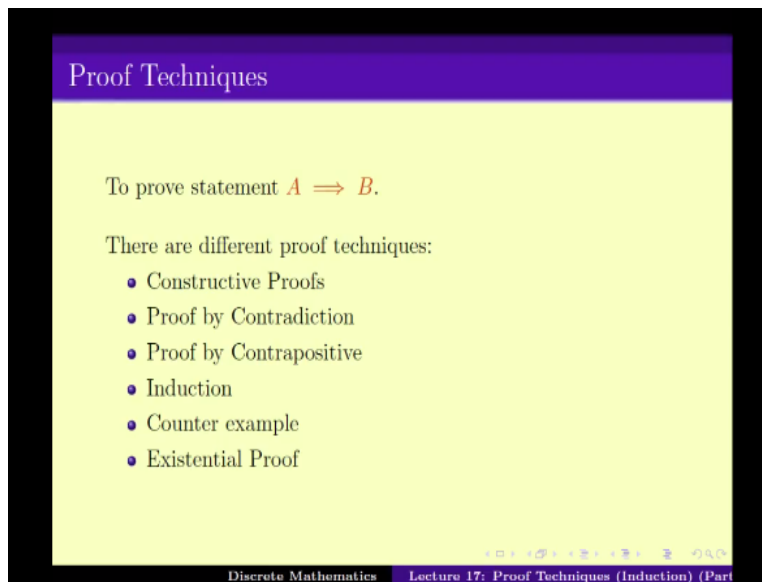


**Discrete Mathematics**  
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**Lecture - 17**  
**Mathematical Induction (Part 4)**

We continue our study of induction. So to recap, we had been looking at the proof techniques.

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we had been looking at various different proof techniques for proving a statement like A implies B.

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Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove  $A \implies B$  and  $B$  can be written as  $B = C \wedge D$  then note that
 
$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$
- (Remove Redundant Assumptions) If  $A \implies B$  then  $A \wedge C$  also implies  $B$ .
 
$$(A \implies B) \implies (A \wedge C \implies B) = True$$
- (Sometimes proving something stronger is easier) If  $C \implies B$  then
 
$$(A \implies C) \implies (A \implies B).$$

Discrete Mathematics    Lecture 17: Proof Techniques (Induction) (Part

We had seen some tricks like how to split a problem into smaller parts, how to remove redundant assumptions and how sometimes proving something stronger can be easier.

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Constructive Proof: Direct Proof

- For proving  $A \implies B$  we can start with the assumption  $A$  and step-by-step prove that  $B$  is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.
- If we have to prove  $(A \implies B)$  then the idea is to simplify  $B$ .
- And if  $C \iff B$  then  $(A \implies B) \equiv (A \implies C)$ .

Discrete Mathematics    Lecture 17: Proof Techniques (Induction) (Part

We also saw some proof techniques mainly a direct proof but you prove  $A$  implies  $B$  directly by working with  $A$  or one can go in the backward direction mainly simplified  $B$  to get some things simple version of  $B$  then the  $C$  and then one can prove  $A$  implies  $C$ . That can be a easier to prove.

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Constructive Proof: Case Studies

- Sometimes the assumption or the premise can be split into different cases. In that case we can split the problem according to cases.
- If  $A = C \vee D$  then

$$(A \implies B) \equiv (C \implies B) \wedge (D \implies B).$$

Discrete Mathematics    Lecture 17: Proof Techniques (Induction) (Part

We also saw another proof technique namely case studies, where if you can split the assumptions in two parts or cases then, this problem can be split into smaller problems based on the number of cases. One thing to note here is that, the number of cases into which you split up this problem is or should be finite.

