Discrete Mathematics

Prof. Sourav Chakraborty
Department of Mathematics
Indian Institute of Technology – Madras

Lecture - 17
Mathematical Induction (Part 4)

We continue our study of induction. So to recap, we had been looking at the proof techniques.

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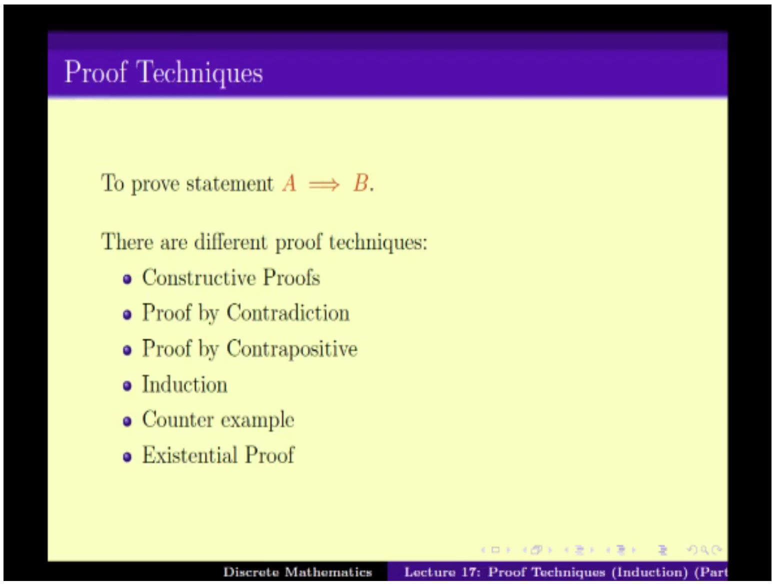

we had been looking at various different proof techniques for proving a statement like A implies B.

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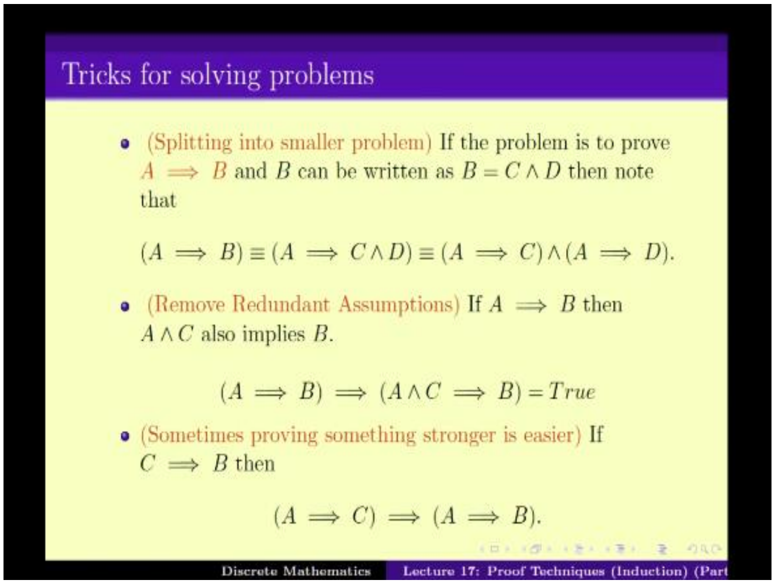

We had seen some tricks like how to split a problem into smaller parts, how to remove redundant assumptions and how sometimes proving something stronger can be easier.

