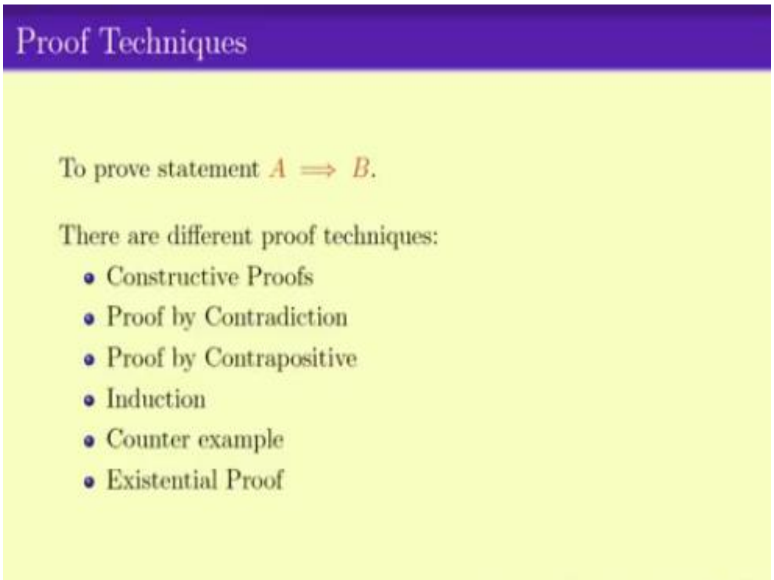
Discrete Mathematics
Prof. Sourav Chakraborty
Department of Mathematics
Indian Institute of Technology – Madras

Lecture - 15
Mathematical Induction (Part 2)

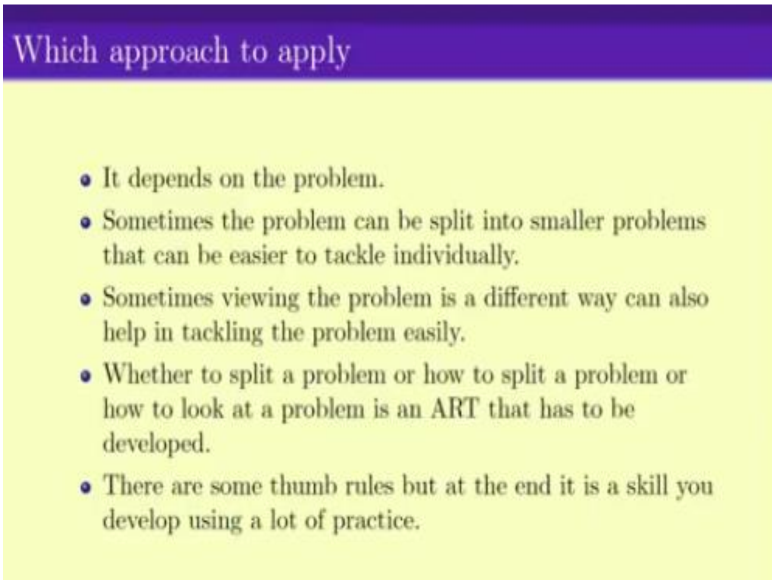
Welcome to the second video lecture in which four of the discrete mathematics, in this video lecture, we will continue with our understanding of induction.

(Refer Slide Time: 00:18)



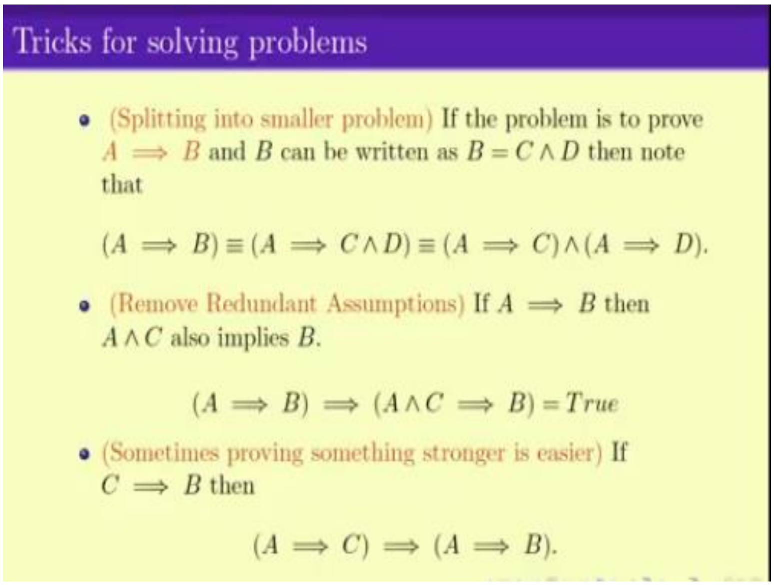
A quick recap, we were looking at the proof techniques, mainly to prove how to prove A implies B, and we have seen that there are quite a number of different proof techniques available was Constructive proof, proof by contradiction, Contrapositive, induction, counter example and existential proof.

