compliment. Now, as I told you, there is one more law other than union, intersection and complement and that is called the Cartesian product.

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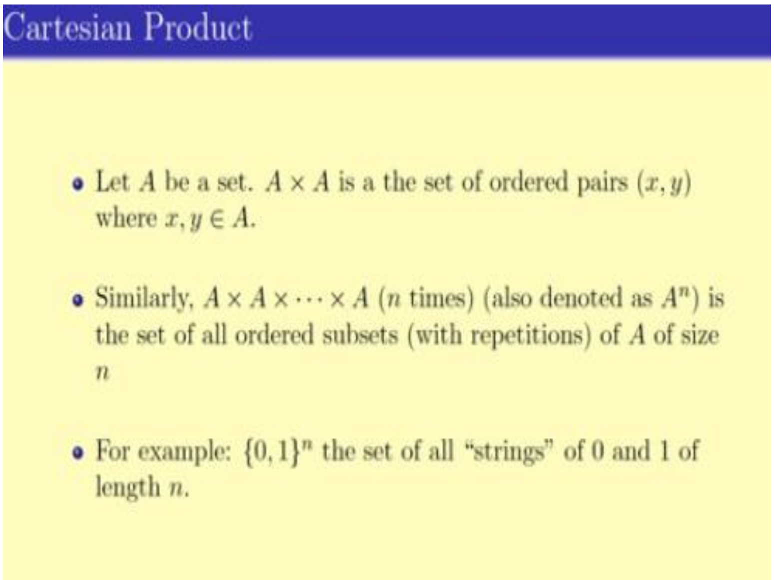

The Cartesian product is another very important operation on sets with A is the set A Cartesian product with A is the set of ordered pairs X, Y but both X and Y (()) (14:49). Here, the ordered pair means that X, Y the ordering of X and Y matters. so X, Y is different from Y, X. Here, I have given you the definition of A Cartesian A, you can also do the definition of A Cartesian Product B where it is the ordered pair X and Y where X is the first element, first set A and Y is in the second set B. So similarly, you can define the Cartesian product between any two sets A and B.

When we have to do something like take Cartesian product of A, a number of times say n times, we sometimes denote it as A Power N. So in other words, in the above example, A Cartesian A can be written as S square. Similarly, if I take the Cartesian product of A n times I get A Power A and what is this if you unfold the definitions of Cartesian product of A with A, you realize this R the strings of elements of A of length n.

In other words, these are the ordered subsets with repetition of A of size n. For example, if A is the set 0, 1 then 0, 1 power n is the string of 0 and 1 of length n. So 0, 1 square if the set comprising of {0, 0}, {0, 1}, {1, 0} and {1, 1}. Similarly, {0, 1} power 3, {0,1} cube is the set comprising of the strings {0, 0, 0}, {0, 0, 1}, {0, 1, 0}, {0, 1, 1}, {1, 0, 0}, {1,0,1} {1,1,0} and {1,1,1}. So to all the strings of length 3 comprising of 0 and 1.

Now this particular object of 0, 1 whole power n is particularly interesting for us, since in computer science, we do work on Boolean things all the time that means, we are only interested in 0 then 1's. So 0, 1 appears again and again. So this set of 0, 1 power n is called the hypercube or the N dimensional hypercube.