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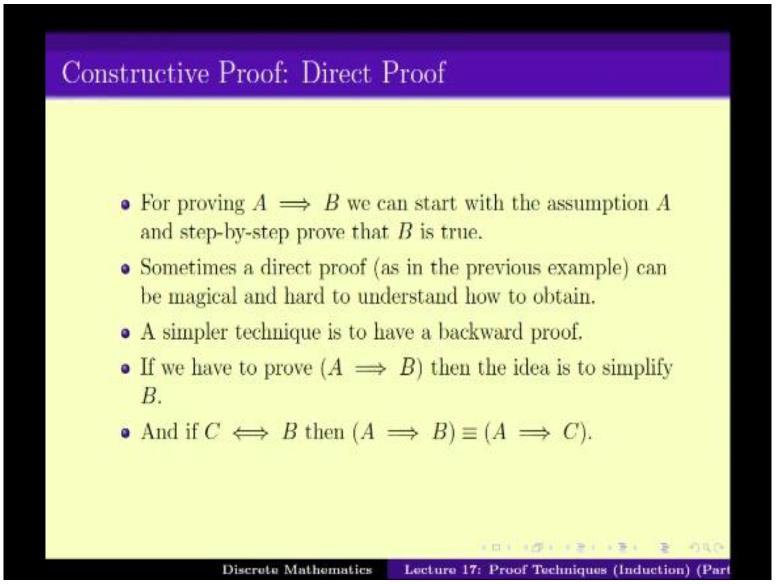

We also saw some proof techniques mainly a direct proof but you prove A implies B directly by working with A or one can go in the backward direction mainly simplified B to get some things simple version of B then the C and then one can prove A implies C. That can be a easier to prove.

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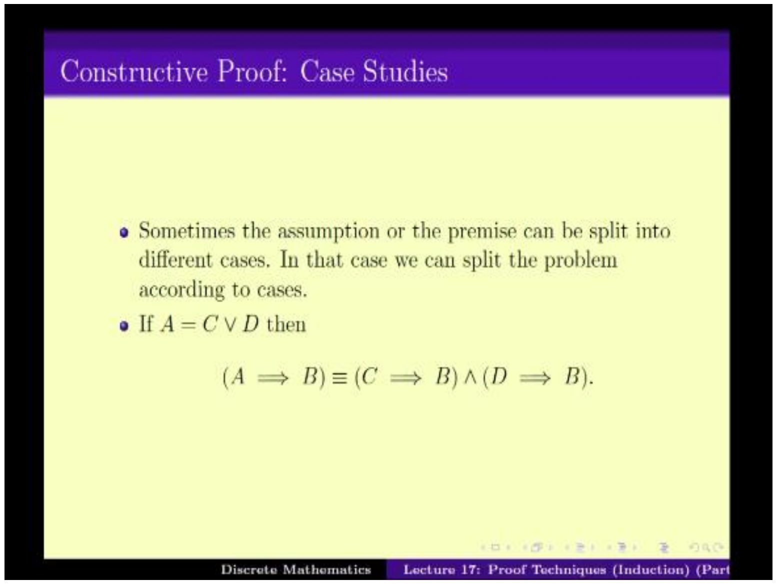

We also saw another proof technique namely case studies, where if you can split the assumptions in two parts or cases then, this problem can be split into smaller problems based on the number of cases. One thing to note here is that, the number of cases into which you split up this problem is or should be finite.

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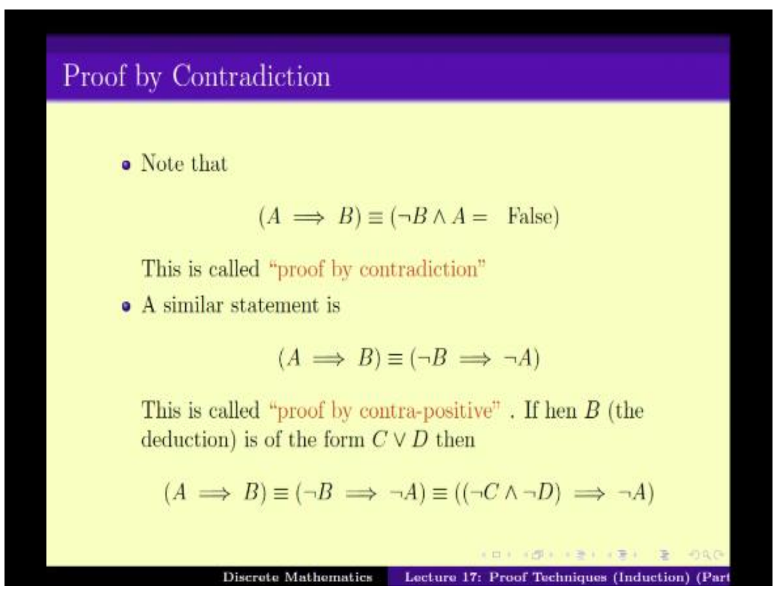

