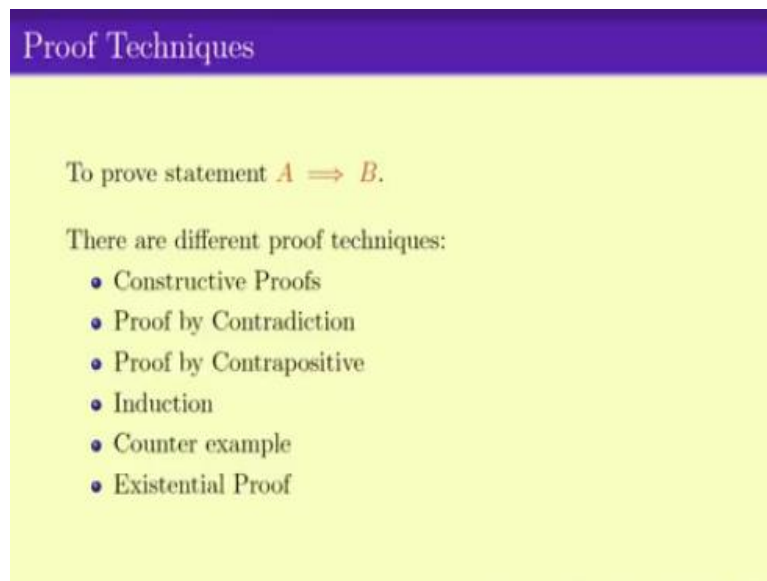


Discrete Mathematics
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Lecture - 15
Mathematical Induction (Part 2)

Welcome to the second video lecture in which four of the discrete mathematics, in this video lecture, we will continue with our understanding of induction.

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The slide has a purple header with the text "Proof Techniques" in white. The main content is on a light yellow background. It starts with the text "To prove statement $A \implies B$." followed by "There are different proof techniques:" and a bulleted list of six techniques.

Proof Techniques

To prove statement $A \implies B$.

There are different proof techniques:

- Constructive Proofs
- Proof by Contradiction
- Proof by Contrapositive
- Induction
- Counter example
- Existential Proof

A quick recap, we were looking at the proof techniques, mainly to prove how to prove A implies B, and we have seen that there are quite a number of different proof techniques available was Constructive proof, proof by contradiction, Contrapositive, induction, counter example and existential proof.

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Which approach to apply

- It depends on the problem.
- Sometimes the problem can be split into smaller problems that can be easier to tackle individually.
- Sometimes viewing the problem in a different way can also help in tackling the problem easily.
- Whether to split a problem or how to split a problem or how to look at a problem is an ART that has to be developed.
- There are some thumb rules but at the end it is a skill you develop using a lot of practice.

Now, this is a slide I have shown you every time, which basically states that there is no rule of which proof technique should be applied to which problem itself, art that we have to develop. Now to quickly recap, whatever of things we have till now.

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Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove $A \implies B$ and B can be written as $B = C \wedge D$ then note that

$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$

- (Remove Redundant Assumptions) If $A \implies B$ then $A \wedge C$ also implies B .

$$(A \implies B) \implies (A \wedge C \implies B) = \text{True}$$

- (Sometimes proving something stronger is easier) If $C \implies B$ then

$$(A \implies C) \implies (A \implies B).$$

We saw some tricks of how to split the problem into smaller problems depending on whether B can be written as C and D . How to remove redundant assumptions and thirdly how to see that sometimes proving something harder or stronger can be easier.

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Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.
- If we have to prove $(A \implies B)$ then the idea is to simplify B .
- And if $C \iff B$ then $(A \implies B) \equiv (A \implies C)$.

We also saw some proof techniques, namely we looked at the direct proof technique where one works with A and then end up proving B . Or one can go backward and can start simplifying B and slowly get to a situation where A implies C can be easier, but C is equivalent of B just a simplified form. So we saw on few examples of this.

