Discrete Mathematics
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Lecture - 18
Mathematical Induction (Part 5)

Welcome to the fifth week of this discrete mathematics course. So this is the first lecture in the fifth week. We have been looking at various proof techniques in the last week, we saw some induction which is one of the most powerful proof technique for proving theorems in discrete mathematics. We will continue with this particular proof technique and see how more, how much more can induction help us proving more complicated theorems.

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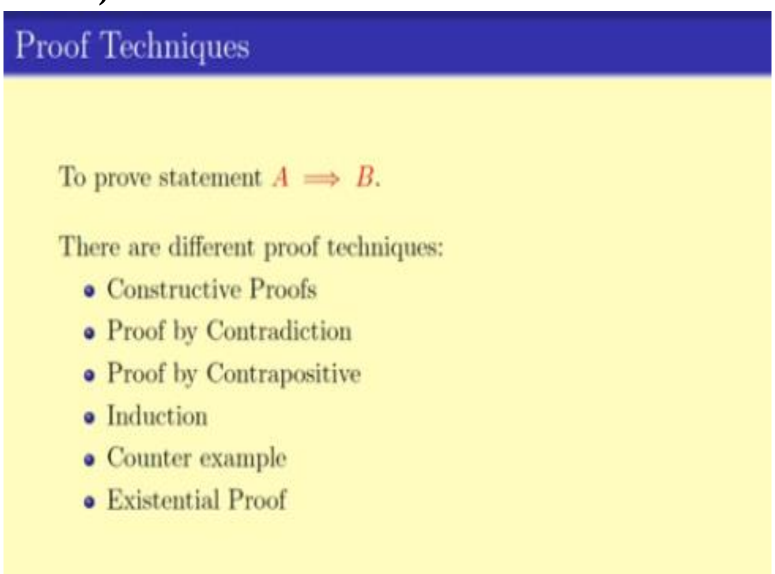

So to quick recap, so we have been trying to look at statements like A implies B and to prove such a statement, we have seen there are quite a number of proof techniques namely constructive proofs, proof by Contradiction, proof by Contrapositive, Induction, Counter example and so on. We have now almost covered most of it and we have been studying the proof by induction. So we started our whole study of proof techniques by looking at how to split a problem into smaller parts.

We took the help of the propositional logic to prove that for certain cases splitting up the problem into smaller parts can be possible.

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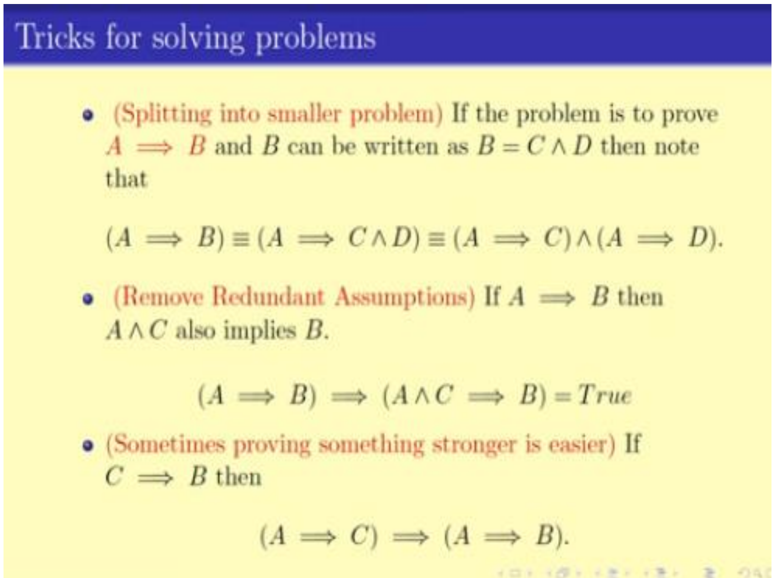

So one of them is in the case of, when you have to prove A implies B, B is written as C and D, in that case we can split up the problem into two smaller parts. Similarly, we saw that removing redundant assumptions is something very useful in getting a simpler statement of this problem, which would be which can be easier to prove. And then we also looked at some examples where proving something stronger is easier, right.