(Refer Slide Time: 00:37)



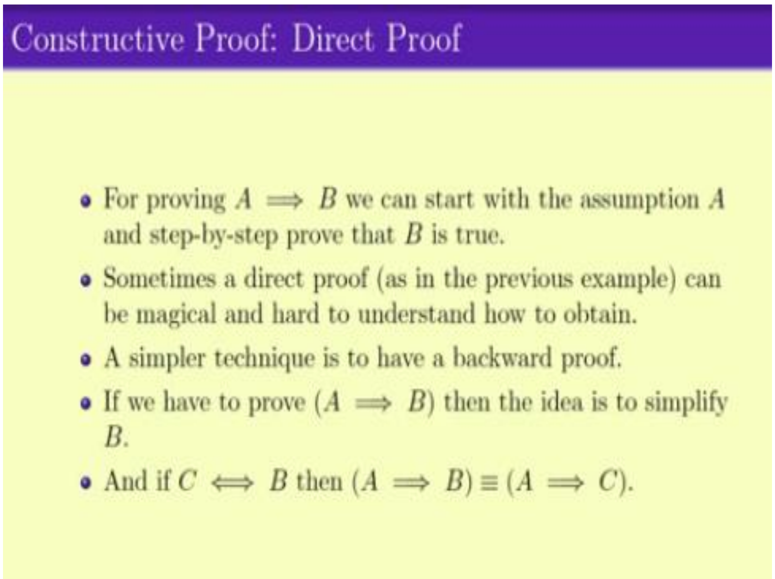
Now, this is a slide I have shown you every time, which basically states that there is no rule of which proof technique should be applied to which problem itself, art that we have to develop. Now to quickly recap, whatever of things we have till now.

(Refer Slide Time: 00:59)



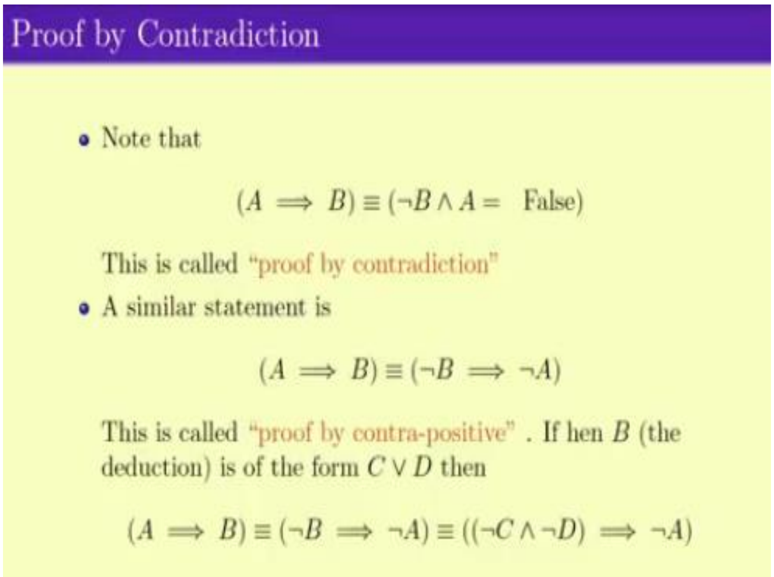
We saw some tricks of how to split the problem into smaller problems depending on whether B can be written as C and D. How to remove redundant assumptions and thirdly how to see that sometimes proving something harder or stronger can be easier.

(Refer Slide Time: 01:34)



We also saw some proof techniques, namely we looked at the direct proof technique where one works with A and then end up proving B. Or one can go backward and can start simplifying B and slowly get to a situation where A implies C can be easier, but C is equivalent of B just a simplified form. So we saw on few examples of this.

