

Discrete Mathematics
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Lecture - 16
Mathematical Induction (Part 3)

Welcome to the third video lecture in week four of discrete mathematics. So we continue with our study of induction. So just a quick recap of what we have done till now. We have been studying various proof techniques for proving a statement like A implies B . We have seen some quick tricks about how to solve split the problem into smaller problems or how removing some redundant assumptions can be helpful or how sometimes proving something order can actually be easier.

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Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.
- If we have to prove $(A \implies B)$ then the idea is to simplify B .
- And if $C \iff B$ then $(A \implies B) \equiv (A \implies C)$.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part 3)

We also looked at the some of the proof techniques namely we looked at the direct proof technique value what is A and prove B . Sometime we go in a backward direction namely with evolve with B and simplify to get something C so that proving A imply C is easier than proving A implies B .

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Proof by Contradiction

- Note that

$$(A \implies B) \equiv (\neg B \wedge A = \text{False})$$

This is called “proof by contradiction”

- A similar statement is

$$(A \implies B) \equiv (\neg B \implies \neg A)$$

This is called “proof by contra-positive”. If then B (the deduction) is of the form $C \vee D$ then

$$(A \implies B) \equiv (\neg B \implies \neg A) \equiv ((\neg C \wedge \neg D) \implies \neg A)$$

We also saw the case study where we can split the assumptions into an on-off of finite number of n and that in turn helps us to split the problem into end of a finite number problem and if A equals to C or D then A implies B is same as C implies B and D implies B . We also saw two of ways at looking at the problem in different ways namely proof by contradiction and proof by contra-positive. In one case we – instead proving A implies B .

We end up proving that not B and A is false and in the other case we proved not B implies not A . For certain cases particularly when B is on the form C or D then the second technique may be prove by contra positive helps.

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But what if the statement is false

- To prove the statement $A \implies B$ is not true what to do?
- If the problem is actually of the form $\forall x, A(x) \implies B(x)$ then the negation of this statement is

$$\exists x, A(x) \not\Rightarrow B(x)$$

- Recall $A \implies B$ is same as $(B \vee \neg A)$. So,

$$\exists x A(x) \not\Rightarrow B(x) \equiv \exists x \neg(B(x) \vee \neg A(x)) \equiv \exists x (\neg B(x) \wedge A(x))$$

- So to prove that the original statement is not true we have to find an x such that $(\neg B(x) \wedge A(x))$ is true.

We also saw how one can prove/disprove a statement particular if the statement is of the form for all $A(x)$ implies $B(x)$ one can just prove it by demonstrating x for which $A(x)$ holds but $B(x)$ does not hold and this is what we called as proof by contra example. So these are the proof technique that are there and we also looked at another one which is topic of this week namely induction.

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But what if the statement is false

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$$\exists x, A(x) \not\Rightarrow B(x)$$
- Recall $A \implies B$ is same as $(B \vee \neg A)$. So,

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- So to prove that the original statement is not true we have to find an x such that $(\neg B(x) \wedge A(x))$ is true.

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Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

So just in the case of the case studies where we can split up the assumptions into finite number of cases here the assumption and we split up into any finite number of cases. And that in turn helps to split up the problem into an infinite number of cases and AND of all that. So the problem of A implies B can be split up into AND of infinitely many problems or sub-problems. The sub-problems are indexed by some parameter of the input.

So this technique A implies B gets written as P_1 and P_2 and dot dot till the infinite. So this is the AND of an infinite number of problem. We have seen how to split up some of the problem into this infinite number of sub-problem naturally.

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For example

Problem
 For all $n \geq 1$ prove that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Let P_k be

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}$$

So the problem can be restated as

Problem
 For all $k \geq 1$ prove that P_k is TRUE.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

In particular, we saw how to split up the sum of first n integers as n into $n+1$ by 2 this can be done by—it is the way of to proving directly for all n this is true. Let us first define for P_k for problem which says that prove that 1 into k sum of 1 is k into 1 $k+1$ by 2 and then the problem can be stated as for all k prove that the statement P_k is true. Similarly, for the case problem of 11 divides 23 power n minus one, one can write it as AND of P_k is when P_k is 11 divides 23 power k minus 1.

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For example

Problem
 Prove that for all $n \geq 1$ and all positive real number a_1, a_2, \dots, a_n we have

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{(a_1 a_2 \dots a_n)}$$

Let P_k be for all positive real numbers a_1, a_2, \dots, a_k

$$\frac{a_1 + a_2 + \dots + a_k}{k} \geq \sqrt[k]{(a_1 a_2 \dots a_k)}$$

So the problem can be restated as

Problem
 For all $k \geq 1$ prove that P_k is TRUE.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

And thirdly, we looked at the (()) (05:10) same statement holds where if we define P_k as sum of the average the arithmetic mean of k of real number is greater than the geometric mean of the k real numbers and you want to prove that for all k the P_k is true. Now this is the way of splitting

the problem into infinitely number-- smaller number so infinitely main number of sub-problems.
Now how do you go about proving it?