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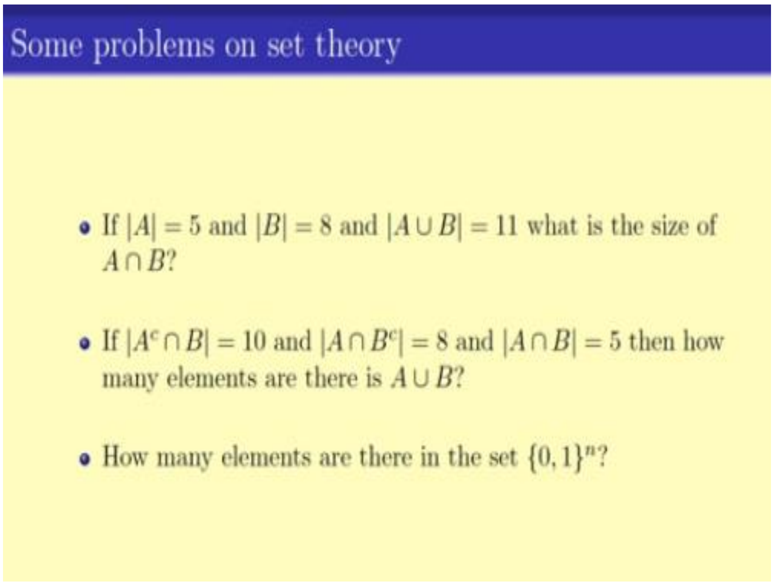

Now, let me end this introduction to set theory by giving you some problems on set theory. I will be requesting you guys to do these problems as a homework for your better understanding of this subject. I will use the video lectures that I dedicate for problem solving to give you solutions to these problems. So here is, the 3 problems that I have listed out for you, you can take down and solve them.

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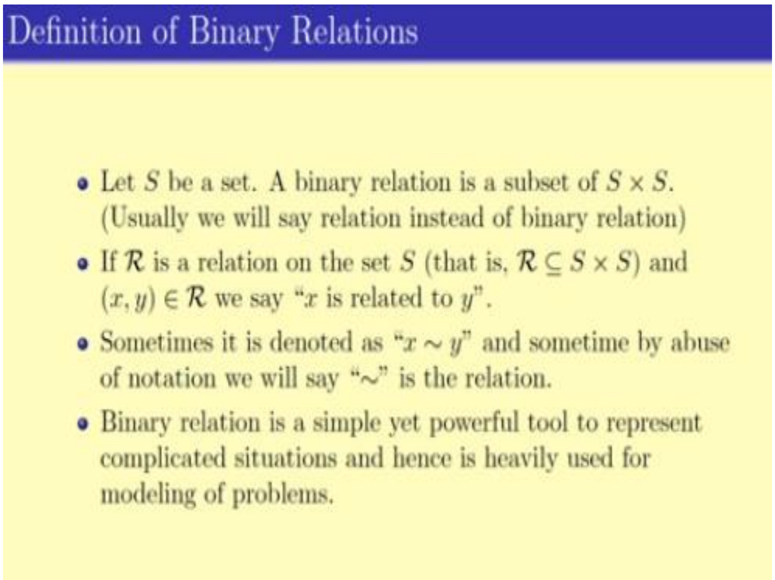

Now, moving on let us move to the next important abstract object, this is called relation. In particular, we will be looking at binary relation. So what is the binary relation? So if S is set, a binary relation is a subset of S x S or S Cartesian product with S, we usually call this one relation though note correctly it should be always be called binary relation. So if R is the relation on the set S that means R is the subset of S Cartesian product S.

And if this ordered pair of X, Y is in this R we say X is related to Y. Sometimes, we denote the relation with this kind of a tilted rotation and this is used as saying that X is related to Y. Binary relations is an outstandingly powerful tool to represent various complicated situations and it is heavily used for modelling of real life problems and mathematical problems. We will see how this binary relation can be used for modelling problems in the due course of this video lectures.

In fact, understanding binary relations is one of this central subject in whole of mathematics and computer science.