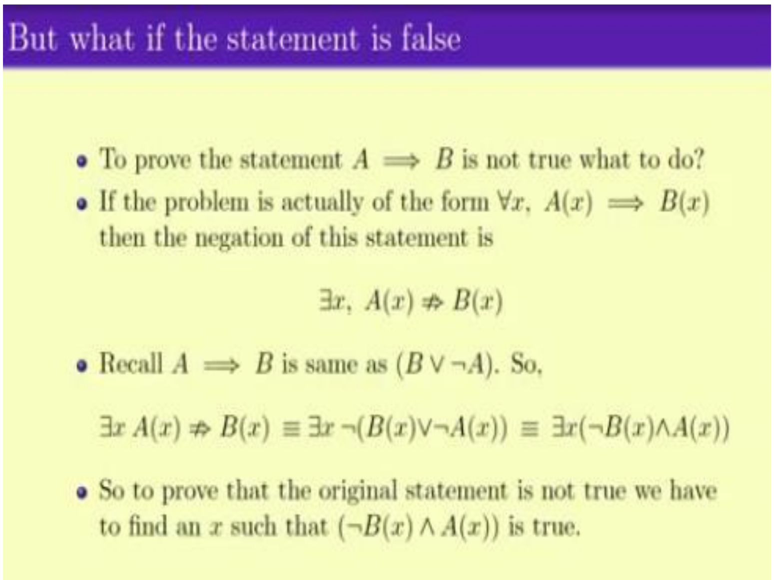
(Refer Slide Time: 02:06)



We also looked at the case study, in this case if we split the assumptions into some constant number of cases, so what happens is that if you write A as C or D, then A implies B get split up as C implies B and D implies B. This particular case study proof is something relevant to the proof of, prove by induction also, we will see very soon.

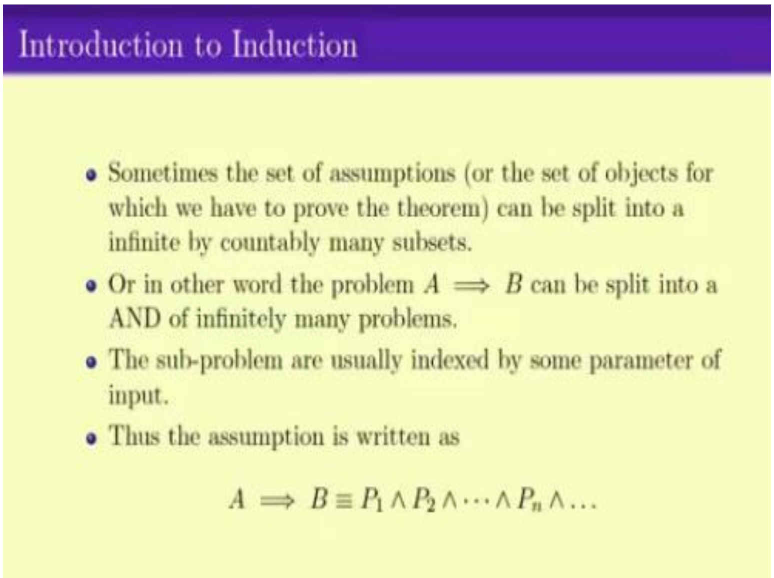
So other than the case study proofs, we also looked at the proof by contradiction, mainly proving A implies B is same as proving not B and A is false. Or in a similar way, one can prove A implies B by proving that not B implies not A. This second one is more like proof by contra-positive and this can be useful particularly when B can be written as C or D, in that case A implies B can be written as not C and not D implies not A.

(Refer Slide Time: 03:20)



We also solved this case of proof by counter example, where if I have given a problem of the form for x for all x prove that, prove or disprove Ax implies B. To disprove the statement, one needs to give that x such that Ax does not implies Bx, or in other words we have given x such that Bx is not true, but Ax is true. So these are the proof technique that we saw last week.