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Proof by Contradiction

- Note that

$$(A \implies B) \equiv (\neg B \wedge A = \text{False})$$

This is called "proof by contradiction"

- A similar statement is

$$(A \implies B) \equiv (\neg B \implies \neg A)$$

This is called "proof by contra-positive". If hen  $B$  (the deduction) is of the form  $C \vee D$  then

$$(A \implies B) \equiv (\neg B \implies \neg A) \equiv ((\neg C \wedge \neg D) \implies \neg A)$$

Discrete Mathematics    Lecture 17: Proof Techniques (Induction) (Part

We also looked at two more proof techniques, namely proof by contradiction and proof by contra-positive and both the cases, we look at the problem in a different way namely proving A implies B is similar to is same as proving nod B and A is false and this is called proof by contradiction. While in the proof of contra-positive, you prove nod B implies nod A. We have

seen various proofs or brief various problems for which this different kind of proof techniques as alike.

So in this week, we had been focusing on another proof technique called induction. So it is kind of similar to the case study proofs to slightly different.

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Introduction to Induction

- Sometimes the set of assumptions (or the set of objects for which we have to prove the theorem) can be split into a infinite by countably many subsets.
- Or in other word the problem  $A \implies B$  can be split into a AND of infinitely many problems.
- The sub-problem are usually indexed by some parameter of input.
- Thus the assumption is written as

$$A \implies B \equiv P_1 \wedge P_2 \wedge \dots \wedge P_n \wedge \dots$$

Discrete Mathematics Lecture 17: Proof Techniques (Induction) (Part 1)

In the fact that, here we split up the set of assumptions, into not finitely number of cases but infinite number of cases, though countable number of cases and that induces or that implies that this proof of A implies B is split up into infinite number of small sub problems and you have to prove all of them. These sub problems are usually indexed by some parameter of input and A implies B is written as P 1 and P 2 and so on till P n and keep goes on.

We have seen in the last two videos some examples where how to split the cases or how to split the problem into the smaller sub problems.

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**Principle of Mathematical Induction**

**Problem**  
*For all  $k \geq 1$  prove that  $P_k$  is TRUE.*

- Since there are infinitely many sub-problems one cannot expect to solve all the sub-problems.
- Idea is to solve the first one, namely
 

Prove that  $P_1$  is TRUE
- And prove that,  $P_1, P_2, P_3, \dots$   
 if for any  $k \geq 1$ ,  $P_k$  is TRUE then  $P_{k+1}$  is TRUE.
- Then for any  $n \geq 1$  the problem  $P_n$  is true and hence proved.

Discrete Mathematics    Lecture 17: Proof Techniques (Induction) (Part 1)

Now, once you split up the problem into the smaller sub problems, we get something of this form that proof this from, that proof for all k greater than 1 proves that P k is true. That is what reduces to when we breakup the main problem into sub problems. Now how to go above proving this statement and that is what the technique of approaching this kind of thing is known as mathematical induction.

So usually the idea is that, since there are infinitely many sub problems one cannot expect to solve all the sub-problems separately. So instead, we start with proving the first problem. After you prove the first problem, the idea is that if we can prove that for any k greater than equal to 1, if P k is true then P k plus one is true, then we would basically prove it for all cases namely same we end of proving a forth the –this first one says P 1 is true, then the second one says that all case P 1 is true therefore P 1 and P 2, if P 2 is true therefore P 3 is true and it goes on like that.

And thus we end up proving that for all n, the problem P n is true which is exactly what we wanted to prove in the first place. Now, why does this proof technique actually work? This work because of the principle of mathematical induction which kind of guarantees us that it will work.

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Principle of Mathematical Induction

$$\forall P, [P_1 \vee (\forall(k \geq 1)P_k \implies P_{k+1})] \implies [\forall(k \geq 1)P_k]$$

- There are different versions that one can use.

Discrete Mathematics Lecture 17: Proof Techniques (Induction) (Part

It is a kind of an axiom and it says that if you can do that mainly first prove this statement for P 1 and then for any k prove that P k implies P k plus 1 then you have proved that P k is true for any change. A few steps help us to prove for all infinite number of cases. Now, we have already seen some applications of this mathematical induction in the last three video lectures, in particular, they are different versions of this mathematical induction that one can use.

