Discrete Mathematics
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Lecture - 16
Mathematical Induction (Part 3)

Welcome to the third video lecture in week four of discrete mathematics. So we continue with our study of induction. So just a quick recap of what we have done till now. We have been studying various proof techniques for proving a statement like A implies B. We have seen some quick tricks about how to solve split the problem into smaller problems or how removing some redundant assumptions can be helpful or how sometimes proving something order can actually be easier.

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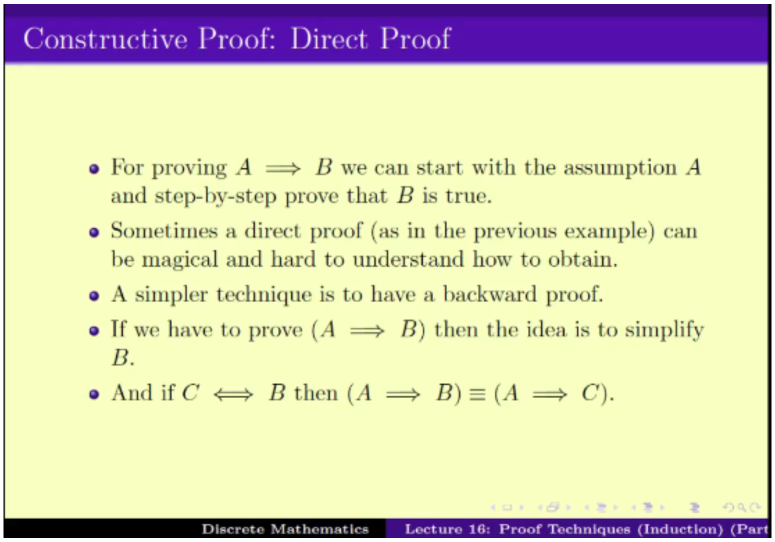

We also looked at the some of the proof techniques namely we looked at the direct proof technique value what is A and prove B. Sometime we go in a backward direction namely with evolve with B and simplify to get something C so that proving A imply C is easier than proving A implies B.

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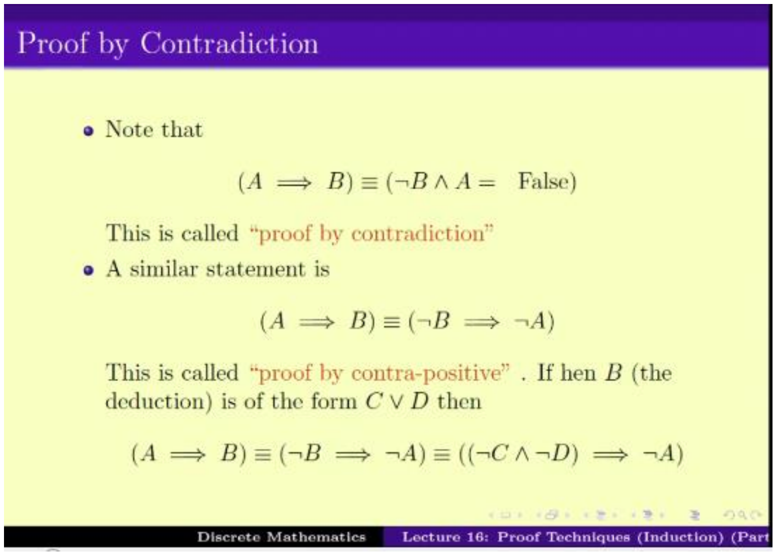

We also saw the case study where we can split the assumptions into an on-off of finite number of n and that in turn helps us to split the problem into end of a finite number problem and if A equals to C or D then A implies B is same as C implies B and D implies B. We also saw two of ways at looking at the problem in different ways namely proof by contradiction and proof by contra-positive. In one case we – instead proving A implies B.

We end up proving that not B and A is false and in the other case we proved not B implies not A. For certain cases particularly when B is on the form C or D then the second technique may be prove by contra positive helps.

