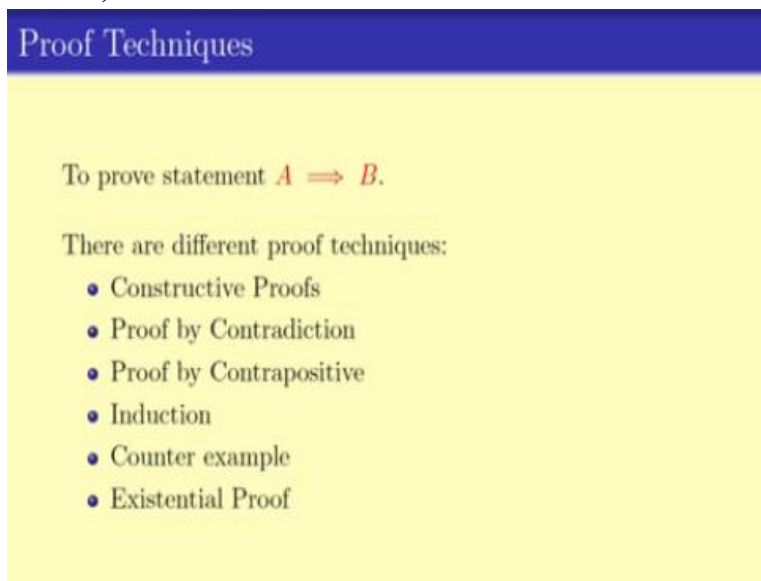


Discrete Mathematics
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Lecture - 18
Mathematical Induction (Part 5)

Welcome to the fifth week of this discrete mathematics course. So this is the first lecture in the fifth week. We have been looking at various proof techniques in the last week, we saw some induction which is one of the most powerful proof technique for proving theorems in discrete mathematics. We will continue with this particular proof technique and see how more, how much more can induction help us proving more complicated theorems.

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The slide has a blue header with the text "Proof Techniques". Below the header, on a yellow background, is the text "To prove statement $A \implies B$." followed by "There are different proof techniques:" and a bulleted list of six proof techniques: Constructive Proofs, Proof by Contradiction, Proof by Contrapositive, Induction, Counter example, and Existential Proof.

So to quick recap, so we have been trying to look at statements like A implies B and to prove such a statement, we have seen there are quite a number of proof techniques namely constructive proofs, proof by Contradiction, proof by Contrapositive, Induction, Counter example and so on. We have now almost covered most of it and we have been studying the proof by induction. So we started our whole study of proof techniques by looking at how to split a problem into smaller parts.

We took the help of the propositional logic to prove that for certain cases splitting up the problem into smaller parts can be possible.

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Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove $A \implies B$ and B can be written as $B = C \wedge D$ then note that

$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$

- (Remove Redundant Assumptions) If $A \implies B$ then $A \wedge C$ also implies B .

$$(A \implies B) \implies (A \wedge C \implies B) = \text{True}$$

- (Sometimes proving something stronger is easier) If $C \implies B$ then

$$(A \implies C) \implies (A \implies B).$$

So one of them is in the case of, when you have to prove A implies B , B is written as C and D , in that case we can split up the problem into two smaller parts. Similarly, we saw that removing redundant assumptions is something very useful in getting a simpler statement of this problem, which would be which can be easier to prove. And then we also looked at some examples where proving something stronger is easier, right.

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Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.
- If we have to prove $(A \implies B)$ then the idea is to simplify B .
- And if $C \iff B$ then $(A \implies B) \equiv (A \implies C)$.

In the different proof techniques, we have seen constructive proof, namely particularly with the direct proofs where you work with A and end up proving B . And why this can be sometimes be magical, we can do a backward proof namely first work with B , simplify it to some other form say C , and then mainly proving A implies C can be easier.

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Proof by Contradiction

- Note that

$$(A \implies B) \equiv (\neg B \wedge A = \text{False})$$

This is called "proof by contradiction"

- A similar statement is

$$(A \implies B) \equiv (\neg B \implies \neg A)$$

This is called "proof by contra-positive". If hen B (the deduction) is of the form $C \vee D$ then

$$(A \implies B) \equiv (\neg B \implies \neg A) \equiv ((\neg C \wedge \neg D) \implies \neg A)$$

We also saw some case studies in which you can split the assumptions into a finite number of cases and for each case you prove it separately. That is, if A can be written as C or D , then you can split up the problem A implies B as C implies B and D implies B . Then there was also the proof by contradiction, the idea is that instead of proving A implies B one can also prove not B and A is false.