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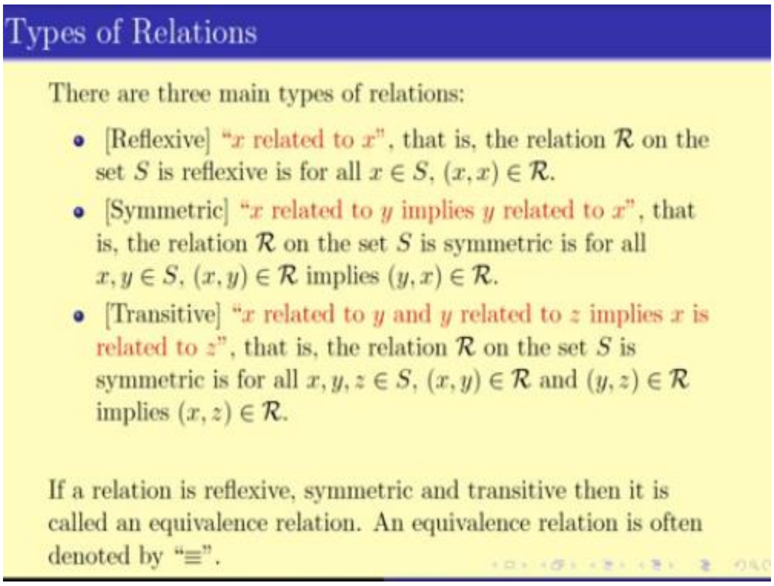

The binary relations depending on whether they have some more structure or not has been classified into various categories. So here are 3 main types of binary relations, number one what we call as reflexive binary relation. So in other words, X is related to X, so X, X is the binary relation R. Second one is the symmetric binary relation. So here if X is related to Y that means Y must be related to X or in other words, X, Y is in R, if and only if Y, X is in R.

The third is the transitive relation but if X, Y is an R and Y, Z is an R then implies that X, Z is also in R. So if X is related to Y and Y is related to Z then X is also related to Z. So these are the 3 main types of relation depending on the structure of the binary relation all the subset of S process that depend - that defines the binary relation. If a binary relation has all the 3 properties that is - it is reflexive, symmetric and transitive.

We call it equivalence relation; equivalence relations are usually denoted by this particular sign of three horizontal lines.

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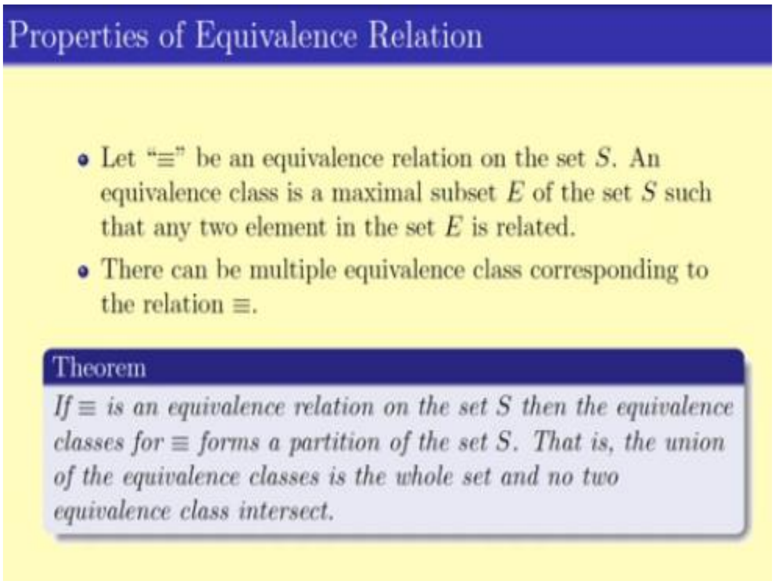

Now let us consider an equivalence relation, now given an equivalence relation, I can define a maximal subset E of the set S such that any 2 elements of the set E is related to each other. It is maximal in the sense that you cannot add any more element to this set E by still maintaining the property that any 2 element in E are related, this maximal, it cannot be grown any further.

So once I get such a set E, I call that E equivalence class, given an equivalence relation, there can be multiple equivalence classes or in other words, there can be more than one set such that any of those sets are maximal and any two element of those in any of the set is related. Here there is a beautiful theorem, it says that the set of equivalence classes partitions this set S that means the union of the equivalence classes give you the set S and no two equivalence class can intersect

This is a theorem, this theorem again follows from the definitions of equivalence relation and equivalence classes, it is not an odd theorem to prove but like many other theorems the implication of it is pretty enormous. Again I leave it as an exercise for you to give a proof of this theorem in the video lectures where I will do the problem sets. I will be giving you the formal proof of this theorem.