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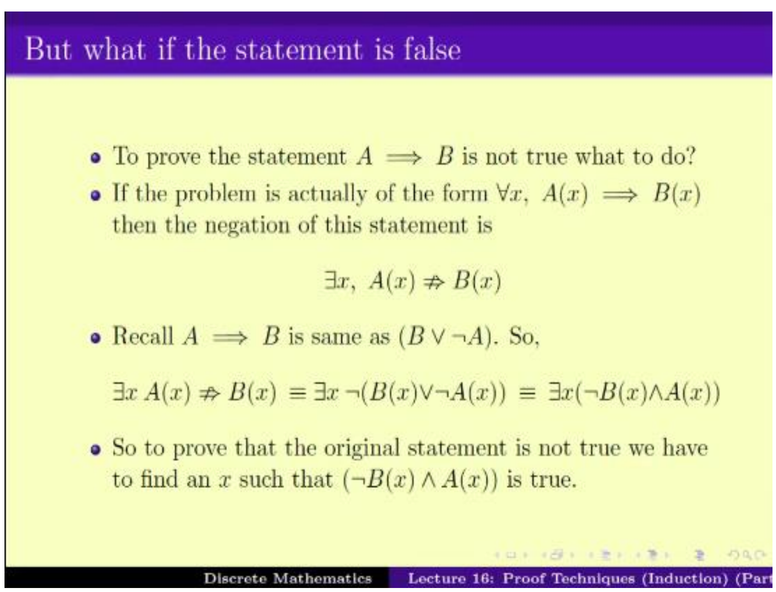

We also saw how one can prove disprove a statement particular if the statement is of the form for all A(x) implies B(x) one can just prove it by demonstrating x for which A(x) holds but B(x) does not hold and this is what we called as proof by contra example. So these are the proof technique that are there and we also looked at another one which is topic of this week namely induction.

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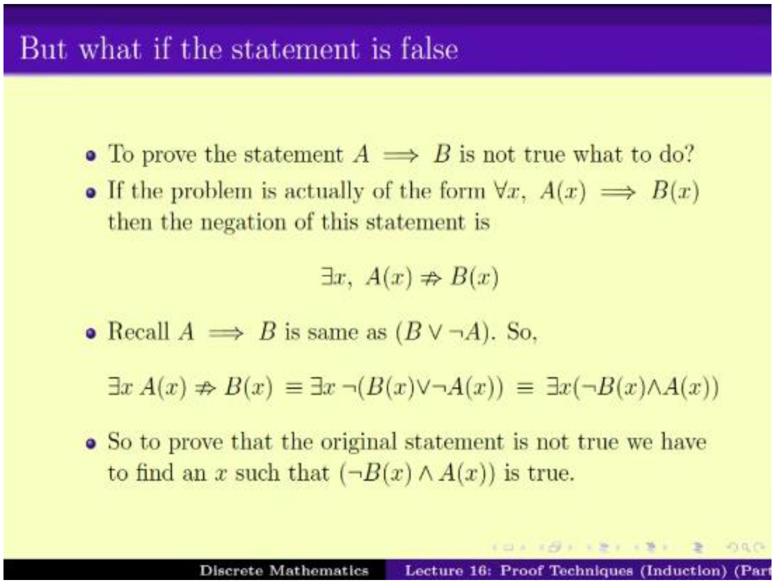

So just in the case of the case studies where we can split up the assumptions into finite number of cases here the assumption and we split up into any finite number of cases. And that in turn helps to split up the problem into an infinite number of cases and AND of all that. So the problem of A implies B can be split up into AND of infinitely many problems or sub-problems. The sub-problems are indexed by some parameter of the input.

So this technique A implies B gets written as P1 and P2 and dot dot till the infinite. So this is the AND of an infinite number of problem. We have seen how to split up some of the problem into this infinite number of sub-problem naturally.