Sometimes, this is a different way of looking at the problem, and can be easier to solve than A implies B . Another particular technique is, instead of proving A implies B , one proves that not B implies not A , they are similar statements or equivalent statements. And this can be useful particularly when B is of the form C or D . So this was the various proof techniques that we saw.

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But what if the statement is false

- To prove the statement $A \implies B$ is not true what to do?
- If the problem is actually of the form $\forall x, A(x) \implies B(x)$ then the negation of this statement is

$$\exists x, A(x) \not\Rightarrow B(x)$$

- Recall $A \implies B$ is same as $(B \vee \neg A)$. So,
$$\exists x A(x) \not\Rightarrow B(x) \equiv \exists x \neg(B(x) \vee \neg A(x)) \equiv \exists x (\neg B(x) \wedge A(x))$$
- So to prove that the original statement is not true we have to find an x such that $(\neg B(x) \wedge A(x))$ is true.

One more proof technique that we were timely going on, one more proof technique that was, we also saw a proof technique called counter example, where if we have to disprove a theorem or statement A implies B. The idea is to produce a proof that A does not implies B, or in other words, if the problem is of the form, for all x A implies B, and we have to prove the negation of it, which is there x is state A not implies B, which in turn becomes there x is state such that Bx does not hold and Ax holds.

So this is what we call as the proof by counter examples. Other than these set of proofs, we also looked at the proof of induction, and that is what we have been doing for the last week.

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Introduction to Induction

- Sometimes the set of assumptions (or the set of objects for which we have to prove the theorem) can be split into a infinite by countably many subsets.
- Or in other word the problem $A \implies B$ can be split into a AND of infinitely many problems.
- The sub-problem are usually indexed by some parameter of input.
- Thus the assumption is written as

$$A \implies B \equiv P_1 \wedge P_2 \wedge \dots \wedge P_n \wedge \dots$$

The idea is to again split up the problems into smaller problems, but here, we will split up the assumptions or the sake of this equation in which we have to prove the theorem. We have to split up in to possibly infinite number of subsets. So in other words, this will imply that this A implies B get split up into infinitely many problems. Now usually the sub problems are indexed by some parameter of the input. So the A implies B has become something like p_1 and p_2 and so on till infinity.

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Principle of Mathematical Induction

Problem
For all $k \geq 1$ prove that P_k is TRUE.

- Since there are infinitely many sub-problems one cannot expect to solve all the sub-problems.
- Idea is to solve the first one, namely
 Prove that P_1 is TRUE
- And prove that,
 if for any $k \geq 1$, P_k is TRUE then P_{k+1} is TRUE.
- Then for any $n \geq 1$ the problem P_n is true and hence proved.

So thus, the problems A implies B bounds down to a problem like this, for all k greater than or equal to one, prove that P_k is true. Unfortunately, we cannot solve all the P_i s individually, because there are infinite number of them. So, we take the help of mathematical induction, which helps us to solve all of them at one sort. The idea is simple, first prove that P_1 is true, then prove that if for some k we prove P_k is true, then using that prove P_k plus one is true. If we can prove that, then the P_n is true for all n .

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Principle of Mathematical Induction

$$\forall P, [P_1 \vee (\forall(k \geq 1)P_k \implies P_{k+1})] \implies [\forall(k \geq 1)P_k]$$

- There are different versions that one can use.

Now, this is quite an accepted in principle, though to ensure that correctness of this principle we do have to use or write a new step, axiom which we call the principle of mathematical induction, which states that this statement or this way of proving is a valid proof. Now, there are different version that one can view of this various of this particular case. We have already seen some of the versions.

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Mathematical Induction: Version 1

Problem

For all $k \geq 1$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_1 is TRUE
- **Induction Hypothesis:** Let P_k be true for some $k \geq 1$
- **Inductive Step:** Assuming Inductive Hyposthesis prove P_{k+1} is TRUE.

So the version one, is the case, which we just discussed, which is that, if we have to prove that for all k greater than or equal to one P_k is true, then we have to first prove that the P_1 is true, because this one is base case. Then, we have the induction hypothesis, which states that let P_k be

true for some k and the inductive step is assuming that, the induction hypothesis true, prove that P_{k+1} is true.

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Mathematical Induction: Version 2

Problem
For all $k \geq r$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_r is TRUE
- **Induction Hypothesis:** Let P_k be true for some $k \geq r$
- **Inductive Step:** Assuming Inductive Hypothesis prove P_{k+1} is TRUE.
- Then for any $n \geq r$ the problem P_n is true and hence proved.