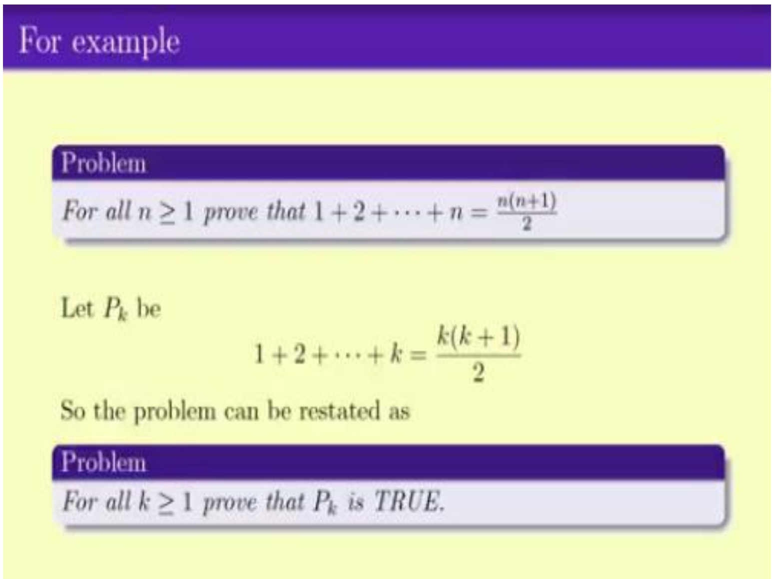
(Refer Slide Time: 04:02)



In the last video lecture, we started with this to proof by induction. The proof of induction is very similar to case study proof, except that in case studies once get the assumptions into constantly many number of cases. And thus, the problem gets split up into a constantly varying and of some small problems. But there are times when one can split up the assumptions into infinitely many, but by countably many number of cases.

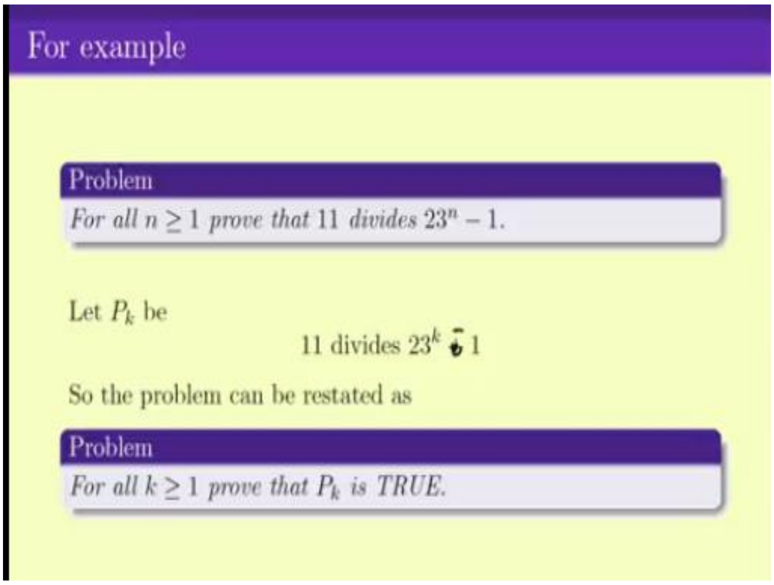
In that case, of course just in the case of case studies, the problem can split up into a AND of infinitely many number of components. The sub problems are usually we do get indexed by some parameter of the input, or intent of the parameter of the input, in other words one would like, one likes this whole thing of A implies B as P1 and P2 and so on as the infinite collection of thing. So thus, to prove A implies B one is to prove, that this Pi is false for all the i's,

(Refer Slide Time: 05:27)



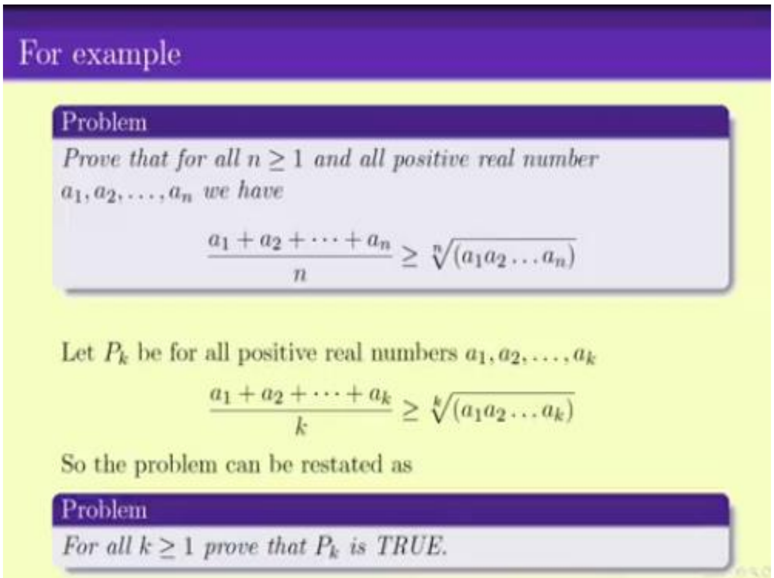
So we saw some few examples, of how to split up the problems. So the example and we saw last time was that for all n, if we have to prove that, the sum of first n integers is n into n plus one by two. We can then split up this problem as for a particular k, the sum of k integers, first k integers is k into k plus one by two, and then we have to prove that this technique is true for all k. So this problem becomes you have to prove is basically and of all the Pi's, wher i being all the possible integers, natural integers.