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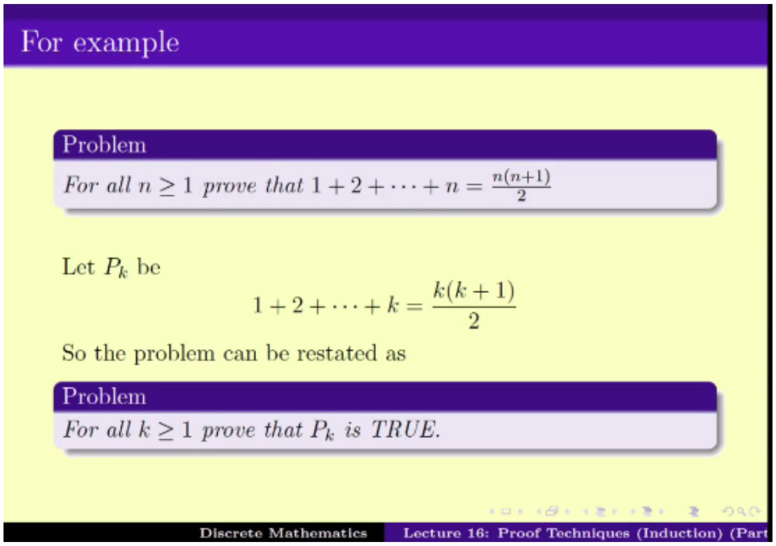

In particular, we saw how to split up the sum of first n integers as n into n+1 by 2 this can be done by—it is the way of to proving directly for all n this is true. Let us first define for Pk for problem which says that prove that 1 into k sum of 1 is k into 1 k+1 by 2 and then the problem can be stated as for all k prove that the statement Pk is true. Similarly, for the case problem of 11 divides 23 power n minus one, one can write it as AND of Pk is when Pk is 11 divides 23 power k minus 1.

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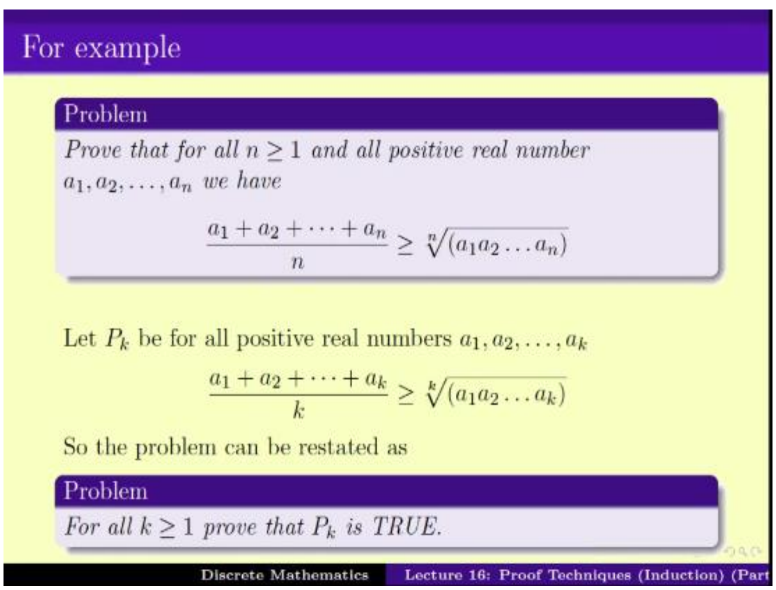

And thirdly, we looked at the (()) (05:10) same statement holds where if we define Pk as sum of the average the arithmetic mean of k of real number is greater than the geometric mean of the k real numbers and you want to prove that for all k the Pk is true. Now this is the way of splitting the problem into infinitely number-- smaller number so infinitely main number of sub-problems. Now how do you go about proving it?

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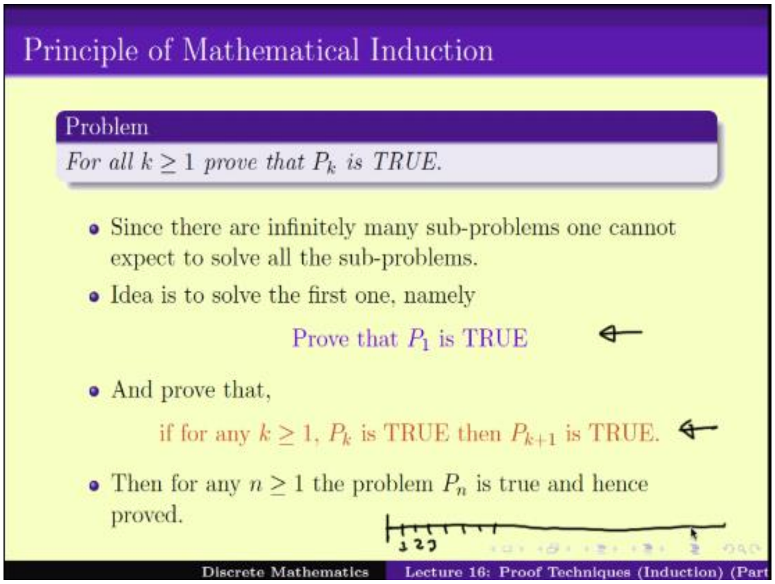