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Mathematical Induction: Version 1

**Problem**  
For all  $k \geq 1$  prove that  $P_k$  is TRUE.

**Proof using Mathematical Induction:**

- **Base Case:** Prove that  $P_1$  is TRUE
- **Induction Hypothesis:** Let  $P_k$  be true for some  $k \geq 1$
- **Inductive Step:** Assuming Inductive Hyposthesis prove  $P_{k+1}$  is TRUE.
- Then for any  $n \geq 1$  the problem  $P_n$  is true and hence proved.

Discrete Mathematics Lecture 17: Proof Techniques (Induction) (Part

The simplest and basic version is, when we have to prove that for all k greater than equal to 1, we have to prove P k is true.

In this case, we start with the base case which says that first prove  $P_1$  is true then induction hypothesis states that let  $P_k$  be true for some  $k$  greater than equal to 1 and induction hypothesis or induction step says that, assuming the inductive hypothesis proves that  $P_{k+1}$  is true and if you can do that, then we end up proving that for all  $n$  greater than or equal to 1 the problem  $P_n$  is true and induction hypothesis guarantees us that if you follow this few steps, base case induction hypothesis definitely states that we get the whole through.

Now the second version is when, if base case does not start from 1, so in another words, It is a slight modification.

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The slide is titled "Mathematical Induction: Version 2" and is set against a yellow background with a purple header. It contains the following text:

**Problem**  
*For all  $k \geq r$  prove that  $P_k$  is TRUE.*

**Proof using Mathematical Induction:**

- **Base Case:** Prove that  $P_r$  is TRUE
- **Induction Hypothesis:** Let  $P_k$  be true for some  $k \geq r$
- **Inductive Step:** Assuming Inductive Hyposthesis prove  $P_{k+1}$  is TRUE.
- Then for any  $n \geq r$  the problem  $P_n$  is true and hence proved.

At the bottom of the slide, there is a footer that reads "Discrete Mathematics Lecture 17: Proof Techniques (Induction) (Part 2)".

While we say that if you have to prove that for all  $k$  greater than or equal to 4, you have to prove  $P_k$  is true, then we start with the base case where  $P_r$  is true and then we follow the induction hypothesis we say that if we assume that  $P_k$  is true for some  $k$  greater than equal to 4 or then inductive by using induction hypothesis we have to prove that  $P_{k+1}$  is true and we end up proving that for all  $n$  greater than equal to  $r$ , the problem  $P_n$  is true.

So this is a slight modification from the basic induction hypothesis and using that one you saw last video how one can solve a problem.

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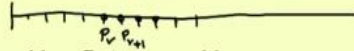
## Mathematical Induction: Version 3

### Problem

For all  $k \geq r$  prove that  $P_k$  is TRUE.

### Proof using Mathematical Induction:

- **Base Case:** Prove that  $P_r$  and  $P_{r+1}$  is TRUE
- **Induction Hypothesis:** Let  $P_k$  be true for some  $k \geq r$
- **Inductive Step:** Assuming Inductive Hypothesis prove  $P_{k+2}$  is TRUE.
- Then for any  $n \geq r$  the problem  $P_n$  is true and hence proved.



So, in this video we look at a third version, so in this version let's start with the same thing, the statement means that for all  $k$  greater than or equal to  $r$  we want to prove that  $P_k$  is true. But now, I start with the different base case namely  $P_r$  and  $P_{r+1}$  is true and induction hypothesis is the same, that inductive state is what makes it pretty interesting. Inductive state says that actually the inductive hypothesis, it means that assuming that  $P_k$  is true for some  $k$  proves that  $P_{k+2}$  is true.

Note that here is no longer  $P_{k+1}$  but  $P_{k+2}$  and the induction hypothesis basically states that, then also we get the whole through. Now why is this true? As is told you earlier, the whole work is to ensure that for all  $i$   $P_i$  is true, so if you think of this real line and say  $P_r$  is somewhere here then  $P_{r+1}$  or plus 2 or plus 3 or plus 4 or plus 5 and so on. What this says that, the base case we have to handle them separately. Maybe we have to prove for  $P_r$ ,  $P_{r+1}$  differently.