(Refer Slide Time: 06:31)



Similarly, if the problem is, for all n greater than or equal to one, prove that 11 divides 23 power n minus 1. We can split up by n inductor n which means that, we can say that, okay, Pk be 11 divides 23 power k minus, it should be minus one, minus one 23 power k minus one and we have to prove that this statement is true for all the Pk's.

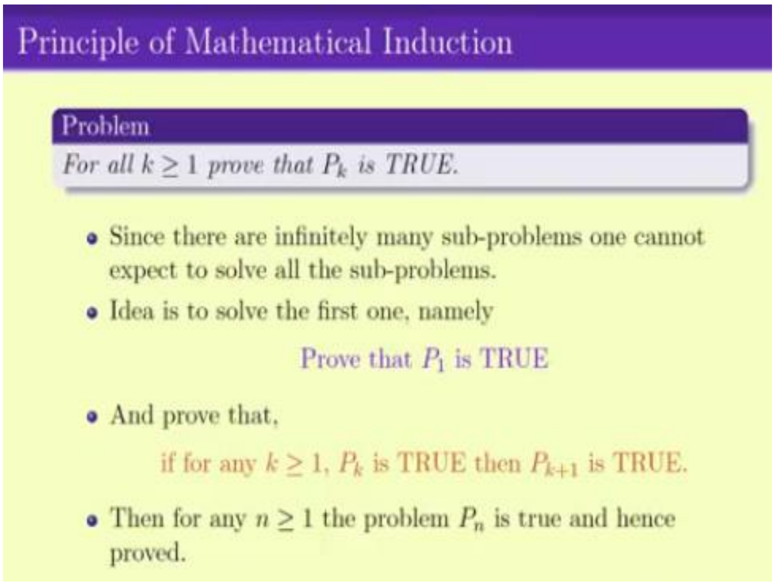
(Refer Slide Time: 07:10)



The third example is the an, bn equality, namely the average of or arithmetic mean of any n positive real numbers is more than or equal to (n th) root of the product of this n real numbers. And again here, we induct of n and thus, we define Pk as n any k positive real numbers then, prove the statement for those a real numbers and then Pn or then the actual problem states that, for all k greater than or equal to one prove that Pk is true.

So these are all examples of how a problem can be split up into an infinite number of sub problems. But once a infinite number of sub problems, how do we solve it?

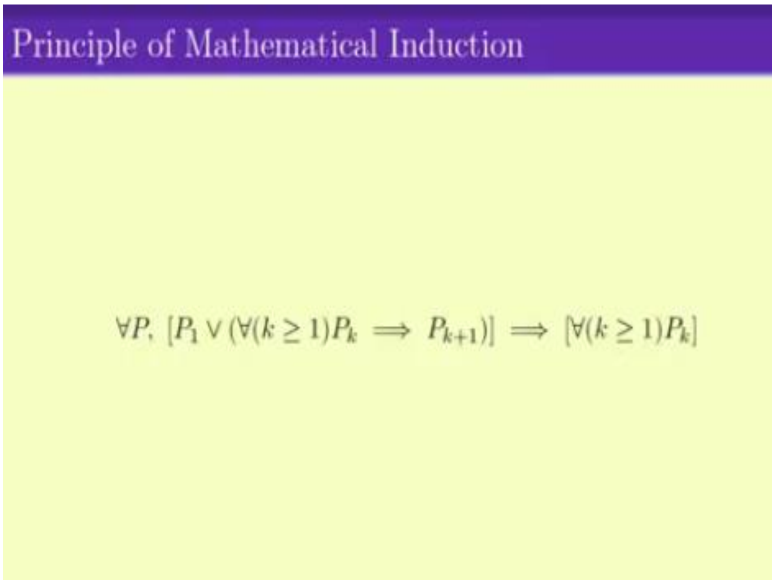
(Refer Slide Time: 08:13)



To prove this infinitely sub problems surely we cannot go on solving every one of them, though they are infinitely given. So one way of getting around it, is first prove, the first one is true, first the P1 is true, then prove that for any k, if Pk is true then Pk plus one is true. If you can solve that we expect that, so the idea is that, if you look at this whole real line, then I have first proved that P1 is true and this statement says that P1 is true, then P2 is true, now if P2 is true, then P3 is true and so on.