So the main idea is of course that we first of all cannot end up proving all the sub-problems because there are infinitely many. But the idea is that we can first prove P1 is true than you can prove that if for some k greater than or equal to 1 Pk is true than you can prove that Pk+1 is true and this will in turn will help us to prove Pn is true for all n. The idea again is that if here is the real line and this says that first one says that okay P1 is true the second statement says okay.

Since 1 is true therefore 2 is true, since 2 is true therefore 3 is true and so on so thus I end up proving this Pi for all the natural numbers. So in other words this state would help us to finish or over all the possible Pi assuming that we manage to make a start which is in this case P1 is true. Right.

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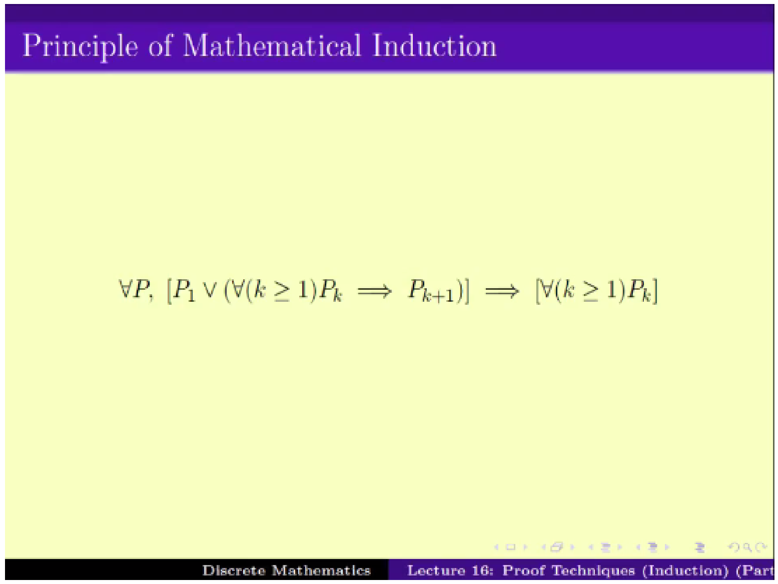

But again the fact that this whole thing do work and this is no (()) (07:11) in the involved in this whole process is guaranteed by what is known as the principle of mathematical induction it is the axiom in the incompetence in math which says that indeed this kind of approach works.

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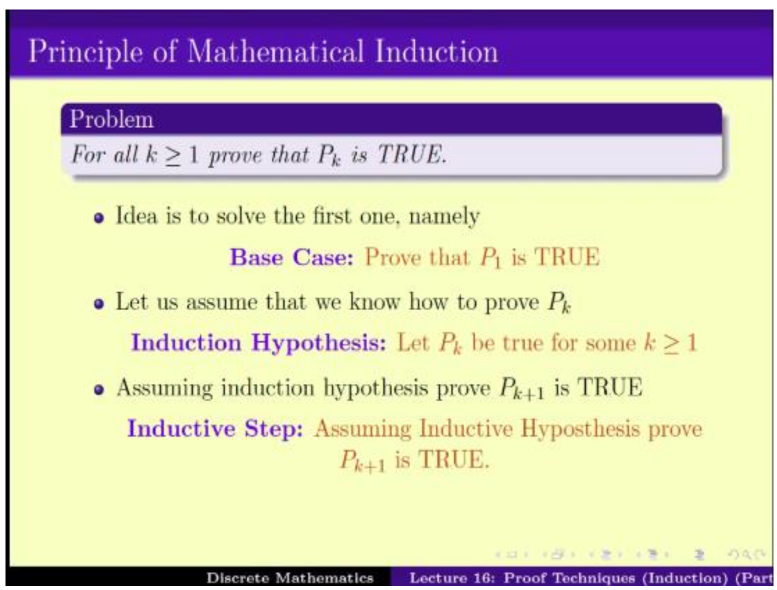

Thus to prove a statement of this form, there are three parts to do it, so to prove that for all k greater than equal to 1 proving that Pk is true first of all, there is a Base Case where you prove Pl is true then second is the Induction Hypothesis where you prove that Pk is true or I think that Pk is true for some k greater than equal to 1 and finally assuming induction hypothesis you prove Pk+1 is true.