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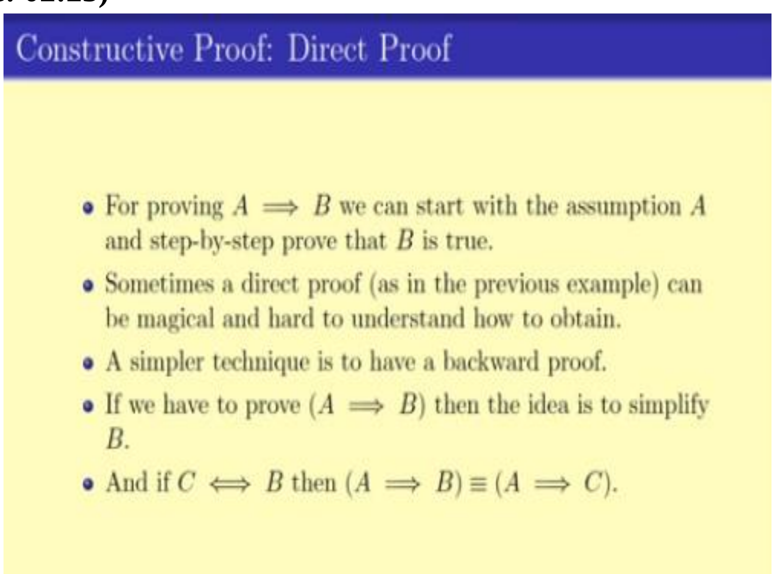

In the different proof techniques, we have seen constructive proof, namely particularly with the direct proofs where you work with A and end up proving B. And why this can be sometimes be magical, we can do a backward proof namely first work with B, simplify it to some other form say C, and then mainly proving A implies C can be easier.

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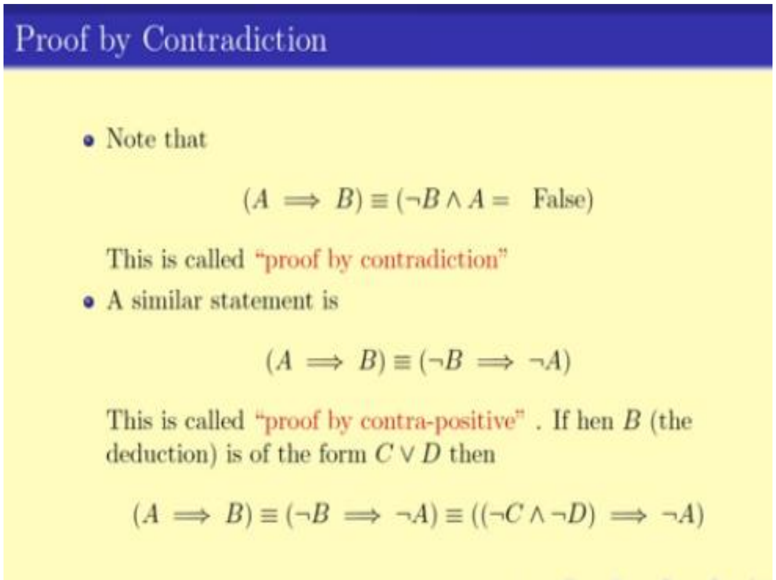

We also saw some case studies in which you can split the assumptions into a finite number of cases and for each case you prove it separately. That is, if A can be written as C or D, then you can split up the problem A implies B as C implies B and D implies B. Then there was also the proof by contradiction, the idea is that instead of proving A implies B one can also prove not B and A is false.

Sometimes, this is a different way of looking at the problem, and can be easier to solve than A implies B. Another particular technique is, instead of proving A implies B, one proves that not B implies not A, they are similar statements or equivalent statements. And this can be useful particularly when B is of the form C or D. So this was the various proof techniques that we saw.