So I can keep on basically, filling up the whole real line, meaning for all k between one to infinity, I will be able to prove that this statement is true. So by doing so, let me first prove P1 is true and then by proving Pk is true, implies Pk plus one is true you will be able to prove that for all n greater than or equal to one the problem Pn is true and hence we will be done.

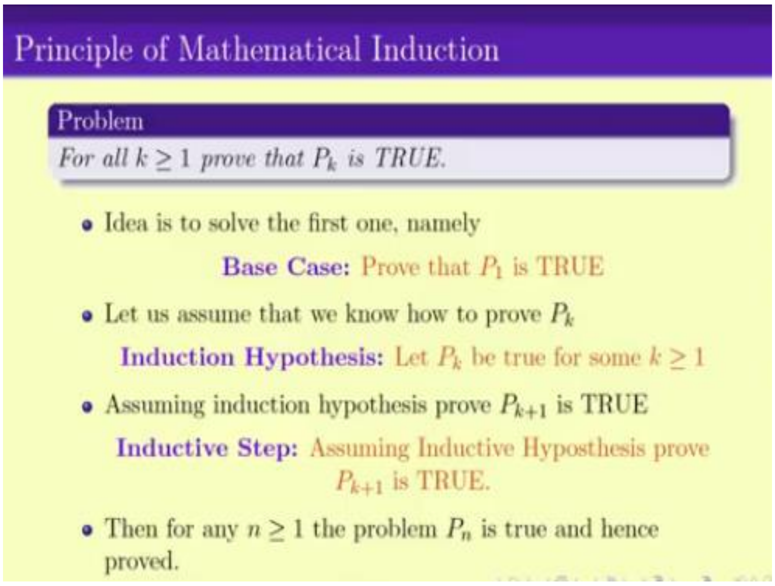
(Refer Slide Time: 09:35)



Now for this one, what we need is a particular action, which states that whatever we are doing is correct and this is what it says is called the principle of mathematical induction. And it says that for any predicate, if you first prove P1 is true and for all k greater than one, if I can prove Pk is true implies Pk plus one is true, then that means that for all k we end up proving Pk is true.

It is a roundabout way of proving that for all the Pk is true. The very powerful technique that we have, we will be seeing more of it in the next couple of weeks.

(Refer Slide Time: 10:27)

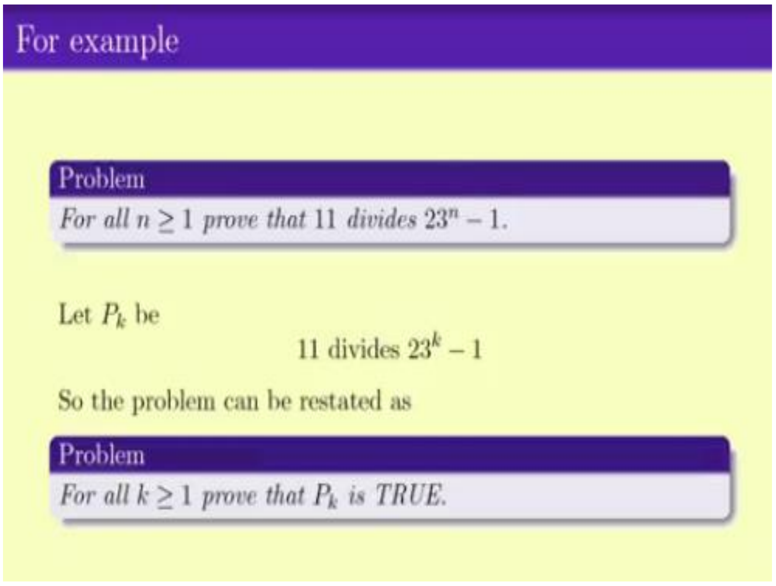


Now, so to prove this statement using the Mathematical Induction there are three basic steps, the first step is what we call as Base case, we basically means that P1 is true. Note that all the three cases is true is important, namely if I do not start with the base case then there is no way of starting whole process, the base case is required we have to first prove that P1 is true.

Second case is that, we have to assume, this is an inductive hypothesis, assume that Pk is true for some K greater than equal one, and say okay, if Pk is true then truly inductive Hypothesis prove that Pk plus one is true. Now this three steps, if you can solve them, then we proved that the whole problem is true. All these three steps are essential. So the steps are basically, P1 is true, then defining the induction hypothesis and then using induction hypothesis prove the next one is true, thus the inductive step.