Till the time, I request you guys to try to solve it for yourselves.

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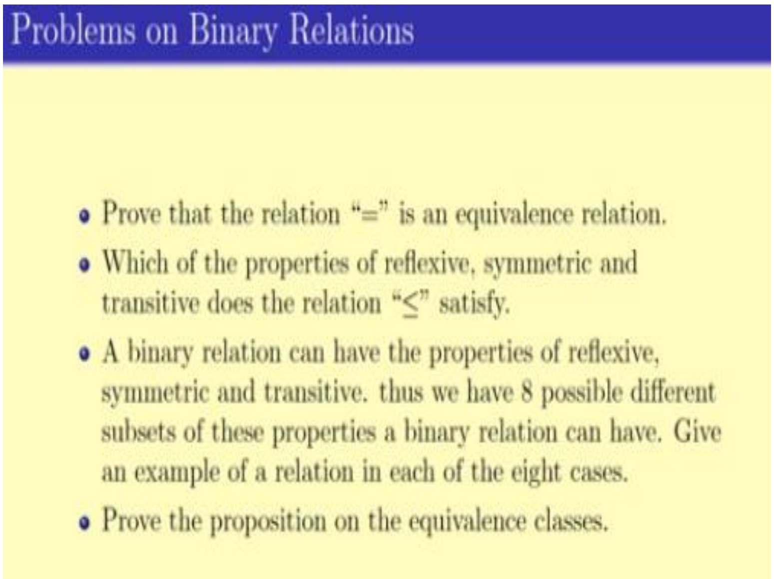

Here is the list of problems on binary relations that you should attempt to solve by yourself, they are all easy and should follow from the definition of binary relations.

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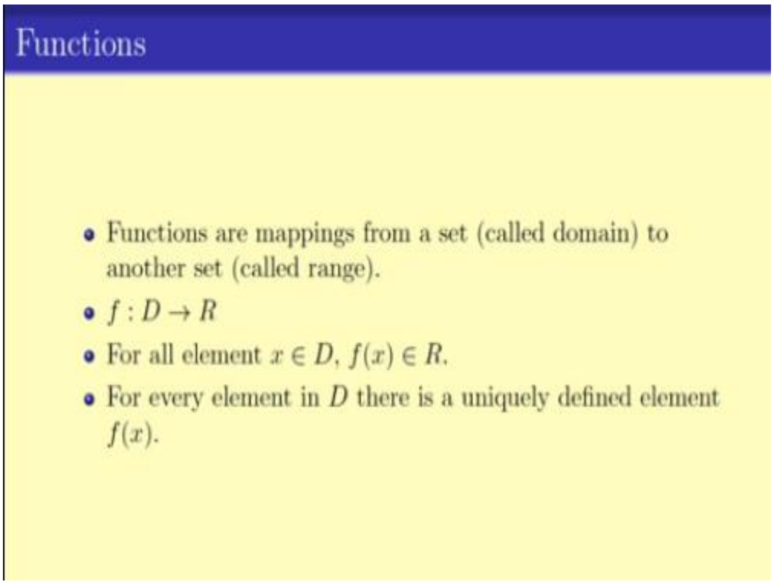

Moving on, let us tackle the third important concept or abstract concept and these are called functions. So what are functions? Functions are mappings from a set which we call a domain to another set if you called range. So it is denoted like this, if from a domain D to a range R.

The function has some nice properties that for every element in the domain X the f of X or the mapping of X is well defined and must be in the range R.

Again another property for every element in D, there is a uniquely defined element f of X that means, you cannot have X mapped to two different elements in the set R. If a mapping satisfies these two properties, we call that a function.