Now, there is another version that we saw, which is the case that, for k , if you have to prove that for all k greater than or equal to r prove that P_r is true, then we just shift the base case, namely we prove that P_r is true, and assuming the same inductive hypothesis can we prove that P_{k+1} is true as well P_k is true. And once you prove that it helps us to say that, for all n greater than or equal to r the problem P_n is true.

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Mathematical Induction: Version 3

Problem
For all $k \geq r$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_r and P_{r+1} is TRUE
- **Induction Hypothesis:** Let P_k be true for some $k \geq r$
- **Inductive Step:** Assuming Inductive Hypothesis prove P_{k+2} is TRUE.
- Then for any $n \geq r$ the problem P_n is true and hence proved.

Let us look at another version, the third version, this one was the case when we have the same kind of condition, where we have to prove that for all k greater than or equal to r P_k is true, see

instead of proving having base case as P_r , we can have the base case as P_r and P_{r+1} . And then, we can have a slightly weaker induction hypothesis, sorry Induction state. So namely the induction hypothesis is same that P_k is true for some k greater than r .

All we need to show is that, that assuming that P_k is true, prove that P_{k+1} is true. As I told you in the last video also, the idea is to ensure that, every possible points are getting proved. So for example, if say this is r , the base case says that, okay, I know how to solve r , I know how to solve $r+1$. Now induction hypothesis says that, okay, if I know r how to prove $r+2$, if I know $r+1$ I know to prove $r+3$ and so on.

And you can convince yourself that we will end up proving all the points greater than just r . So in other words, this technique will help us prove that for all n greater than r the problem P_n is true and hence proved. Now, one thing to remember here is that, already we have seen this version one, two and three, particular version two and three. Solve the same problem for all k greater than or equal to r . Now, which version to use?

Now, which version to use, of course depend on the problem. For some problem applying version three will be easier mainly proving $k+2$ is true assuming P_k will be an easier thing. In some cases, proving P_{k+1} is true assuming P_k will be an easier thing and in that respect, in that case we use version two. So which version of induction hypothesis to use, depends fully on the problem in hand.

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Mathematical Induction: Version 4

Problem

For all $k \geq r$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_r and P_{r+1} is TRUE
- **Induction Hypothesis:** Let P_k and P_{k+1} be true.
- **Inductive Step:** Assuming Inductive Hypothesis prove P_{k+2} is TRUE.
- Then for any $n \geq r$ the problem P_n is true and hence proved.

Now there is a zillion more versions that can be done, I will give you one more version. Here, for the same problem, where for all k greater than or equal to r you want to prove P_k is true. Now we have the same base case that we will prove that P_r and P_r plus one is true. We have the same induction hypothesis of late, sorry induction hypothesis changes here, instead of having the hypothesis that P_k is true.

We assume that both P_k and P_k plus one is true, and using that can you prove that P_k plus two is true, right. If we can prove that, again that will solve the whole thing, let us try to see how we can ensure. So, if this is r , and this is r plus one, the base case says that, we know how to prove r and r plus one, the induction hypothesis says that, okay, since I know P_r and P_r plus one, so I will be knowing P_r plus two.

Now, again since I know P_r plus one and P_r plus two, I would be knowing P_r plus three and so on. Thus, this way continuing we would be able to prove, see P_k for all k greater than or equal to r . So this is also a valid induction hypothesis, right.

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For example

Problem

Let $\{a_n\}$ be a sequence of number such that $a_1 = 5$ and $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$ then prove that $a_n = 2^n + 3^n$.

Let P_k be "Let $\{a_n\}$ be a sequence of number such that $a_1 = 5$ and $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ then prove that $a_k = 2^k + 3^k$."

So the problem can be restated as

Problem

For all $k \geq 1$ prove that P_k is TRUE.

So let us see how can one apply this particular version, say let us look at this particular problem. So, a is the sequence of numbers such that a_1 is five, a_2 is thirteen and we have been told that for all n greater than or equal to one a_{n+2} is equal to five times a_{n+1} minus six times a_n . Then prove that a_n is equals to two power n plus three power n . Now, as we have done in other cases also, we have to split them up into smaller cases, right.

So, here of course, let P_k be the case that, if a is the sequence, then a_k is two power k plus three power k . And we have to prove that for all k greater than or equal to one, P_k is true right.

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Example: Recurrence

Problem

Let $\{a_n\}$ be a sequence of number such that $a_1 = 5$ and $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$ then prove that $a_n = 2^n + 3^n$.