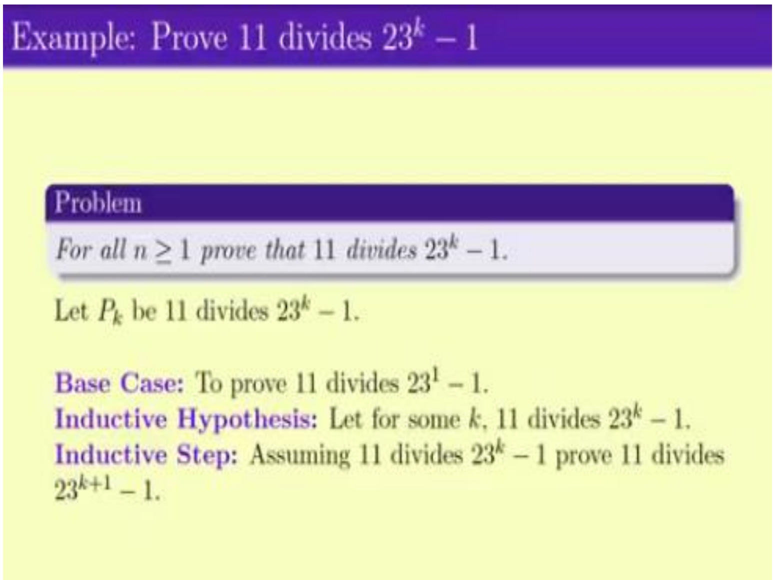
In last class, we saw one particular example of how to use induction, mathematical inductions for proving the sum of first n integer is n into n plus one by two. In this video let us look at the second one.

(Refer Slide Time: 12:03)



Now to prove this statement, this problem of 11 divides 23 n minus one, of course we have to found the three base steps, namely give a base case, induction hypothesis and inductive statement.

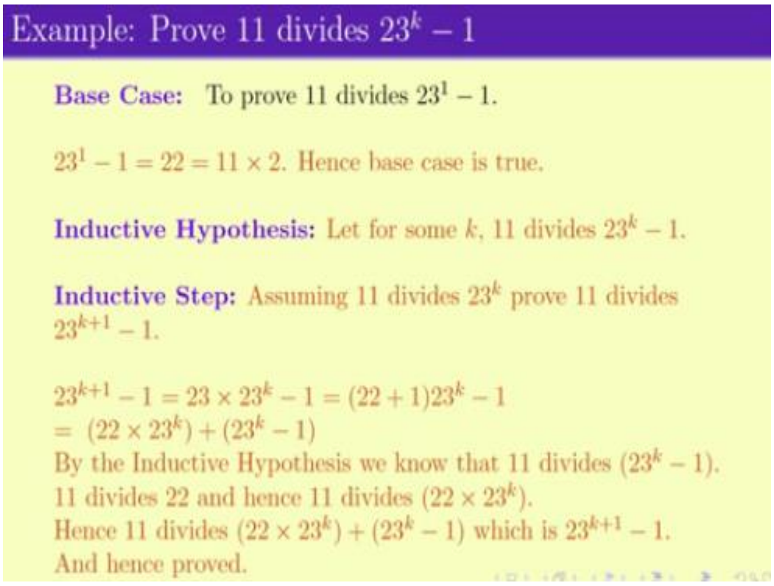
(Refer Slide Time: 12:22)



So, in other words, this is the problem the Pk says that 11 divides 23 k minus one and you do the base case, then P1 is true in the Inductive hypothesis, let us assume for some Pk for some k Pk is true, and inductive step I assume, Pk is true prove that Pk plus one is true. Now putting values of Pk or statements of Pk and Pk plus one in this set up, the thing that we have to prove is that the base case becomes 11 divides 23 power one minus one.

Inductive hypothesis says that for some k 11 divides 23 k minus one and assuming that 11 divides 23 k minus one prove that 11 divides 23 power k plus one minus one and now let us see how to prove that.