We also looked at two more proof techniques, namely proof by contradiction and proof by contra-positive and both the cases, we look at the problem in a different way namely proving A implies B is similar to is same as proving nod B and A is false and this is called proof by contradiction. While in the proof of contra-positive, you prove nod B implies nod A. We have seen various proofs or brief various problems for which this different kind of proof techniques as alike.

So in this week, we had been focusing on another proof technique called induction. So it is kind of similar to the case study proofs to slightly different.

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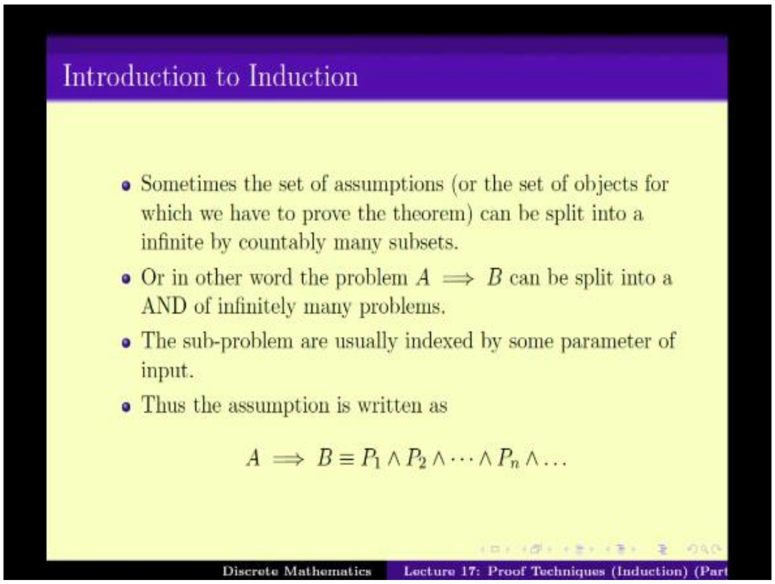

In the fact that, here we split up the set of assumptions, into not finitely number of cases but infinite number of cases, though countable number of cases and that induces or that implies that this proof of A implies B is split up into infinite number of small sub problems and you have to prove all of them. These sub problems are usually indexed by some parameter of input and A implies B is written as P 1 and P 2 and so on till P n and keep goes on.

We have seen in the last two videos some examples where how to split the cases or how to split the problem into the smaller sub problems.

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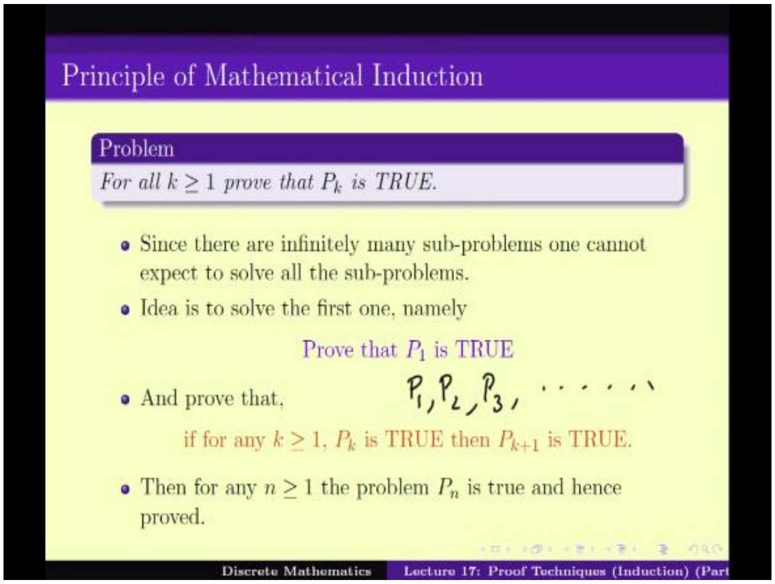