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Proof by Contradiction

- Note that

$$(A \implies B) \equiv (\neg B \wedge A = \text{False})$$

This is called "proof by contradiction"

- A similar statement is

$$(A \implies B) \equiv (\neg B \implies \neg A)$$

This is called "proof by contra-positive". If hen B (the deduction) is of the form $C \vee D$ then

$$(A \implies B) \equiv (\neg B \implies \neg A) \equiv ((\neg C \wedge \neg D) \implies \neg A)$$

We also looked at the case study, in this case if we split the assumptions into some constant number of cases, so what happens is that if you write A as C or D , then A implies B get split up as C implies B and D implies B . This particular case study proof is something relevant to the proof of, prove by induction also, we will see very soon.

So other than the case study proofs, we also looked at the proof by contradiction, mainly proving A implies B is same as proving not B and A is false. Or in a similar way, one can

prove A implies B by proving that not B implies not A. This second one is more like proof by contra-positive and this can be useful particularly when B can be written as C or D, in that case A implies B can be written as not C and not D implies not A.

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But what if the statement is false

- To prove the statement $A \implies B$ is not true what to do?
- If the problem is actually of the form $\forall x, A(x) \implies B(x)$ then the negation of this statement is

$$\exists x, A(x) \not\Rightarrow B(x)$$
- Recall $A \implies B$ is same as $(B \vee \neg A)$. So,

$$\exists x A(x) \not\Rightarrow B(x) \equiv \exists x \neg(B(x) \vee \neg A(x)) \equiv \exists x(\neg B(x) \wedge A(x))$$
- So to prove that the original statement is not true we have to find an x such that $(\neg B(x) \wedge A(x))$ is true.

We also solved this case of proof by counter example, where if I have given a problem of the form for x for all x prove that, prove or disprove Ax implies B . To disprove the statement, one needs to give that x such that Ax does not implies Bx , or in other words we have given x such that Bx is not true, but Ax is true. So these are the proof technique that we saw last week.

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Introduction to Induction

- Sometimes the set of assumptions (or the set of objects for which we have to prove the theorem) can be split into a infinite by countably many subsets.
- Or in other word the problem $A \implies B$ can be split into a AND of infinitely many problems.
- The sub-problem are usually indexed by some parameter of input.
- Thus the assumption is written as

$$A \implies B \equiv P_1 \wedge P_2 \wedge \dots \wedge P_n \wedge \dots$$

In the last video lecture, we started with this to proof by induction. The proof of induction is very similar to case study proof, except that in case studies once get the assumptions into

constantly many number of cases. And thus, the problem gets split up into a constantly varying and of some small problems. But there are times when one can split up the assumptions into infinitely many, but by countably many number of cases.

In that case, of course just in the case of case studies, the problem can split up into a AND of infinitely many number of components. The sub problems are usually we do get indexed by some parameter of the input, or intent of the parameter of the input, in other words one would like, one likes this whole thing of A implies B as P1 and P2 and so on as the infinite collection of thing. So thus, to prove A implies B one is to prove, that this Pi is false for all the i's,

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The slide has a purple header with the text "For example". Below the header, there is a light yellow background. At the top of this background is a dark purple box with the word "Problem" in white. Below this box, the text reads "For all $n \geq 1$ prove that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ ". Below this is the text "Let P_k be" followed by the equation $1 + 2 + \dots + k = \frac{k(k+1)}{2}$. Below the equation is the text "So the problem can be restated as". At the bottom of the slide is another dark purple box with the word "Problem" in white. Below this box, the text reads "For all $k \geq 1$ prove that P_k is TRUE."

So we saw some few examples, of how to split up the problems. So the example and we saw last time was that for all n, if we have to prove that, the sum of first n integers is n into n plus one by two. We can then split up this problem as for a particular k, the sum of k integers, first k integers is k into k plus one by two, and then we have to prove that this technique is true for all k. So this problem becomes you have to prove is basically and of all the Pi's, wher i being all the possible integers, natural integers.

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For example

Problem
For all $n \geq 1$ prove that 11 divides $23^n - 1$.

Let P_k be
 $11 \text{ divides } 23^k - 1$

So the problem can be restated as

Problem
For all $k \geq 1$ prove that P_k is TRUE.