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The slide is titled "Principle of Mathematical Induction" in a purple header. Below the header, a purple box labeled "Problem" contains the text "For all $k \geq 1$ prove that P_k is TRUE." Below this, three bullet points describe the solution strategy. The first bullet point states that since there are infinitely many sub-problems, one cannot expect to solve all of them. The second bullet point suggests solving the first one, P_1 , which is true, with a handwritten arrow pointing left. The third bullet point states that one should prove that if P_k is true for any $k \geq 1$, then P_{k+1} is true, with a handwritten arrow pointing left. The final bullet point concludes that for any $n \geq 1$, the problem P_n is true and hence proved. At the bottom of the slide, there is a number line diagram with tick marks and the numbers 1, 2, 3, and 4. The footer of the slide reads "Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part 1)".

So the main idea is of course that we first of all cannot end up proving all the sub-problems because there are infinitely many. But the idea is that we can first prove P_1 is true then you can prove that if for some k greater than or equal to 1 P_k is true then you can prove that P_{k+1} is true and this will in turn will help us to prove P_n is true for all n . The idea again is that if here is the real line and this says that first one says that okay P_1 is true the second statement says okay.

Since 1 is true therefore 2 is true, since 2 is true therefore 3 is true and so on so thus I end up proving this P_i for all the natural numbers. So in other words this state would help us to finish or over all the possible P_i assuming that we manage to make a start which is in this case P_1 is true. Right.

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Principle of Mathematical Induction

$$\forall P, [P_1 \vee (\forall(k \geq 1)P_k \implies P_{k+1})] \implies [\forall(k \geq 1)P_k]$$

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

But again the fact that this whole thing do work and this is no (()) (07:11) in the involved in this whole process is guaranteed by what is known as the principle of mathematical induction it is the axiom in the incompetence in math which says that indeed this kind of approach works.

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Principle of Mathematical Induction

Problem
For all $k \geq 1$ prove that P_k is TRUE.

- Idea is to solve the first one, namely
Base Case: Prove that P_1 is TRUE
- Let us assume that we know how to prove P_k
Induction Hypothesis: Let P_k be true for some $k \geq 1$
- Assuming induction hypothesis prove P_{k+1} is TRUE
Inductive Step: Assuming Inductive Hyposthesis prove P_{k+1} is TRUE.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

Thus to prove a statement of this form, there are three parts to do it, so to prove that for all k greater than equal to 1 proving that P_k is true first of all, there is a Base Case where you prove P_1 is true then second is the Induction Hypothesis where you prove that P_k is true or I think that P_k is true for some k greater than equal to 1 and finally assuming induction hypothesis you prove P_{k+1} is true.

Now we saw a couple of example last couple of videos on how to you use this particular mathematical induction to solve problems. Now let us see their different versions of mathematical induction. There are quite a number of different versions of mathematical induction and let us look at the second different version.

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Different Variants of the Induction Hypothesis: Version 2

Problem
For all $k \geq r$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_r is TRUE
- **Induction Hypothesis:** Let P_k be true for some $k \geq r$
- **Inductive Step:** Assuming Inductive Hyposthesis prove P_{k+1} is TRUE.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

So in the second different version is like instead of-- so till now we have looked at a problems which are of the form for all k greater than or equal to 1 P_k is true. But now instead of this 1 if I replace it with some any other integer r than what happen. So if P_k -- if for k greater than equal to r ; if I ask P_k is true than how do you solve it? The idea is again kind of similar that we have the real line here is r , I want to prove that P_r is true, I want to prove that P_{r+1} is true and so on and so forth.

So the idea is again similar the only difference is that you have to engage the Base Case. So the Base Case here it will become that P_r , the r n place is true. And of course the induction hypothesis says that P_k is true for some k greater than or equal to r then we want to use the inductive step to prove that you use inductive hypothesis to prove that P_{k+1} is true. Note that if I end up doing it by the Base Case I know r is true and by this Inductive Step I know okay.

Since r is true therefore $r+1$ is true if $r+1$ —since the $r+1$ is true then $r+2$ is true and so on and so forth. So it goes on and on for proving all the P_n for all the n . Now let us see how one can apply this particular version to get another proof of a problem.

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For example

Problem
For all $n \geq 5$ prove that $2^n > n^2$.