Now, once you split up the problem into the smaller sub problems, we get something of this form that proof this from, that proof for all k greater than 1 proves that P k is true. That is what reduces to when we breakup the main problem into sub problems. Now how to go above proving this statement and that is what the technique of approaching this kind of thing is known as mathematical induction.

So usually the idea is that, since there are infinitely many sub problems one cannot expect to solve all the sub-problems separately. So instead, we start with proving the first problem. After you prove the first problem, the idea is that if we can prove that for any k greater than equal to 1, if P k is true then P k plus one is true, then we would basically prove it for all cases namely same we end of proving a forth the –this first one says P 1 is true, then the second one says that all case P 1 is true therefore P 1 and P 2, if P 2 is true therefore P 3 is true and it goes on like that.

And thus we end up proving that for all n, the problem P n is true which is exactly what we wanted to prove in the first place. Now, why does this proof technique actually work? This work because of the principle of mathematical induction which kind of guarantees us that it will work.

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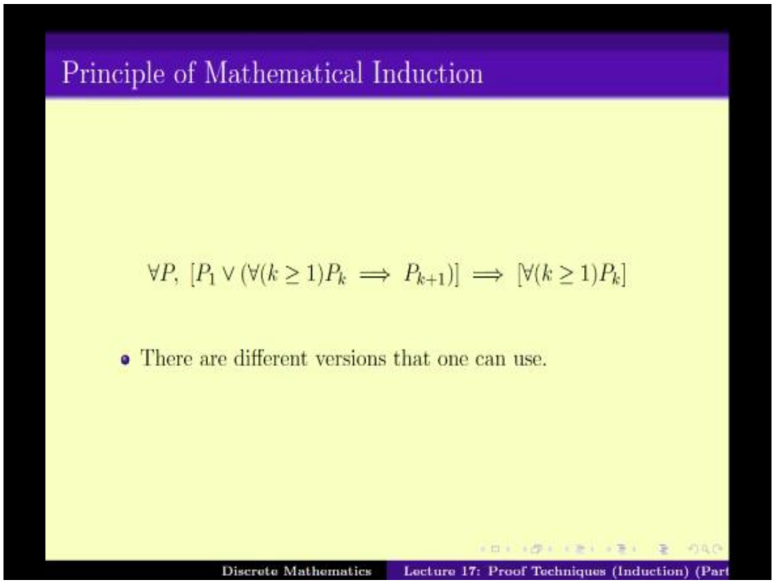

It is a kind of an axiom and it says that if you can do that mainly first prove this statement for P 1 and then for any k prove that P k implies P k plus 1 then you have proved that P k is true for any change. A few steps help us to prove for all infinite number of cases. Now, we have already seen some applications of this mathematical induction in the last three video lectures, in particular, they are different versions of this mathematical induction that one can use.