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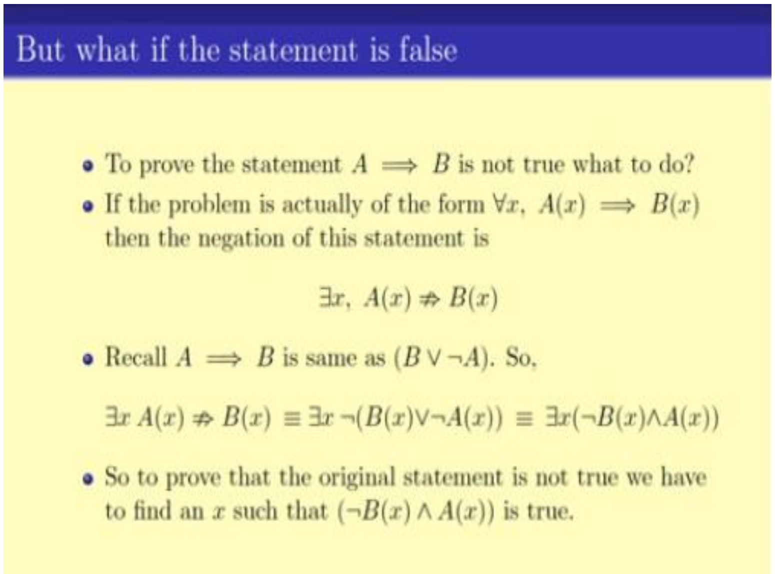

One more proof technique that we were timely going on, one more proof technique that was, we also saw a proof technique called counter example, where if we have to disprove a theorem or statement A implies B. The idea is to produce a proof that A does not implies B, or in other words, if the problem is of the form, for all x A implies B, and we have to prove the negation of it, which is there x is state A not implies B, which in turn becomes there x is state such that Bx does not hold and Ax holds.

So this is what we call as the proof by counter examples. Other than these set of proofs, we also looked at the proof of induction, and that is what we have been doing for the last week.

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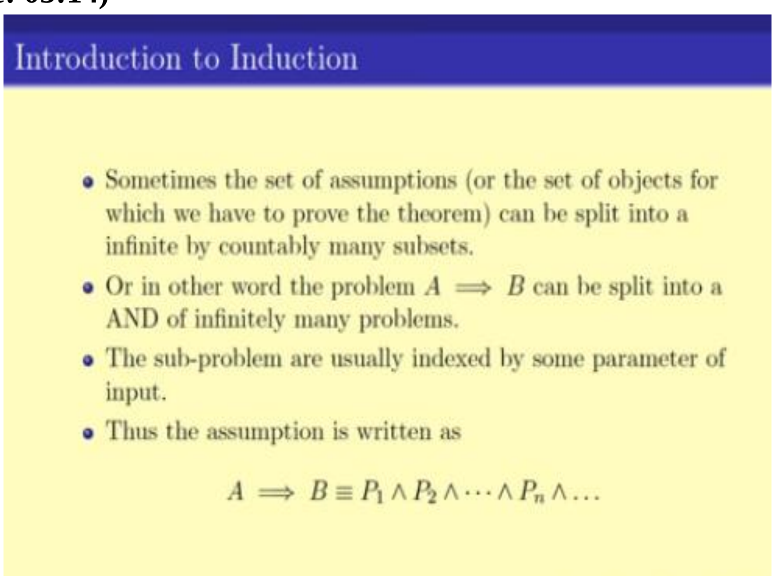

The idea is to again split up the problems into smaller problems, but here, we will split up the assumptions or the sake of this equation in which we have to prove the theorem. We have to split up in to possibly infinite number of subsets. So in other words, this will imply that this A implies B get split up into infinitely many problems. Now usually the sub problems are indexed by some parameter of the input. So the A implies B has become something like p1 and p2 and so on till infinity.

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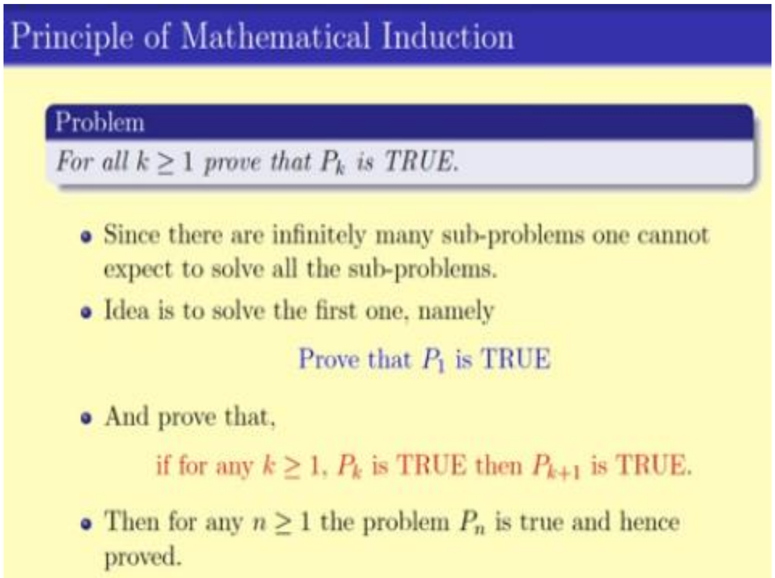