Now we saw a couple of example last couple of videos on how to you use this particular mathematical induction to solve problems. Now let us see their different versions of mathematical induction. There are quite a number of different versions of mathematical induction and let us look at the second different version.

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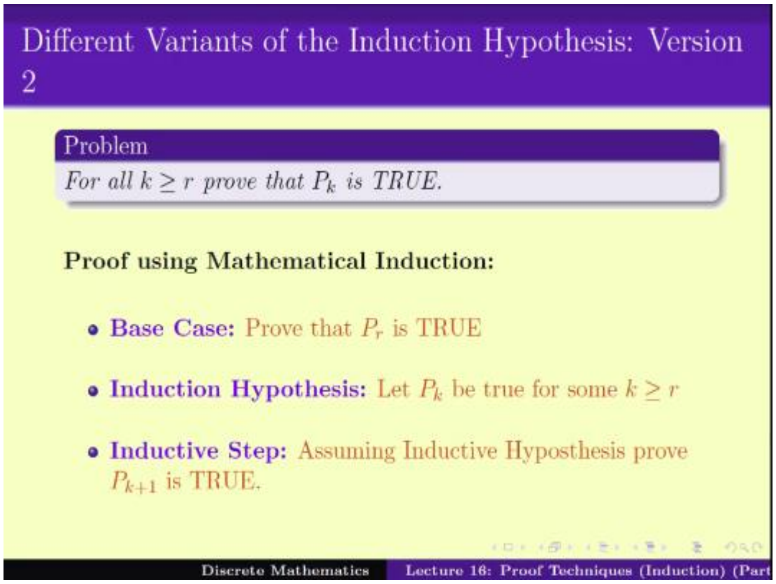

So in the second different version is like instead of-- so till now we have looked at a problems which are of the form for all k greater than or equal to 1 Pk is true. But now instead of this 1 if I replace it with some any other integer r than what happen. So if Pk-- if for k greater than equal to r; if I ask Pk is true than how do you solve it? The idea is again kind of similar that we have the real line here is r, I want to prove that Pr is true, I want to prove that Pr + 1 is true and so on and so forth.

So the idea is again similar the only difference is that you have to engage the Base Case. So the Base Case here it will become that Pr, the r n place is true. And of course the induction hypothesis says that Pk is true for some k greater than or equal to r then we want to use the inductive step to prove that you use inductive hypothesis to prove that Pk+1 is true. Note that if I end up doing it by the Base Case I know r is true and by this Inductive Step I know okay.

Since r is true therefore r+1 is true if r+—since the r +1 is true then r+2 is true and so on and so forth. So it goes on and on for proving all the Pn for all the n. Now let us see how one can apply this particular version to get an another proof of a problem.

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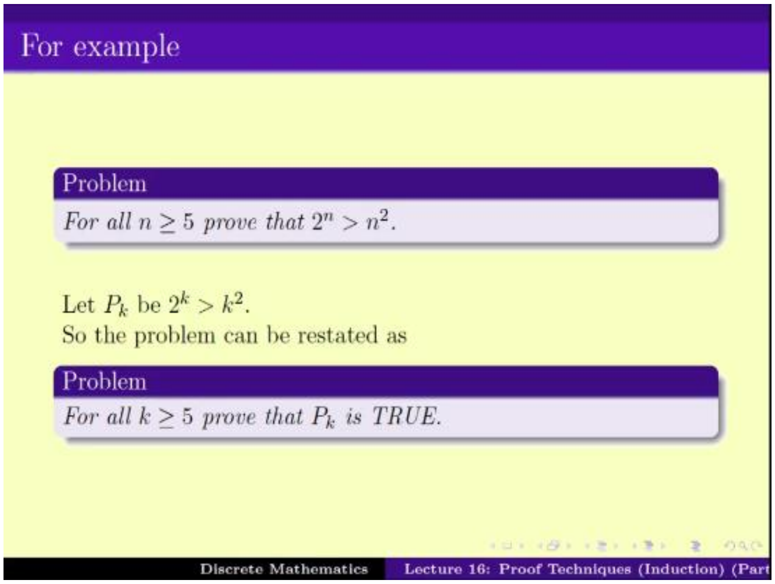