So, we have to prove this and this. Now, we have to ensure that for all the terms which are on the right side of  $P_r$  meaning all these terms we are able to prove that  $P_k$  is true. Inductive state says that if  $P_k$  is true from some  $k$  then  $P_{k+2}$  is true. Among that if  $P_r$  is true, then  $P_{r+2}$  is true. Now, this  $P_{r+1}$  is true, so  $P_{r+3}$  is true again if this one is true  $P_{r+2}$ ,  $P_{r+4}$  is true and so on and you can see that we will end up proving slowly all of them are true.



So in short, we have to somehow ensure that all this proof technique or all the dots or all the integer points or all the  $P_k$  or  $k$  greater than or equal to  $r$  is covered by this step. And once you follow that thing, it works fine. So this is a kind of division why the inductive with particular induction version works. We will see many other induction versions in the coming week and they will use this failed idea of filling up or somehow  $(\cdot)$  (11:11) all the  $k$ 's or  $k$  greater than equal to  $r$ . Now, let us see how can we use this one for solving a problem.

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For example

**Problem**  
*For all  $n \geq 1$  prove that there exists distinct natural numbers  $x, y, z$  for which  $x^2 + y^2 + z^2 = 14^n$ .*

Let  $P_k$  be “prove that there exists distinct natural numbers  $x, y, z$  for which  $x^2 + y^2 + z^2 = 14^k$ .”  
 So the problem can be restated as

**Problem**  
*For all  $k \geq 1$  prove that  $P_k$  is TRUE.*

Discrete Mathematics Lecture 17: Proof Techniques (Induction) (Part)

So here is the problem. It says that, for all  $n$  greater than or equal to 1 proves that there exist distinct natural numbers  $x$ ,  $y$  and  $z$  such that  $x$  square plus  $y$  square equals to 4 power  $n$ . Now, first of all we have to split this problem up into this cases right. So what is the  $k$ 's case? Of nice we are putting this  $k$ 's case will be of course put  $n$  equals to  $k$  and then it says that  $P_k$  is the problem which says that there exist distinct natural numbers  $x$ ,  $y$ ,  $z$  such that  $x$  square plus  $y$  square plus  $z$  square is 14 power  $k$ .

And if we can prove this particular statement with  $P_k$  for all  $k$ , then we are done right. So by now for fully you will have some idea of how to split the problem into various cases. And we will now apply the inversion three of the induction hypothesis okay.

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## Example: $14^n$ as sum of three squares

### Problem

For all  $n \geq 1$  prove that there exists distinct natural numbers  $x, y, z$  for which  $x^2 + y^2 + z^2 = 14^n$ .

Let  $P_k$  be "prove that there exists distinct natural numbers  $x, y, z$  for which  $x^2 + y^2 + z^2 = 14^k$ ."

**Base Case:** To prove  $14$  can be written as sum of three squares and  $14^2$  can be written as sum of three squares.

**Inductive Hypothesis:** Let for some  $k$ , there exists distinct natural numbers  $x, y, z$  for which  $x^2 + y^2 + z^2 = 14^k$ .

**Inductive Step:** Assuming  $14^k$  can be written as sum of three squares prove that  $14^{k+2}$  can be written as sum of three squares.

So, in another words, you have to first prove base case, it says that  $P_1$  and  $P_2$  are true. In base case, we have to prove that  $P_1$  and  $P_2$  are true. Induction hypothesis of course says that lets for some case, let's assume that  $P_k$  is true and the inductive hypothesis we will be proving that if  $P_k$  is true then,  $P_{k+2}$  is true right. It is not  $P_{k+1}$ ,  $P_{k+2}$  is true. And induction hypothesis, the principle of mathematical induction will then help us to state that it is true for all  $n$ .