So thus, the problems A implies B bounds down to a problem like this, for all k greater than or equal to one, prove that Pk is true. Unfortunately, we cannot solve all the Pis individually, because there are infinite number of them. So, we take the help of mathematical induction, which helps us to solve all of them at one sort. The idea is simple, first prove that P1 is true, then prove that if for some k we prove Pk is true, then using that prove Pk plus one is true. If we can prove that, then the Pn is true for all n.

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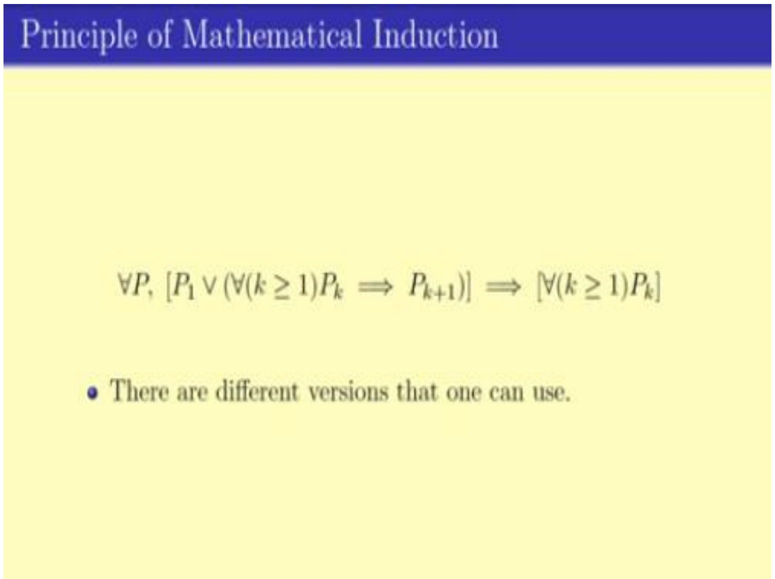

Now, this is quite an accepted in principle, though to ensure that correctness of this principle we do have to use or write a new step, axiom which we call the principle of mathematical induction, which states that this statement or this way of proving is a valid proof. Now, there are different version that one can view of this various of this particular case. We have already seen some of the versions.

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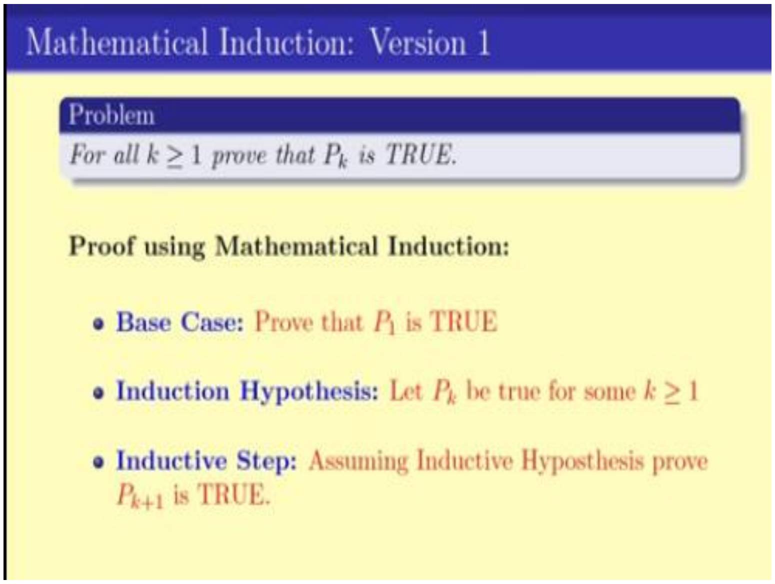

So the version one, is the case, which we just discussed, which is that, if we have to prove that for all k greater than or equal to one Pk is true, then we have to first prove that the P1 is true, because this one is base case. Then, we have the induction hypothesis, which states that let Pk be true for some k and the inductive step is assuming that, the induction hypothesis true, prove that Pk plus one is true.