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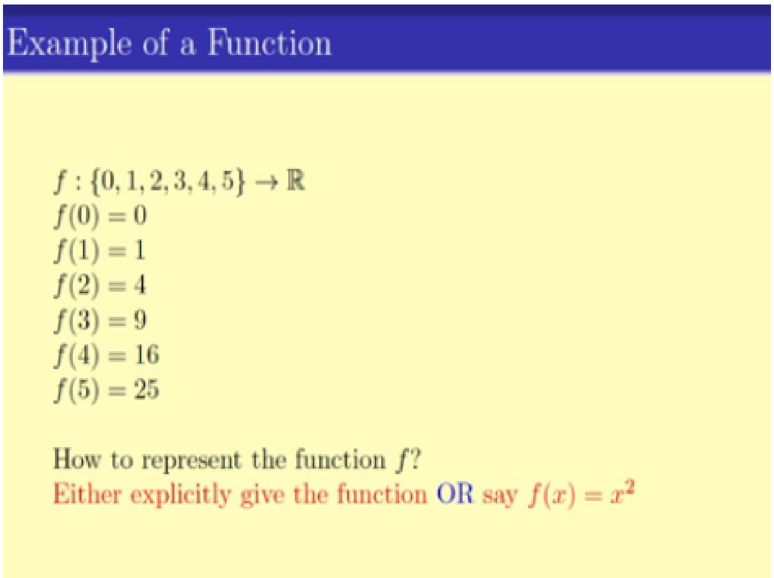

For example, here take this function mapping from the set {0, 1, 2, 3, 4 and 5} to the set of real numbers. For example, f of 0 is 0, f of 1 is 1, f of 2 is 4, f of 3 is 9, f of 4 is 16 and f of 5 is 25. You can check that f of any element of the domain is well defined and unique and hence it is the function. Now how do you represent this function?

Note that I have given you this function explicitly that means for every element x in the domain set, I have specified what is the f of x, that you can represent a function explicitly by giving the function or you can come up with some pretty clever ways of defining it. For example, here you can check that I could have just told f of x is x square, so you can either explicitly give the function or implicitly give a nice equation for the function.

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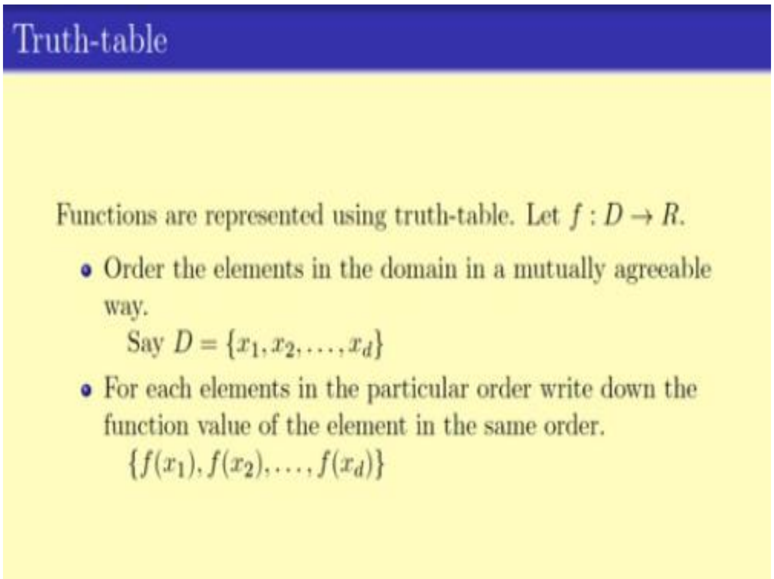

When somebody gives you the function explicitly, we call it a truth-table. The functions are represented using truth tables. So how do I represent the function is in truth table? So for example, if I have to describe a function to you, one way is that we mutually agree, ordering of the domains, the domain set say let us say that ok see the ordering of the domain set is x1, x2, x3, …. xd, now want that ordering has been predetermined.

I can just give you f of x1, f of x2, f of x3 …. f of xd and you will immediately understand that when I the third element in the ordering that I give you is a mapping of x 3,2 the range. Similarly, maybe the tenth element here is the mapping of x 10, this particular description of the S by giving you the f of x in a predetermined order is called the truth table. Thus, the size of the truth table is always equals to the size of the domain.