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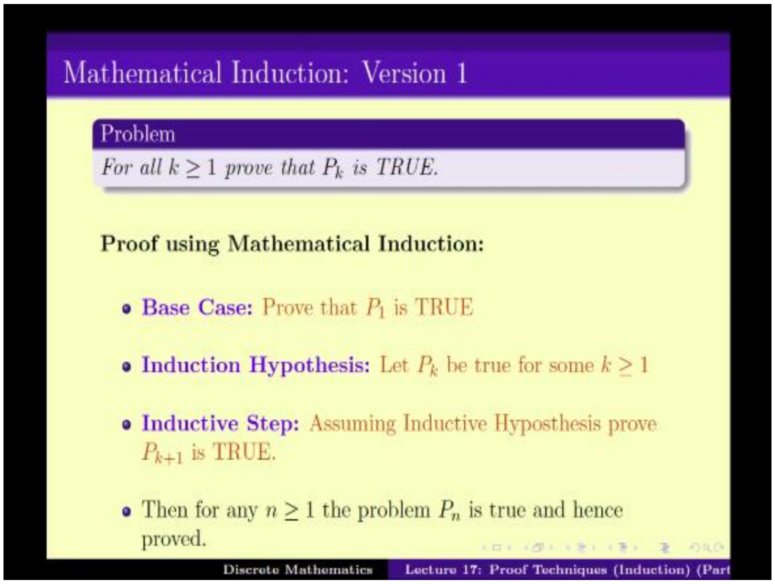

The simplest and basic version is, when we have to prove that for all k greater than equal to 1, we have to prove P k is true.

In this case, we start with the base case which says that first prove P 1 is true then induction hypothesis states that let P k be true for some k greater that equal to 1 and induction hypothesis or induction step says that, assuming the inductive hypothesis proves that P k plus 1 is true and if you can do that, then we end up proving that for all n greater than or equal to 1 the problem P n is true and induction hypothesis guarantees us that if you follow this few steps, base case induction hypothesis definitely states that we get the whole through.

Now the second version is when, if base case does not start from 1, so in another words, It is a slight modification.

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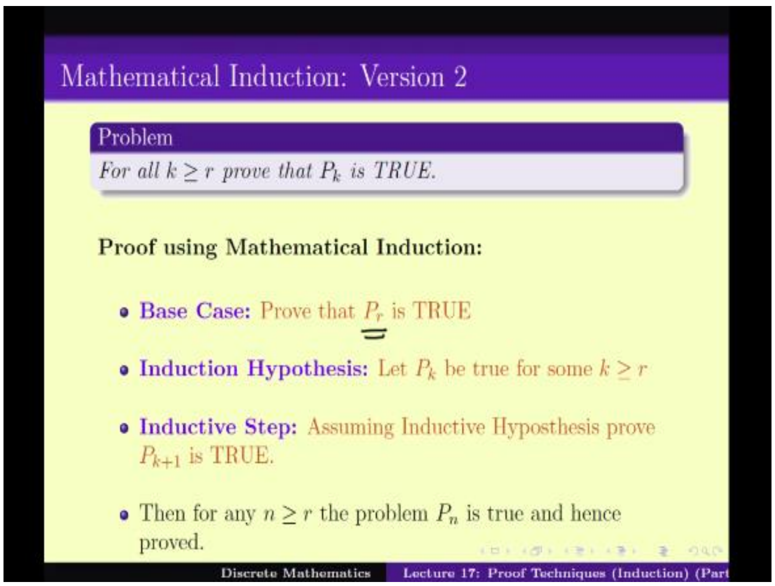

While we say that if you have to prove that for all k greater than or equal to 4, you have to prove P k is true, then we start with the base case where P r is true and then we follow the induction hypothesis we say that if we assume that P k is true for some k greater than equal to 4 or then inductive by using induction hypothesis we have to prove that P k plus 1 is true and we end up proving that for all n greater than equal to r, the problem P n is true.

So this is a slight modification from the basic induction hypothesis and using that one you saw last video how one can solve a problem.

