Discrete Mathematics
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Lecture - 14
Mathematical Induction (Part 1)

Welcome to the fourth week of this Discrete Mathematics course. So till now we have been looking at various proof techniques and we have looked at some of the most interesting proof technique namely direct proof by contradiction and proof by contrapositive. In this week and in the next week we will be looking at one of the most powerful proof technique that will available to us in this field of discrete mathematics namely, Induction.

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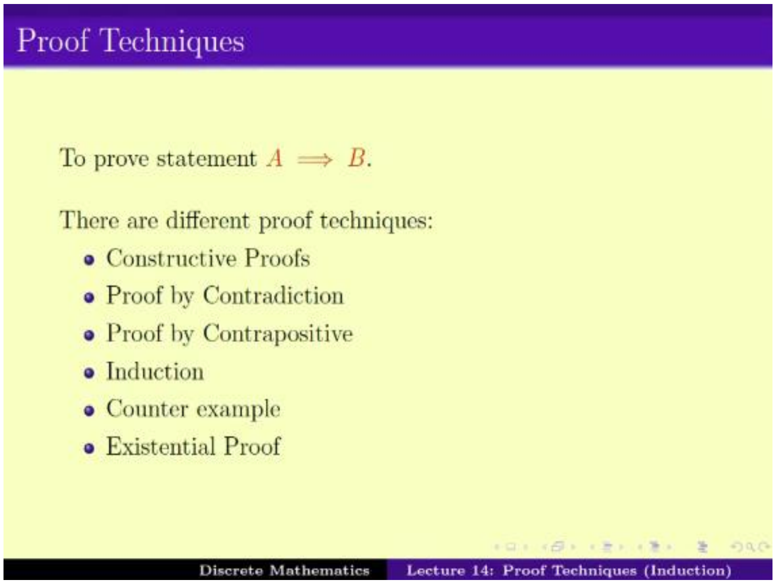

So quickly to recap to prove a statement like A implies B there are different kinds of proof techniques. Constructive Proofs, Proof by Contradiction, Proof by Contrapositive, Induction, Counter example, Existential Proof and so on. We have still now looked at some of the proof techniques.

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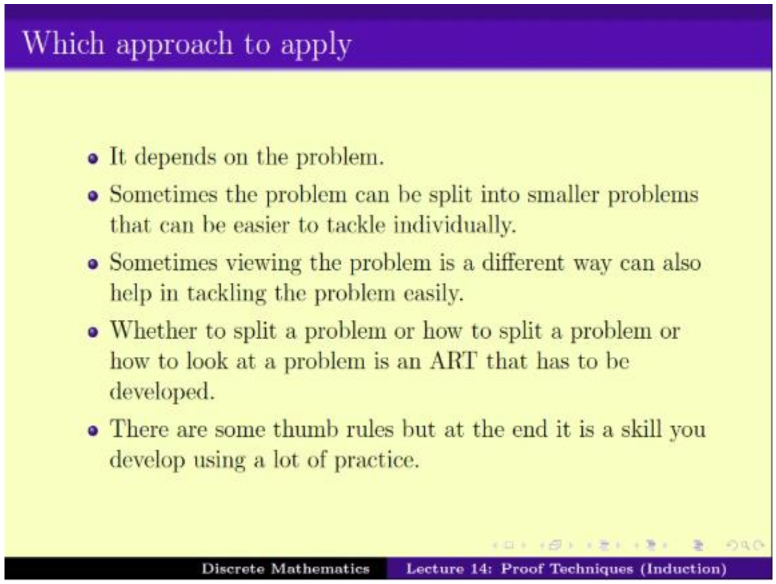

So this is something that I have told you again and again but I repeat it all once again once more time. Namely, given a problem which proof technique to apply? Now there is no fix rule about that. Which proof technique to apply depends upon the problem and your understanding of the subject. So there are some problems that can be split into smaller problem; that can be easier to handle while for some problems one can view to problem exactly different way which can make the problem easy.

But how to split a problem and when to split a problem or when to look at in a defined way, all of these depends upon your understanding of this subject. It is an art that has to be developed. There are some thumb rules which we have been discussing and we will keep on continue to discuss but in the end of the day you have to decide which rules to apply for which problem. Now till now we have seen a few simple tricks that can be applied.

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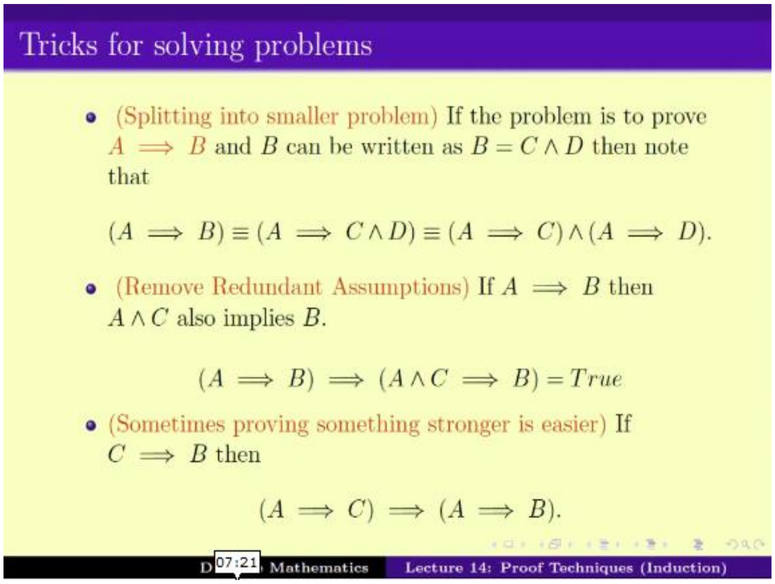