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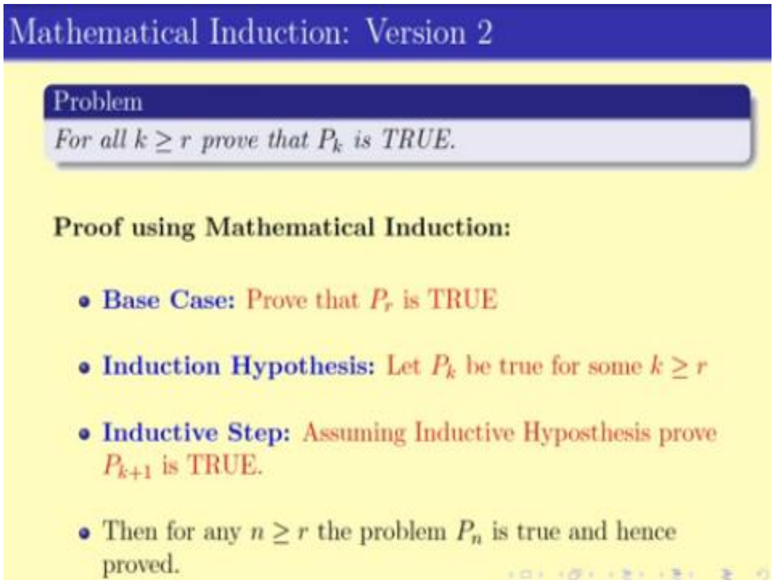

Now, there is another version that we saw, which is the case that, for k, if you have to prove that for all k greater than or equal to r prove that Pr is true, then we just shift the base case, namely we prove that Pr is true, and assuming the same inductive hypothesis can we prove that Pk plus one is true as well Pk is true. And once you prove that it helps us to say that, for all n greater than or equal to r the problem Pn is true.

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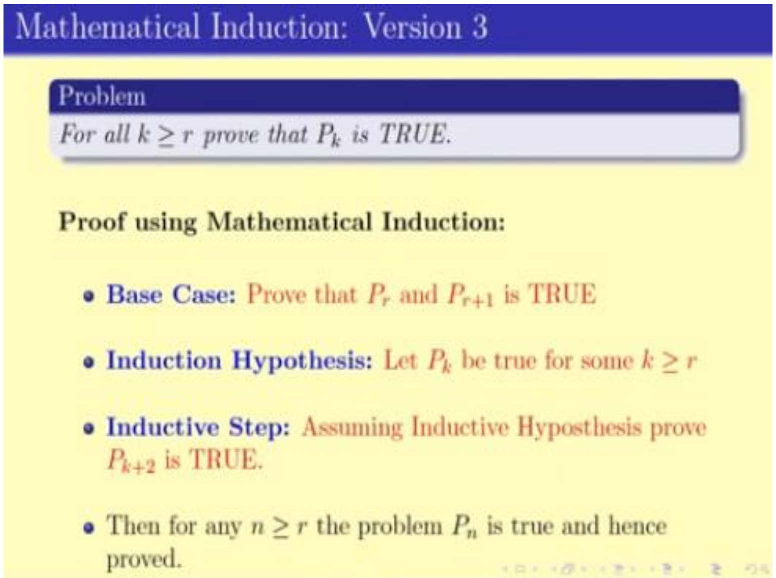

Let us look an another version, the third version, this one was the case when we have the same kind of condition, where we have to prove that for all k greater than or equal to r Pk is true, see instead of proving having base case as Pr, we can have the base case as Pr and Pr plus one. And then, we can have a slightly weaker induction hypothesis, sorry Induction state. So namely the induction hypothesis is same that Pk is true for some k greater than r.