Similarly, if the problem is, for all n greater than or equal to one, prove that 11 divides $23^n - 1$. We can split up by n inductor n which means that, we can say that, okay, P_k be 11 divides $23^k - 1$, it should be minus one, minus one $23^k - 1$ and we have to prove that this statement is true for all the P_k 's.

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For example

Problem
Prove that for all $n \geq 1$ and all positive real number a_1, a_2, \dots, a_n we have

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{(a_1 a_2 \dots a_n)}$$

Let P_k be for all positive real numbers a_1, a_2, \dots, a_k

$$\frac{a_1 + a_2 + \dots + a_k}{k} \geq \sqrt[k]{(a_1 a_2 \dots a_k)}$$

So the problem can be restated as

Problem
For all $k \geq 1$ prove that P_k is TRUE.

The third example is the a_n, b_n equality, namely the average of or arithmetic mean of any n positive real numbers is more than or equal to (n^{th}) root of the product of this n real numbers. And again here, we induct of n and thus, we define P_k as n any k positive real numbers then, prove the statement for those a real numbers and then P_n or then the actual problem states that, for all k greater than or equal to one prove that P_k is true.

So these are all examples of how a problem can be split up into an infinite number of sub problems. But once a infinite number of sub problems, how do we solve it?

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The slide is titled "Principle of Mathematical Induction" in a purple header. Below the header, there is a box labeled "Problem" containing the text "For all $k \geq 1$ prove that P_k is TRUE." Below this box, there are four bullet points: 1. "Since there are infinitely many sub-problems one cannot expect to solve all the sub-problems." 2. "Idea is to solve the first one, namely" followed by the centered text "Prove that P_1 is TRUE". 3. "And prove that," followed by the centered text "if for any $k \geq 1$, P_k is TRUE then P_{k+1} is TRUE." 4. "Then for any $n \geq 1$ the problem P_n is true and hence proved."

To prove this infinitely sub problems surely we cannot go on solving every one of them, though they are infinitely given. So one way of getting around it, is first prove, the first one is true, first the P_1 is true, then prove that for any k , if P_k is true then P_k plus one is true. If you can solve that we expect that, so the idea is that, if you look at this whole real line, then I have first proved that P_1 is true and this statement says that P_1 is true, then P_2 is true, now if P_2 is true, then P_3 is true and so on.

So I can keep on basically, filling up the whole real line, meaning for all k between one to infinity, I will be able to prove that this statement is true. So by doing so, let me first prove P_1 is true and then by proving P_k is true, implies P_k plus one is true you will be able to prove that for all n greater than or equal to one the problem P_n is true and hence we will be done.

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Principle of Mathematical Induction

$$\forall P, [P_1 \vee (\forall(k \geq 1)P_k \Rightarrow P_{k+1})] \Rightarrow [\forall(k \geq 1)P_k]$$

Now for this one, what we need is a particular action, which states that whatever we are doing is correct and this is what it says is called the principle of mathematical induction. And it says that for any predicate, if you first prove P_1 is true and for all k greater than one, if I can prove P_k is true implies P_k plus one is true, then that means that for all k we end up proving P_k is true.

It is a roundabout way of proving that for all the P_k is true. The very powerful technique that we have, we will be seeing more of it in the next couple of weeks.

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Principle of Mathematical Induction

Problem

For all $k \geq 1$ prove that P_k is TRUE.

- Idea is to solve the first one, namely

Base Case: Prove that P_1 is TRUE

- Let us assume that we know how to prove P_k

Induction Hypothesis: Let P_k be true for some $k \geq 1$

- Assuming induction hypothesis prove P_{k+1} is TRUE

Inductive Step: Assuming Inductive Hypothesis prove P_{k+1} is TRUE.

- Then for any $n \geq 1$ the problem P_n is true and hence proved.