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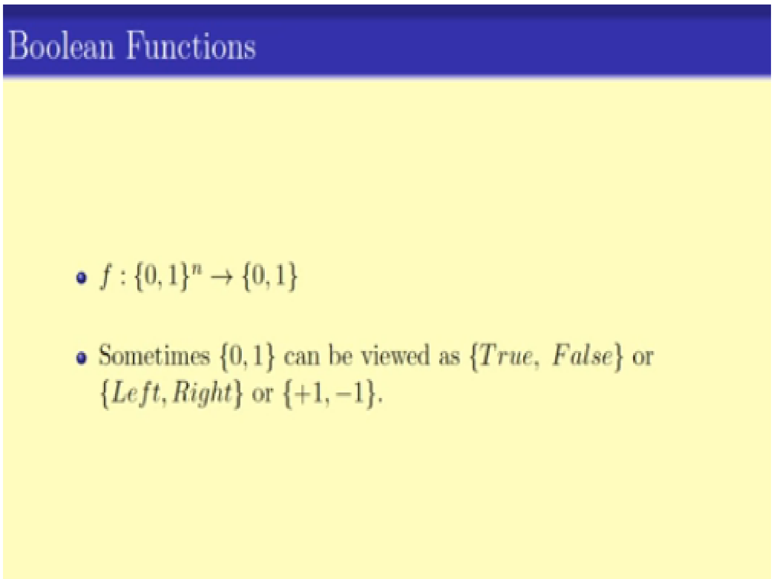

Now, in this course, there will be one particular type of functions, we will be looking at bit more carefully, these are called Boolean functions. So Boolean functions are functions that take values or the domain set is 0, 1 to the n, the n dimensional hypercube and it outputs 0,1. This 0,1 can be viewed in different ways you can think of it as 0 representing true 1 representing false or 0 representing left, 1 representing right and so on and so forth.

Depending on need, and the context of the problem, we will interpret the 0 and 1 differently.

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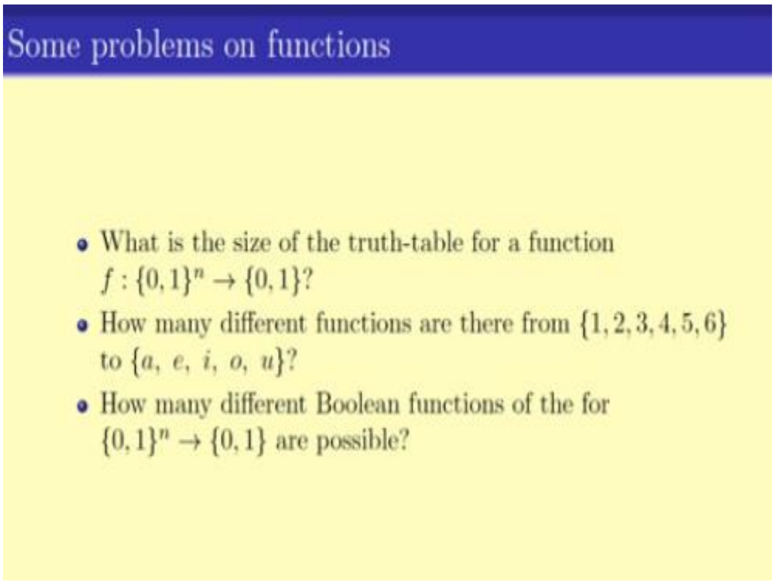

So this brings us to the end of the definition of functions. Here are some of the problems on functions again, you should do them for your practice. I will go over these problems in a - in one of the problem solving videos. With this, we come to an end of our introduction to the definitions and notations of sets, relations and functions. Three of the most important abstract objects that we will be using in this course.

In this next video lecture, we will be moving on to doing propositional logic and predicate logic and that would set us up for solve how to solve problems in general. Thank you.