All we need to show is that, that assuming that Pk is true, prove that Pk plus one is true. As I told you in the last video also, the idea is to ensure that, every possible points are getting proved. So for example, if say this is r, the base case says that, okay, I know how to solve r, I know how to solve r plus one. Now induction hypothesis says that, okay, if I know r how to prove r plus two, if I know r plus one I know to prove r plus three and so on.

And you can convince yourself that we will end up proving all the points greater than just r. So in other words, this technique will help us prove that for all n greater than r the problem Pn is true and hence proved. Now, one thing to remember here is that, already we have seen this version one, two and three, particular version two and three. Solve the same problem for all k greater than or equal to r. Now, which version to use?

Now, which version to use, of course depend on the problem. For some problem applying version three will be easier mainly proving k plus two is true assuming Pk will be an easier thing. In some cases, proving Pk plus one is true assuming Pk will be an easier thing and in that respect, in that case we use version two. So which version of induction hypothesis to use, depends fully on the problem in hand.

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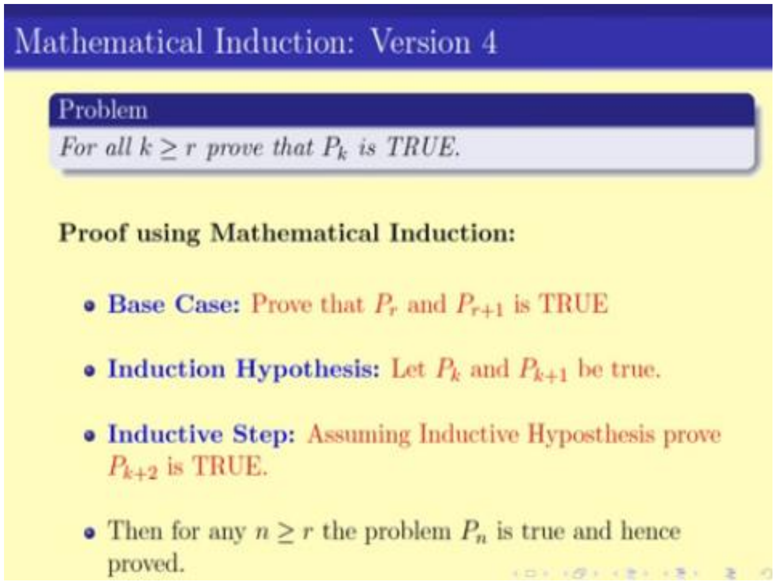

Now there is a zillion more versions that can be done, I will give you one more version. Here, for the same problem, where for all k greater than or equal to r you want to prove Pk is true. Now we have the same base case that we will prove that Pr and Pr plus one is true. We have the same induction hypothesis of late, sorry induction hypothesis changes here, instead of having the hypothesis that Pk is true.

We assume that both Pk and Pk plus one is true, and using that can you prove that Pk plus two is true, right. If we can prove that, again that will solve the whole thing, let us try to see how we can ensure. So, if this is r, and this is r plus one, the base case says that, we know how to prove r and r plus one, the induction hypothesis says that, okay, since I know Pr and Pr plus one, so I will be knowing Pr plus two.

Now, again since I know Pr plus one and Pr plus two, I would be knowing Pr plus three and so on. Thus, this way continuing we would be able to prove, see Pk for all k greater than or equal to r. So this is also a valid induction hypothesis, right.

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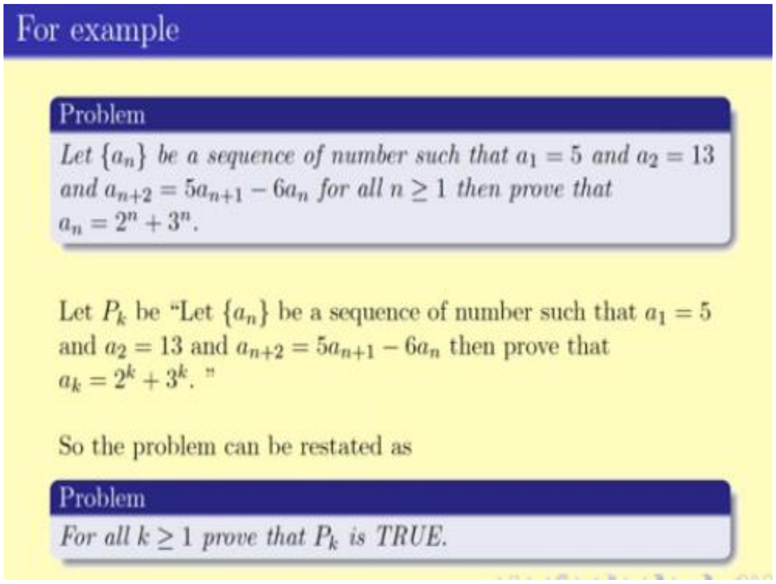