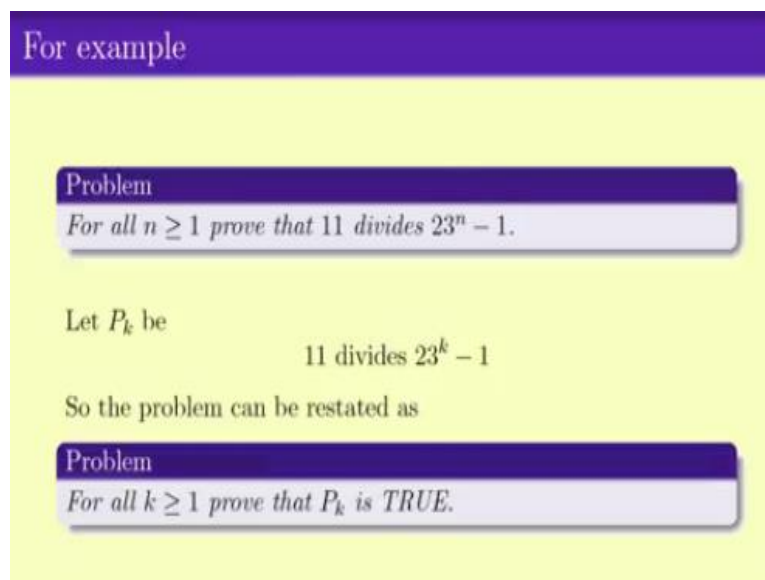
Now, so to prove this statement using the Mathematical Induction there are three basic steps, the first step is what we call as Base case, we basically means that P_1 is true. Note that all the

three cases is true is important, namely if I do not start with the base case then there is no way of starting whole process, the base case is required we have to first prove that P_1 is true.

Second case is that, we have to assume, this is an inductive hypothesis, assume that P_k is true for some K greater than equal one, and say okay, if P_k is true then truly inductive Hypothesis prove that P_k plus one is true. Now this three steps, if you can solve them, then we proved that the whole problem is true. All these three steps are essential. So the steps are basically, P_1 is true, then defining the induction hypothesis and then using induction hypothesis prove the next one is true, thus the inductive step.

In last class, we saw one particular example of how to use induction, mathematical inductions for proving the sum of first n integer is n into n plus one by two. In this video let us look at the second one.

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For example

Problem
For all $n \geq 1$ prove that 11 divides $23^n - 1$.

Let P_k be

11 divides $23^k - 1$

So the problem can be restated as

Problem
For all $k \geq 1$ prove that P_k is TRUE.

Now to prove this statement, this problem of 11 divides $23^n - 1$, of course we have to found the three base steps, namely give a base case, induction hypothesis and inductive statement.

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Example: Prove 11 divides $23^k - 1$

Problem

For all $n \geq 1$ prove that 11 divides $23^k - 1$.

Let P_k be 11 divides $23^k - 1$.

Base Case: To prove 11 divides $23^1 - 1$.

Inductive Hypothesis: Let for some k , 11 divides $23^k - 1$.

Inductive Step: Assuming 11 divides $23^k - 1$ prove 11 divides $23^{k+1} - 1$.

So, in other words, this is the problem the P_k says that 11 divides $23^k - 1$ and you do the base case, then P_1 is true in the Inductive hypothesis, let us assume for some P_k for some k P_k is true, and inductive step I assume, P_k is true prove that P_{k+1} is true. Now putting values of P_k or statements of P_k and P_{k+1} in this set up, the thing that we have to prove is that the base case becomes 11 divides $23^1 - 1$.

Inductive hypothesis says that for some k 11 divides $23^k - 1$ and assuming that 11 divides $23^k - 1$ prove that 11 divides $23^{k+1} - 1$ and now let us see how to prove that.

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Example: Prove 11 divides $23^k - 1$

Base Case: To prove 11 divides $23^1 - 1$.

$23^1 - 1 = 22 = 11 \times 2$. Hence base case is true.

Inductive Hypothesis: Let for some k , 11 divides $23^k - 1$.