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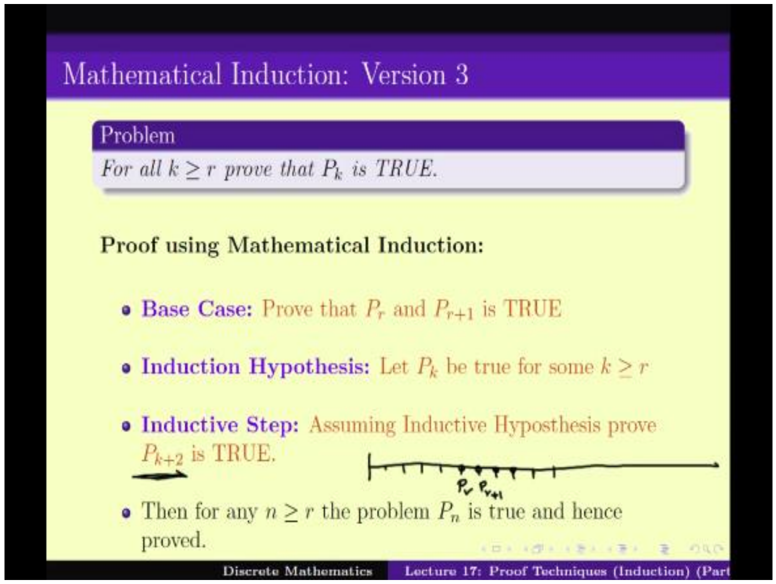

So, in this video we look at a third version, so in this version let's start with the same thing, the statement mean that for all k greater than equal to r we want to prove that P k is true. But now, I start with the different base case namely P r and P r plus 1 is true and induction hypothesis is same, that inductive state is what makes it pretty interesting. Inductive state says that actually the inductive hypothesis, it means that assuming that P k is true for some k proves that P k plus 2 is true.

Note that here is no longer P k plus 1 but P k plus 2 and the induction hypothesis basically states that, then also we get the whole through. Now why is this true? As is told you earlier, the whole work is to ensure that for all i P i is true, so if you think of this real line and say P r is somewhere here then P r plus 1 or plus 2 or plus 3 or plus 4 or plus 5 and so on. What this says that, the base case we have to handle them separately. May be we have to prove for P r, P r plus 1 differently.

So, we have to prove this and this. Now, we have to ensure that for all the terms which are on the right side of P r meaning all this terms we are able to prove that P k is true. Inductive state says that if P k is true from some k then P k plus 2 is true. Among that if P r is true, then P r plus 2 is true. Now, this P r plus 1 is true, so P r plus 3 is true again if this one is true P r plus 2 , P r plus 4 is true and so on and you can see that we will end up proving slowly all of them are true.

So in short, we have to somehow ensure that all this proof technique or all the dots or all the integer points or all the P k or k greater than or equal to r is covered by this step. And once you follow that thing, it works fine. So this is a kind of division why the inductive with particular induction version works. We will see many other induction versions in the coming week and they will use this failed ethic idea of filling up or somehow (()) (11:11) all the k's or k greater than equal to r. Now, let us see how can we use this one for solving a problem.

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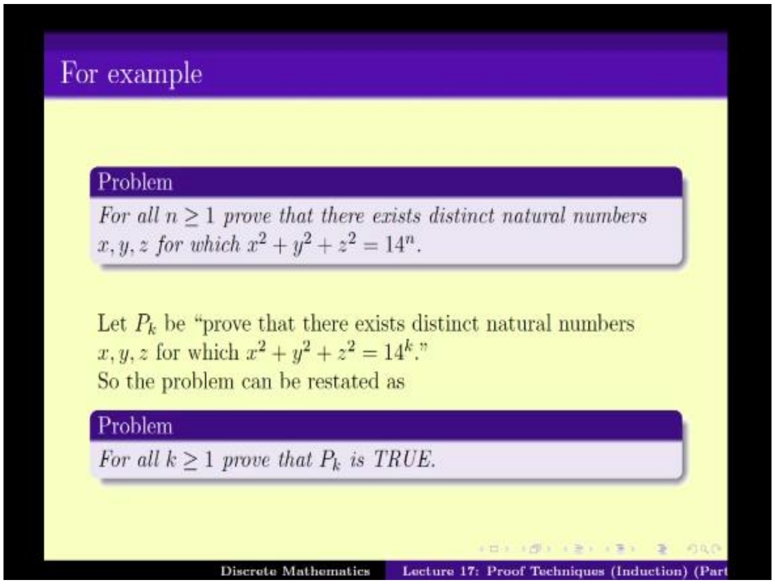

So here is the problem. It says that, for all n greater than or equal to 1 proves that there exist distinct natural numbers x, y and z such that x square plus y square equals to 4 power n. Now, first of all we have to split this problem up into this cases right. So what is the k's case? Of nice we are putting this k's case will be of course put n equals to k and then it says that P k is the problem which says that there exist distinct natural numbers x, y, z such that x square plus y square plus z square is 14 power k.