So let us see how can one apply this particular version, say let us look at this particular problem. So, a is the sequence of numbers such that a1 is five, a2 is thirteen and we have been told that for all n greater than or equal to one a n plus two is equal to five times a n plus one minus six times a n. Then prove that a n is equals to two power n plus three power n. Now, as we have done in other cases also, we have to split them up into smaller cases, right.

So, here of course, let Pk be the case that, if a is the sequence, then ak is two power k plus three power k. And we have to prove that for all k greater than or equal to one, Pk is true right.

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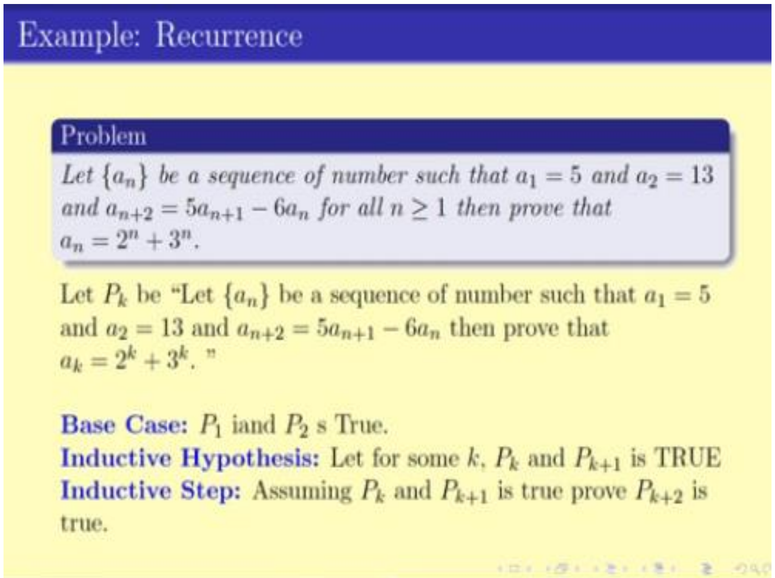

Now, okay, if I have to solve this problem, now what we have to do we have to do the three cases, the base case, which is say we will prove P1 and P2 is true. We have inductive hypothesis, which will be, let Pk and Pk plus one is true and using that we will be proving Pk plus two is true.

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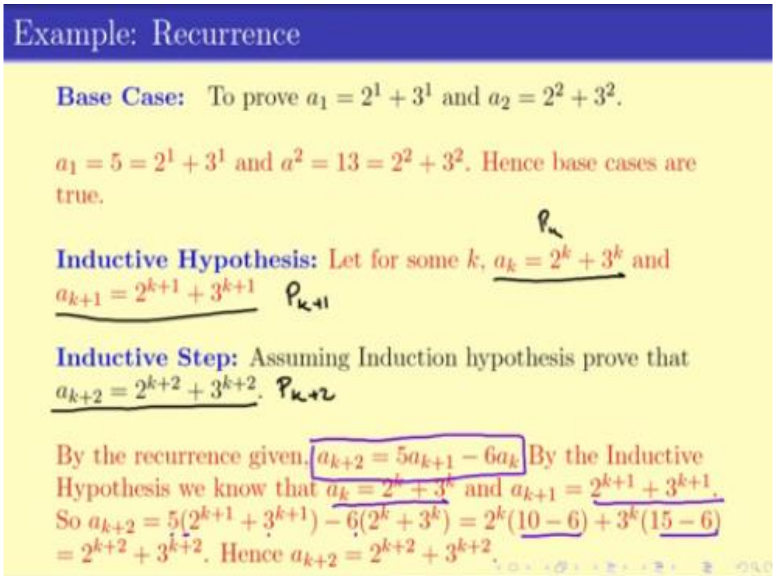

So the base case is that, we have to prove that a1 equals to two power one plus three power one, and a2 equals to two square plus three square. And this is not that hard to prove, because it can be checked or verified, right. Now, the induction hypothesis we have that, let us assume that remember what was Pk, Pk was ak plus one. So this one was Pk, and this is Pk plus one.