Inductive Step: Assuming 11 divides $23^k - 1$ prove 11 divides $23^{k+1} - 1$.

$$\begin{aligned} 23^{k+1} - 1 &= 23 \times 23^k - 1 = (22 + 1)23^k - 1 \\ &= (22 \times 23^k) + (23^k - 1) \end{aligned}$$

By the Inductive Hypothesis we know that 11 divides $(23^k - 1)$.

11 divides 22 and hence 11 divides (22×23^k) .

Hence 11 divides $(22 \times 23^k) + (23^k - 1)$ which is $23^{k+1} - 1$.

And hence proved.

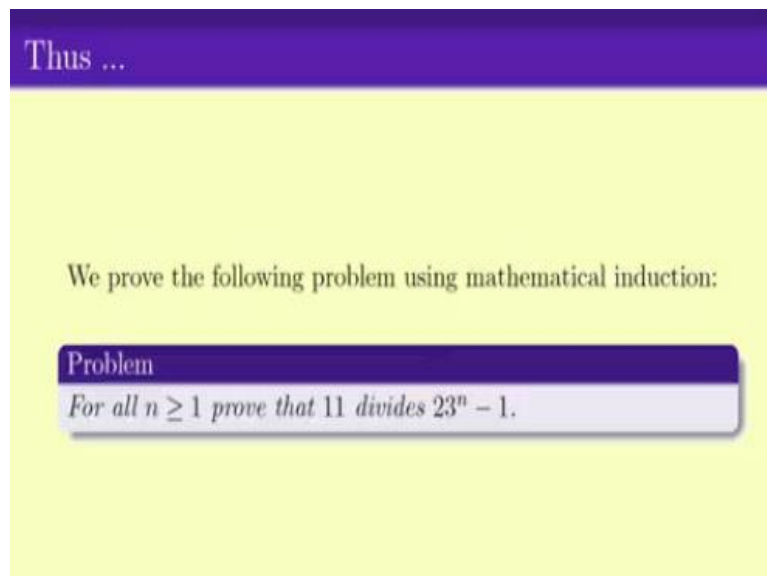
So the base case 11 divides $23^1 - 1$, of this is obvious, because $23^1 - 1$ is 22 which is 11 times two. And the inductive hypothesis states that 11

divides $23^k - 1$. Now assuming this, 11 divides $23^k - 1$ we have to prove that 11 divides $23^{k+1} - 1$. Now let us see how to solve that, so $23^{k+1} - 1$ is nothing but 23 times $23^k - 1$.

If I make it $23 + 1$, we get this number, which is 22 times $23^k + 23^k - 1$. Now by induction hypothesis 11 divides this $23^k - 1$. And the first term which is the 22 times 23^k is divisible by 11 because 11 divides 22 . So thus, 11 divides both this term and this term and thus 11 divides sum of this stuff which is $23^{k+1} - 1$.

So thus, as you can see that the inductive state is not a hard thing to prove, one can easily get the inductive state, if you follow it correctly. We have to apply this usual technique of direct proof of proof by contradiction. But this inductive state is the base case of induction hypothesis, along with the, of course the principal mathematical induction helps us to prove that this statement is true for all k or in other words for all k 11 divides $23^k - 1$.

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Thus ...

We prove the following problem using mathematical induction:

Problem
For all $n \geq 1$ prove that 11 divides $23^n - 1$.

Thus, we have proved that for all n 11 divides $23^n - 1$. Again I ask you guys to prove this statement or try to prove the statement without using induction. Now proving some statement like this without using induction can be quite a pretty job.

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Some exercises

- Prove that for all $n \geq 1$, $1 + 3 + 5 + \dots + (2n - 1) = n^2$.
- Let $x > -1$. Prove that for any $n \geq 1$, $(1 + x)^n \geq 1 + nx$.

I will finish this video today, leaving two exercises, the first one is prove that the sum of one plus three plus five till two n minus one is n square for all n. In other words, the sum of first n odd numbers is n square. And the second one is that, if x is greater than minus one, prove that for all n, one plus x power n is greater than one plus nx. So these are the 2 exercises, which can be solved using the induction technique that we have seen so far.

In the next video, we will see interesting versions of this induction hypothesis which will help us to solve it more interesting problems. Thank you.