So consider this problem for all n greater than or equal to 5 2 power n is greater than n square. Now if I just go back one slide you can realize that this problem fits very much in this framework that for all k greater than r instead of that we have for all n greater than equal to five P-- two power n is greater than n square. So first of all how to split it up, what are the Pi. Of course the Pi of the form, Pk is this that prove that two power k is greater than k square.

And the problem is restated as for all k greater than five Pk is true. Now we will apply the mathematical induction the version two of that for obtaining the proof of this. Okay so what are all things we have to prove for that?

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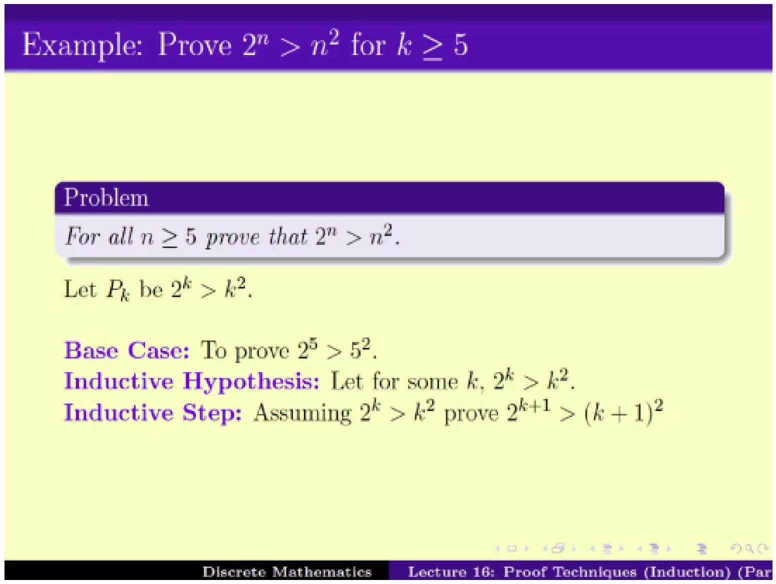

First of all, we have to prove the Base Case namely P5 is true, the Induction hypothesis which says that for all k or for some k Pk is true and then using that assuming Pk is true prove that Pk+1 is true. Once you get in this form it should be a standard step forward proof from now. So let us just put everything in prospective plugging in the what are the statements of Pk Pk+1 and P5. So the Base Case is to prove 2 power 5 is greater than 5 square.

Induction hypothesis-- of course we let for some k 2 power k is greater than k square and then Inductive Step assuming 2 power k is greater than k square prove that 2 power k plus one is greater than k plus 1 whole square.

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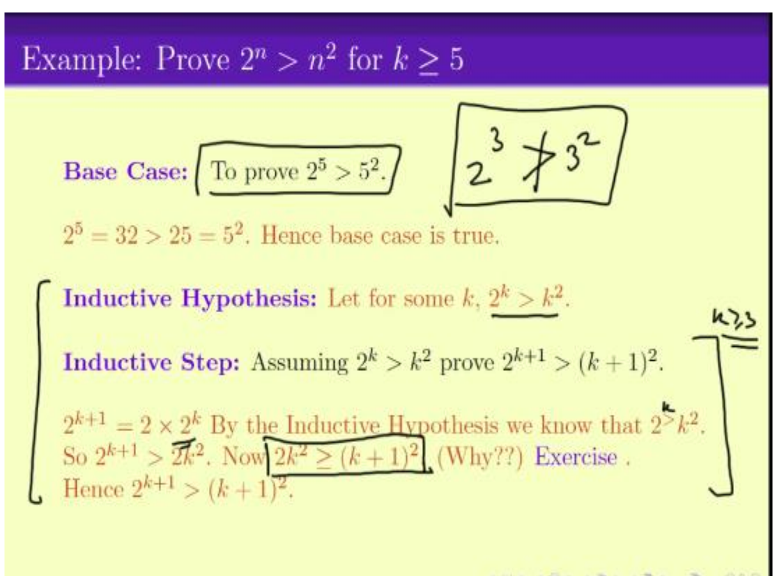