So we have that, ak equals to two power k plus three power k, and ak plus one is two power k plus one three power k plus one. And what we have to do is that, assuming the inductive hypothesis we have to prove, of course this statement which is Pk plus two. Now, let us see how can we solve it, now quickly recall that, we were given this particular recurrence, relation, we were given this relation.

So we were given this relation, ak plus two equals to five times ak plus one minus six k. Now, once we have that, so by inductive hypothesis I know that, this ak is equals to two power k plus three power k and ak plus one is two power k plus one plus three power k plus one. So, we can plug it there and what we get, we get that ak plus two is five times this ak plus one this value minus six times this ak.

And now a little of arithmetic will show that, if we take this 2k out 2 power k out, we get two times five from here in this term and six from here, so two power k times ten minus six plus three power k and in fifteen, in three from here five from here fifteen minus six, and which, as you can see this will equals to four and this is equals to four and this is equals to nine. So we have, this is equals to two power k plus two plus three power k plus two.

And hence ak plus two equals to two power k plus two plus three power k plus two. Thus, we have the proof that Pk plus two is correct, assuming both ak and ak plus one is true. So, using induction hypothesis, and this particular version of it, we have been able to prove this particular sequence, problem.

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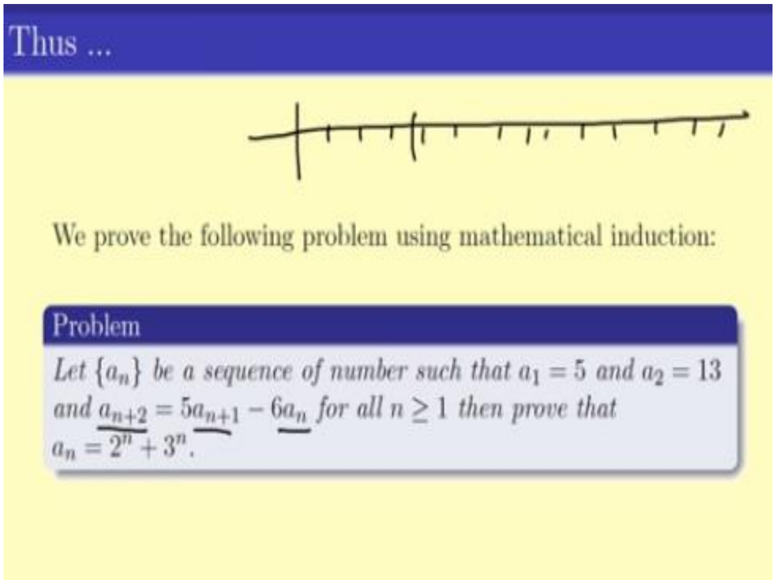

Now this problem, either just one of the examples of what is we call as recurrences, where we are given some term, in as a linear combination of its lower terms. Now, as you can see I can also have a problem where, I can define a n plus three as a n plus one plus a n plus two. In that case, we would have to use some other induction hypothesis, in particular we have to use the induction hypothesis, where assuming Pk, Pk plus one and Pk plus two is correct, prove that Pk plus three is correct.

So, we have seen a few of the induction versions of the mathematical induction, while this is just a finite collection of them. One can come up with a lot of other variance of induction hypothesis.

The idea is simple that, if you write down in the real line all the possible points for Pq have to solves Pk, somehow ensure that, you will be able to prove all of them one by one in some way and this gives us the particular problem or a particular way of typing this induction hypothesis.

So, this brings us to the end of this video lecture. In the next video lecture, we will be looking at some more generalization of this mathematical induction. Thank you.