Let P_k be $2^k > k^2$.
So the problem can be restated as

Problem
For all $k \geq 5$ prove that P_k is TRUE.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part

So consider this problem for all n greater than or equal to 5 2^n is greater than n^2 . Now if I just go back one slide you can realize that this problem fits very much in this framework that for all k greater than r instead of that we have for all n greater than equal to five 2^n is greater than n^2 . So first of all how to split it up, what are the P_i . Of course the P_i of the form, P_k is this that prove that 2^k is greater than k^2 .

And the problem is restated as for all k greater than five P_k is true. Now we will apply the mathematical induction the version two of that for obtaining the proof of this. Okay so what are all things we have to prove for that?

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Example: Prove $2^n > n^2$ for $k \geq 5$

Problem

For all $n \geq 5$ prove that $2^n > n^2$.

Let P_k be $2^k > k^2$.

Base Case: To prove $2^5 > 5^2$.

Inductive Hypothesis: Let for some k , $2^k > k^2$.

Inductive Step: Assuming $2^k > k^2$ prove $2^{k+1} > (k+1)^2$

First of all, we have to prove the Base Case namely P_5 is true, the Induction hypothesis which says that for all k or for some k P_k is true and then using that assuming P_k is true prove that P_{k+1} is true. Once you get in this form it should be a standard step forward proof from now. So let us just put everything in prospective plugging in the what are the statements of P_k P_{k+1} and P_5 . So the Base Case is to prove 2 power 5 is greater than 5 square.

Induction hypothesis-- of course we let for some k 2 power k is greater than k square and then Inductive Step assuming 2 power k is greater than k square prove that 2 power k plus one is greater than k plus 1 whole square.

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Example: Prove $2^n > n^2$ for $k \geq 5$

Base Case: To prove $2^5 > 5^2$.

$2^5 = 32 > 25 = 5^2$. Hence base case is true.

Inductive Hypothesis: Let for some k , $2^k > k^2$.

Inductive Step: Assuming $2^k > k^2$ prove $2^{k+1} > (k+1)^2$.

$2^{k+1} = 2 \times 2^k$ By the Inductive Hypothesis we know that $2^k > k^2$.

So $2^{k+1} > 2k^2$. Now $2k^2 \geq (k+1)^2$ (Why??) Exercise .

Hence $2^{k+1} > (k+1)^2$.

$k \geq 3$

Now to start with the Base Case how do you prove the Base Case? This is kind of standard easy thing to verify that 2^5 is 32 is greater than 25 which is equal to 5^2 and Base Case is true. The induction hypothesis says that for some k let for some k 2^k is bigger than k^2 . Now assuming this one we have to go ahead and prove the Inductive Step, so we have to prove that 2^{k+2} is bigger than $(k+1)^2$.

In other words, this 2^{k+1} is equal to two times 2^k . Now this 2^k is of course greater than k^2 by Induction Hypothesis. So by induction hypothesis-- sorry this should be 2^k is bigger than k^2 . So 2^{k+1} is bigger than two times k^2 . Now note that this is what we have to prove or this is where is enough to prove that 2^{k+1} is bigger than $(k+1)^2$.

Now why is this one true? I leave it to you guys to check why this is true. You have to apply all these techniques that you have learnt so called in the last couple of weeks and prove that for any k 2^k is greater than k^2 . Right. And thus we will be getting that 2^{k+1} is greater than $(k+1)^2$. Now things to note here is that first of all here we-- the Induction Hypothesis or principle of mathematical induction helps to solve this problem by converting it to some very simple three steps process.

First of all, Base Case has the right Induction Hypothesis and then use it to prove the Inductive Step. Proving the Inductive Step, yes many times quite simple straightforward proof or-- one of these standard techniques namely direct proof or proof by contradiction or case studies will work. Now as I pointed out that the Base Case is very important. One thing to note is that in this case the Inductive Step once you get the proof of this 2^k is greater than k^2 .

You can get that the Inductive Step will work for any k greater than equal to 2-- I mean greater than equal to 3 for any k greater than equal to 3, this is true. But the Base Case will not be true for k equals to 2 so 2^2 is not bigger than 2^2 , 2^3 is 8 which is not bigger than nine. So although the Inductive Step might not be the need case that k bigger than 5 all it needs is greater than 3.

But since the Base Case cannot hold for k equal to 3 our proof cannot solve it for k equal greater than equal to 3. In other words, what I am trying to say is that the Base Case is a very important place just proving Inductive Step and Inductive Hypothesis will not give you.

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Thus ...

We prove the following problem using mathematical induction:

Problem

For all $n \geq 5$ prove that $2^n > n^2$.

Discrete Mathematics Lecture 16: Proof Techniques (Induction) (Part 2)

With this we end-- comes to the end of this video lecture. We have proved that for n greater than equal to 5 2^n is greater than n^2 . We have used the new version of the mathematical induction to prove that 2^n is greater than n^2 for n greater than equal to 5. In the next video lecture, we will be looking at some more versions of mathematical induction. Thank you.