(Refer Slide Time: 13:26)

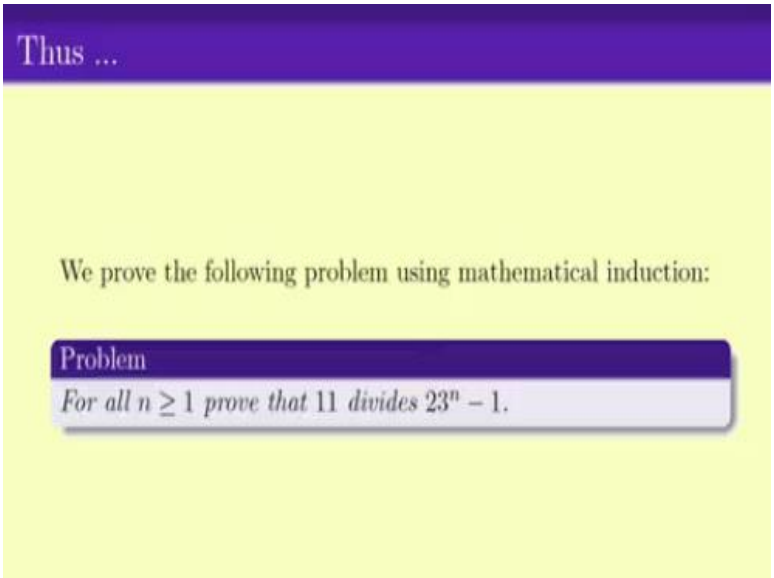


So the base case 11 divides 23 power one minus one, of this is obvious, because 23 power one minus one is 22 which is 11 times two. And the inductive hypothesis states that 11 divides 23 k minus one. Now assuming this, 11 divides 23 k minus one we have to prove that 11 divides 23 k plus one minus one. Now let us see how to solve that, so 23 power k plus one minus one is nothing but 23 times 23 power k minus one.

If I make it 23 plus one, we get this number, which is 22 times 23 power k plus 23 power k minus one. Now by induction hypothesis 11 divides this 23 k minus one. And the first time which is the 22 times 23 power k is divisible by 11 because 11 divides 22. So thus, 11 divides both this term and this term and thus 11 divides sum of this stuff which is 23 power k plus one minus one.

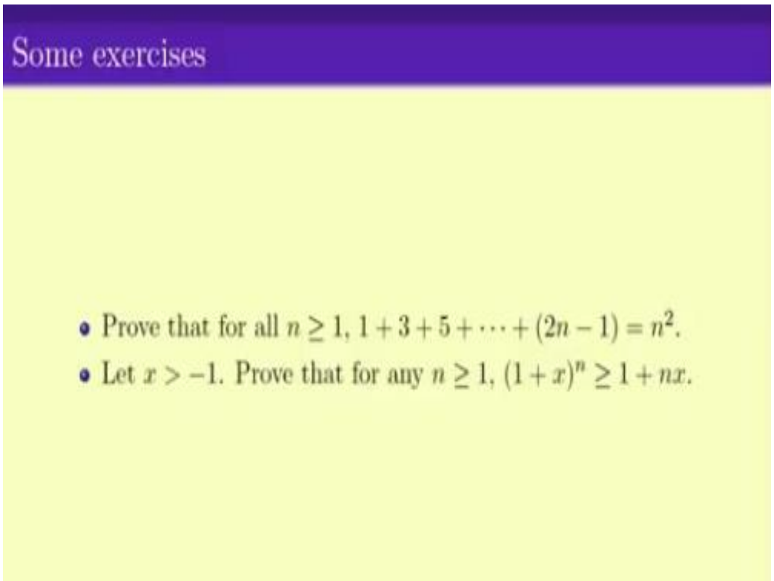
So thus, as you can see that the inductive state is not a hard thing to prove, one can easily get the inductive state, if you follow it correctly. We have to apply this usual technique of direct proof of proof by contradiction. But this inductive state is the base case of induction hypothesis, along with the, of course the principal mathematical induction helps us to prove that this statement is true for all k or in other words for all k 11 divides 23 k minus one.

(Refer Slide Time: 15:23)



Thus, we have proved that for all n 11 divides to 23 n minus one. Again I ask you guys to prove this statement or try to prove the statement without using induction. Now proving some statement like this without using induction can be quite a pretty job.

(Refer Slide Time: 15:48)



I will finish this video today, leaving two exercises, the first one is prove that the sum of one plus three plus five till two n minus one is n square for all n. In other words, the sum of first n odd numbers is n square. And the second one is that, if x is greater than minus one, prove that for all n, one plus x power n is greater than one plus nx. So these are the 2 exercises, which can be solved using the induction technique that we have seen so far.

In the next video, we will see interesting versions of this induction hypothesis which will help us to solve it more interesting problems. Thank you.