Let P_k be "Let $\{a_n\}$ be a sequence of number such that $a_1 = 5$ and $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ then prove that $a_k = 2^k + 3^k$."

Base Case: P_1 and P_2 s True.

Inductive Hypothesis: Let for some k , P_k and P_{k+1} is TRUE

Inductive Step: Assuming P_k and P_{k+1} is true prove P_{k+2} is true.

Now, okay, if I have to solve this problem, now what we have to do we have to do the three cases, the base case, which is say we will prove P1 and P2 is true. We have inductive hypothesis, which will be, let Pk and Pk plus one is true and using that we will be proving Pk plus two is true.

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Example: Recurrence

Base Case: To prove $a_1 = 2^1 + 3^1$ and $a_2 = 2^2 + 3^2$.

$a_1 = 5 = 2^1 + 3^1$ and $a_2 = 13 = 2^2 + 3^2$. Hence base cases are true.

Inductive Hypothesis: Let for some k , $a_k = 2^k + 3^k$ and $a_{k+1} = 2^{k+1} + 3^{k+1}$.

Inductive Step: Assuming Induction hypothesis prove that $a_{k+2} = 2^{k+2} + 3^{k+2}$.

By the recurrence given, $a_{k+2} = 5a_{k+1} - 6a_k$. By the Inductive Hypothesis we know that $a_k = 2^k + 3^k$ and $a_{k+1} = 2^{k+1} + 3^{k+1}$. So $a_{k+2} = 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k) = 2^k(10 - 6) + 3^k(15 - 6) = 2^{k+2} + 3^{k+2}$. Hence $a_{k+2} = 2^{k+2} + 3^{k+2}$.

So the base case is that, we have to prove that a1 equals to two power one plus three power one, and a2 equals to two square plus three square. And this is not that hard to prove, because it can be checked or verified, right. Now, the induction hypothesis we have that, let us assume that remember what was Pk, Pk was ak plus one. So this one was Pk, and this is Pk plus one.

So we have that, ak equals to two power k plus three power k, and ak plus one is two power k plus one three power k plus one. And what we have to do is that, assuming the inductive hypothesis we have to prove, of course this statement which is Pk plus two. Now, let us see how can we solve it, now quickly recall that, we were given this particular recurrence, relation, we were given this relation.

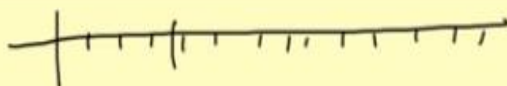
So we were given this relation, ak plus two equals to five times ak plus one minus six k. Now, once we have that, so by inductive hypothesis I know that, this ak is equals to two power k plus three power k and ak plus one is two power k plus one plus three power k plus one. So, we can plug it there and what we get, we get that ak plus two is five times this ak plus one this value minus six times this ak.

And now a little of arithmetic will show that, if we take this 2^k out 2^k out, we get two times five from here in this term and six from here, so 2^k times ten minus six plus three 2^k and in fifteen, in three from here five from here fifteen minus six, and which, as you can see this will equal to four and this equals to four and this equals to nine. So we have, this equals to 2^k plus two plus three 2^k plus two.

And hence a_k plus two equals to 2^k plus two plus three 2^k plus two. Thus, we have the proof that P_k plus two is correct, assuming both a_k and a_{k+1} is true. So, using induction hypothesis, and this particular version of it, we have been able to prove this particular sequence, problem.

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Thus ...



We prove the following problem using mathematical induction:

Problem
 Let $\{a_n\}$ be a sequence of number such that $a_1 = 5$ and $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$ then prove that $a_n = 2^n + 3^n$.

Now this problem, either just one of the examples of what is we call as recurrences, where we are given some term, in as a linear combination of its lower terms. Now, as you can see I can also have a problem where, I can define a_{n+3} as $a_{n+1} + a_{n+2}$. In that case, we would have to use some other induction hypothesis, in particular we have to use the induction hypothesis, where assuming P_k , P_{k+1} and P_{k+2} is correct, prove that P_{k+3} is correct.

So, we have seen a few of the induction versions of the mathematical induction, while this is just a finite collection of them. One can come up with a lot of other variance of induction hypothesis.

The idea is simple that, if you write down in the real line all the possible points for P_q have to solves P_k , somehow ensure that, you will be able to prove all of them one by one in some way and this gives us the particular problem or a particular way of typing this induction hypothesis.

So, this brings us to the end of this video lecture. In the next video lecture, we will be looking at some more generalization of this mathematical induction. Thank you.