So, if I put that this thing in the base case, what does it mean? It means that, first of all the base case will say that the  $14^1$  and  $14^2$  can be written as sum of three squares. Then assuming that the  $14^k$  is written as some of three squares, meaning  $14^k$  equals to  $x^2 + y^2 + z^2$  for some  $x, y$  and  $z$ .  $14^{k+2}$  can be written as sum of three squares right. So, now our problem is taking care of.

We have a base case and we have to prove the inductive step. Let's go ahead and solve prove them.

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### Example: $14^n$ as sum of three squares

**Base Case:** To prove 14 can be written as sum of three squares and  $14^2$  can be written as sum of three squares.

$$14 = 1^2 + 2^2 + 3^2 \text{ and } 14^2 = 196 = 4^2 + 6^2 + 12^2. \text{ Hence true.}$$

**Inductive Hypothesis:** Let for some  $k$ ,  $14^k = x^2 + y^2 + z^2$  for some integers  $x, y, z$ .

**Inductive Step:** Assuming  $14^k$  be written as sum of three squares prove that  $14^{k+2}$  can be written as sum of three squares.

$$\begin{aligned} 14^{k+2} &= 14^2 \times 14^k \text{ By the Inductive Hypothesis we know that} \\ 14^k &= x^2 + y^2 + z^2. \text{ So} \\ 14^{k+2} &= 14^2(x^2 + y^2 + z^2) = (14x)^2 + (14y)^2 + (14z)^2 \\ \text{Hence } 14^{k+2} &\text{ can be written as sum of three squares.} \end{aligned}$$

Now, how to prove the base case? Base case says that 14 can be written as sum of three squares and 14 square can be written as sum of three squares. Unfortunately, there is no particular proof technique here that one can apply except for finding the  $x, y, z$  for each of the cases. So note that 14 can be written as 1 square plus 2 square plus 3 square. While as 14 square can be written as 4 square plus 6 square plus 12 square. These are some obvious observations that one has to do.

So it means that this problem gets correct for the base cases of P 1 and P 2. Now, inductive hypothesis states that for some  $k$ ,  $14$  power  $k$  can be written as  $x$  square plus  $y$  square plus  $z$  square for some  $x, y, z$ . and we have to prove that  $14$  power  $k$  plus 2 term also be done by that way. Let's see how can we get that. Now  $14$  power  $k$  plus 2 is nothing but  $14$  square times  $14$  power  $k$ , but this  $14$  power  $k$  of course can be written as  $x$  square plus  $y$  square plus  $z$  square for some  $x, y, z$  that is what you are guaranteed by the induction hypothesis.

So, we have  $14$  power  $k$  plus 2 is  $14$  square, this  $14$  square times  $x$  square plus  $y$  square plus  $z$  square, which is  $14x$  square,  $14$  square  $x$  square means  $14x$  whole square, plus  $14y$  whole square plus  $14z$  whole square. Now of course in  $x, y, z$  are integers so  $14x, 14y, 14z$  are integers and this ends are proving that  $14$  power  $k$  plus 2 can be written as a sum of three squares hence proved.

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Thus ...

We prove the following problem using mathematical induction:

**Problem**

*For all  $n \geq 5$  prove that there exists distinct natural numbers  $x, y, z$  for which  $x^2 + y^2 + z^2 = 14^n$ .*

Discrete Mathematics Lecture 17: Proof Techniques (Induction) (Part 1)

Note that this statement really required the base case to be both of them. If base case has not both of them, then it could not have powered all the integer points meaning all the cases, and hence we would not have got it. So, when we apply a particular mathematical induction technique or proof principle, we just have to ensure that all this points what we want to prove or all the cases are covered. So, that means for all  $k$  we are able to.

Thus we have proved that for all  $n$  greater than 1 actually  $14$  power  $n$  can be written as  $x$  square plus  $y$  square plus  $z$  square, where  $x, y, z$  are some integers. This brings us to the end of this particular video lecture, in the next video lectures, and particularly next weeks, we will be looking at much more complicated induction hypothesis and how we can use them to put more interesting problems. Thank you.