Now to start with the Base Case how do you prove the Base Case? This is kind of standard easy thing to verify that 2 power five is 32 is greater than 25 which is equal to 5 square and Base Case is true. The induction hypothesis says that for some k let for some k 2 power k is bigger than k square. Now assuming this one we have to go ahead and prove the Inductive Step, so we have to prove that 2 power k+2 is bigger than k+1 whole square.

In other words, this 2 power k+1 is equal to two times 2 power k. Now this 2 power k is of course greater than k square by Induction Hypothesis. So by induction hypothesis-- sorry this should 2 power k—2 power k is bigger than k square. So 2 power k+1 is bigger than two times k square. Now note that this is what we have to prove or this is where is enough to prove that 2 power k square is bigger than k+1 whole square.

Now why is this one true? I leave it to you guys to check why this is true. You have to apply all these techniques that you have learnt so called in the last couple of weeks and prove that for any k 2 power k square is greater k+1 whole square. Right. And thus we will be getting that 2 power k+1 is greater than k+1 whole square. Now things to note here is that first of all here we-- the Induction Hypothesis or principle of mathematical induction helps to solve this problem by converting it to some very simple three steps process.

First of all, Base Case has the right Induction Hypothesis and then use it to prove the Inductive Step. Proving the Inductive Step, yes many times quite simple straightforward proof or-- one of these standard techniques namely direct proof or proof by contradiction or case studies will what. Now as I pointed out that the Base Case is very important. One thing to note is that in this case the Inductive Step once you get the proof of this 2 k square greater then k+1 whole square.

You can get that the Inductive Step will work for any k greater than equal to 2-- I mean greater than equal to 3 for any k greater than equal to 3, this is true. But the Base Case will not be true for k equals to 3 so 2 power three is not bigger than 3 square, 2 power 3 is 8 which is not bigger than nine. So although the Inductive Step might not the need case that k bigger than 5 all it needs is greater than 3.

But since the Base Case cannot hold for k equal to 3 our proof cannot solve it for k equal greater than equal to 3. In other words, what I am trying to say is that the Base Case is a very important place just proving Inductive Step and Inductive Hypothesis will not give you.

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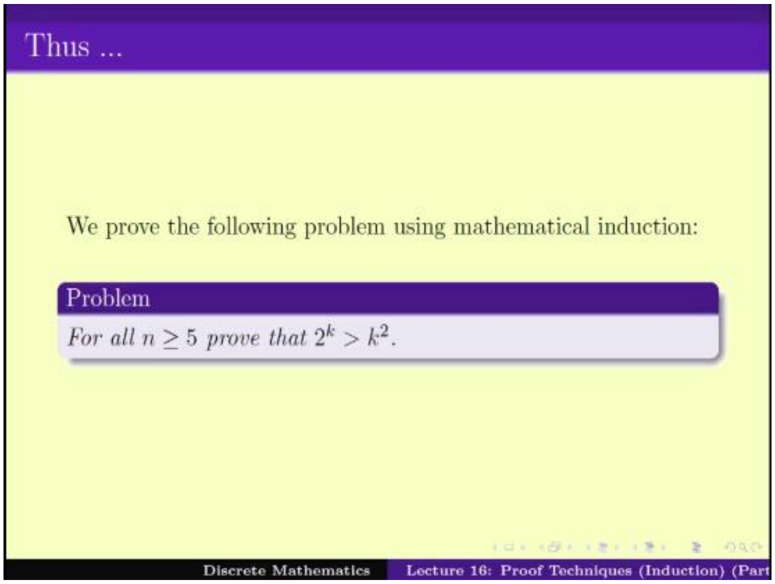

With this we end-- comes to the end of this video lecture. We have proved that for n greater than equal to 5 2 power k is greater than k square. We have used the new version of the mathematical induction to prove that 2k is greater than—2 power k is greater than k square for k greater then equal to 5. In the next video lecture, we will be looking at some more versions of mathematical induction. Thank you.