And if we can prove this particular statement with P k for all k, then we are done right. So by now for fully you will have some idea of how to split the problem into various cases. And we will now apply the inversion three of the induction hypothesis okay.

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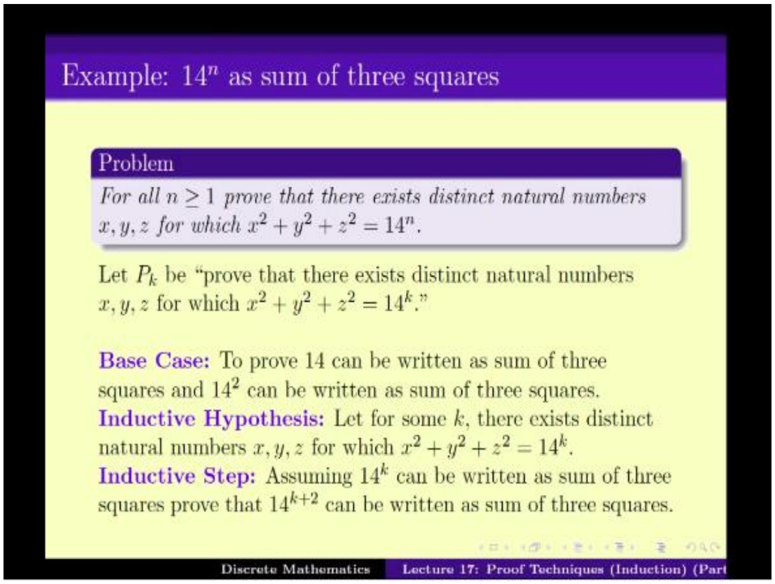

So, in another words, you have to first prove base case, it says that P 1 and P 2 are true. In base case, we have to prove that P 1 and P 2 are true. Induction hypothesis of course says that lets for some case, let's assume that P k is true and the inductive hypothesis we will be proving that if P k is true then, P k plus 2 is true right. It is not P k plus 1, P k plus 2 is true. And induction hypothesis, the principle of mathematical induction will then help us to state that it is true for all n.

So, if I put that this thing in the base case, what does it mean? It means that, first of all the base case will say that the 14 power 1 and 14 power 2 can be written as sum of three squares. Then assuming that the 14 power k is written as some of three squares, meaning 14 power k equals to x square plus y square plus z square for some x, y and z. 14 power k plus 2 can be written as sum of three squares right. So, now our problem is taking care of.

We have a base case and we have to prove the inductive step. Let's go ahead and solve prove them.

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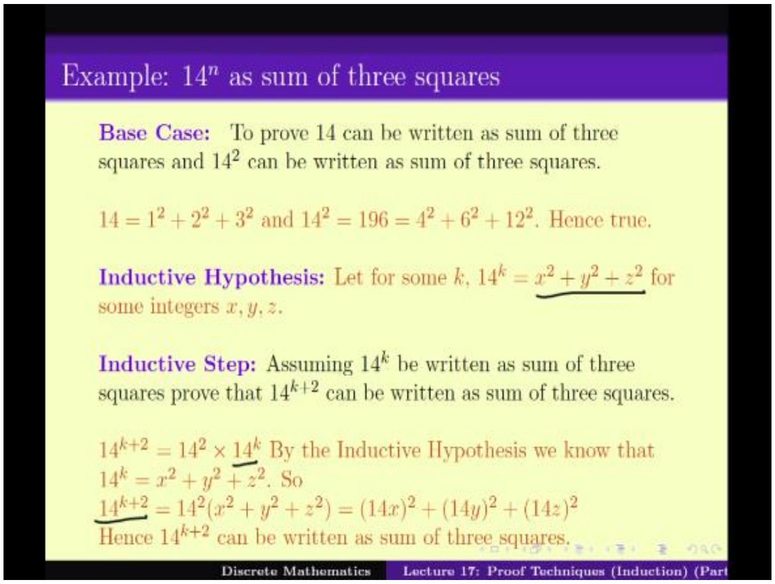

Now, how to prove the base case? Base case says that 14 can be written as sum of three squares and 14 square can be written as sum of three squares. Unfortunately, there is no particular proof technique here that one can apply except for finding the x, y, z for each of the cases. So note that 14 can be written as 1 square plus 2 square plus 3 square. While as 14 square can be written as 4 square plus 6 square plus 12 square. These are some obvious observations that one has to do.

So it means that this problem gets correct for the base cases of P 1 and P 2. Now, inductive hypothesis states that for some k, 14 power k can be written as x square plus y square plus z square for some x, y, z. and we have to prove that 14 power k plus 2 term also be done by that way. Let's see how can we get that. Now 14 power k plus 2 is nothing but 14 square times 14 power k, but this 14 power k of course can be written as x square plus y square plus z square for some x, y, z that is what you are guaranteed by the induction hypothesis.

So, we have 14 power k plus 2 is 14 square, this 14 square times x square plus y square plus z square, which is 14 x square, 14 square x square means 14 x whole square, plus 14 y whole square plus 14 z whole square. Now of course in x, y, z are integers so 14 x, 14 y, 14 z are integers and this ends are proving that 14 power k plus 2 can be written as a sum of three squares hence proved.

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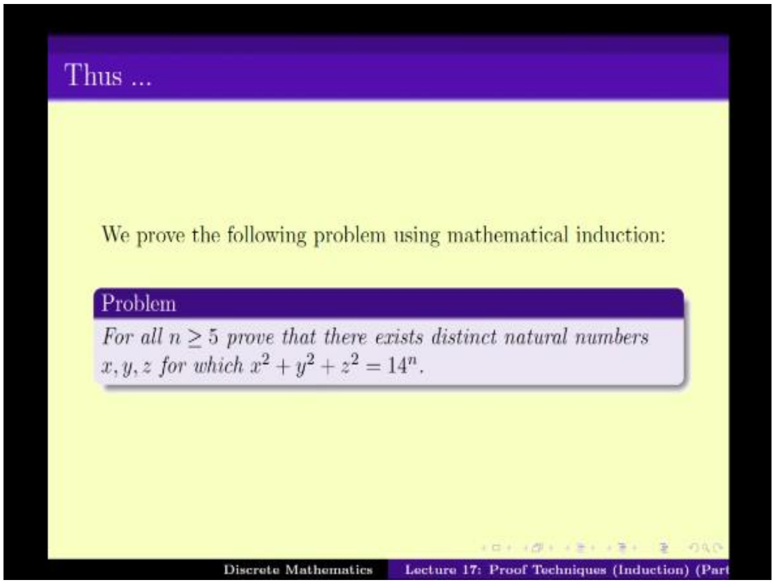

Note that this statement really required the base case to be both of them. If base case has not both of them, then it could not have powered all the integer points meaning all the cases, and hence we would not have got it. So, when we apply a particular mathematical induction technique or proof principle, we just have to ensure that all this points what we want to prove or all the cases are covered. So, that means for all k we are able to.

Thus we have proved that for all n greater than 1 actually 14 power n can be written as x square plus y square plus z square, where x, y, z are some integers. This brings us to the end of this particular video lecture, in the next video lectures, and particularly next weeks, we will be looking at much more complicated induction hypothesis and how we can use them to put more interesting problems. Thank you.