To start we need to look that how to split a problem into smaller part if when to prove A implies B, B can be written as C and D in that way it can be split into A implies C and A implies D each of this can be solved individually and make it little-- slightly easier problem. The second option is removing some kind of a redundancy in the assumptions. Namely if A implies B and that would imply that A as any other assumptions is also imply B.

So if given that the assumptions if A and C one might want to remove the redundant assumptions and that would make this problem easier, neater and hence easier to solve. The third interesting problem – sorry third interesting trick is that sometimes proving something stronger can actually be easier. Namely if you have to prove A implies B, but we know that C implies D and it might need the case that A implies C is easier to prove that A implies D.

And in that case one would like to false the A implies C instead of A implies B although A implies C is our stronger statement that A implies B.

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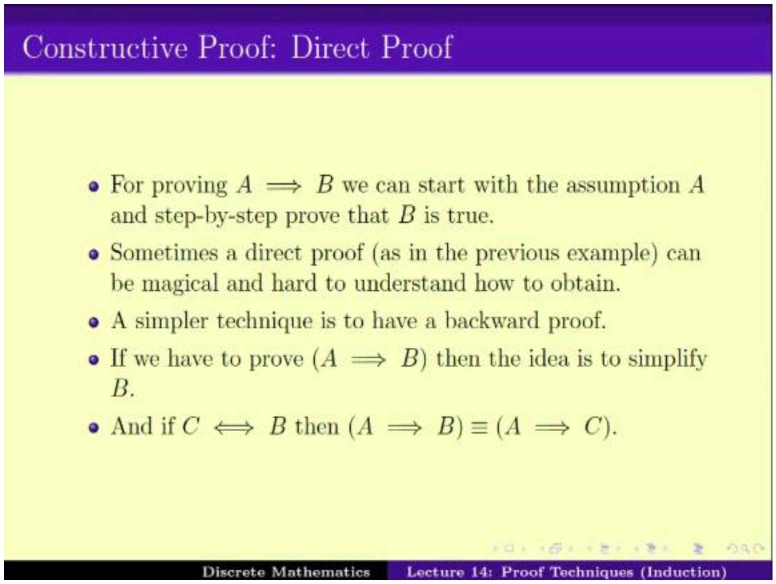

Other than these three tricks for solving problems we also looked at some of the proof technique in particular we looked at the direct proof technique, so idea is that to prove A implies B we start with an assumption A and step-by-step prove B. But sometimes getting such a proof can be magical and hence difficult to understand how to obtain. So there is another technique for attacking this problem in this form namely going by via a backward proof or other words to prove A implies B start with B simplify.

Now if you can simplify B to something called C than A implies B, the same as to be A implies C and in that case A implies C can be easier to prove. So this called the constructive proof or direct proof.

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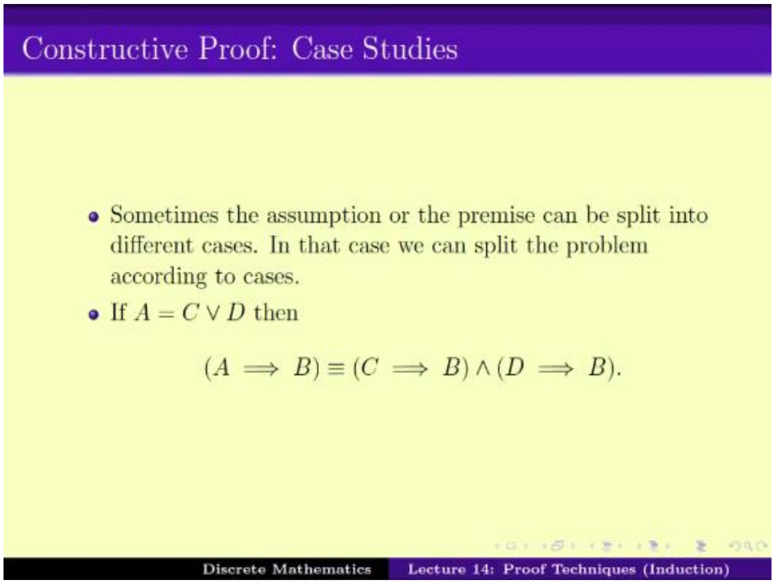

There is one more technique which is also called the constructive proof which we called Case Studies. The ideas are that if you can split up the assumptions into a finite number of cases then you can false them in that case by case stages. So thus, if you can write A at C or D, that A implies B, if same as to be C implies B and D implies B so that you can split into two smaller problems depending on the cases.

Again, how to break up the assumption in two cases? Of course depend upon on the problems itself. We have seen quite number of example in this regard.

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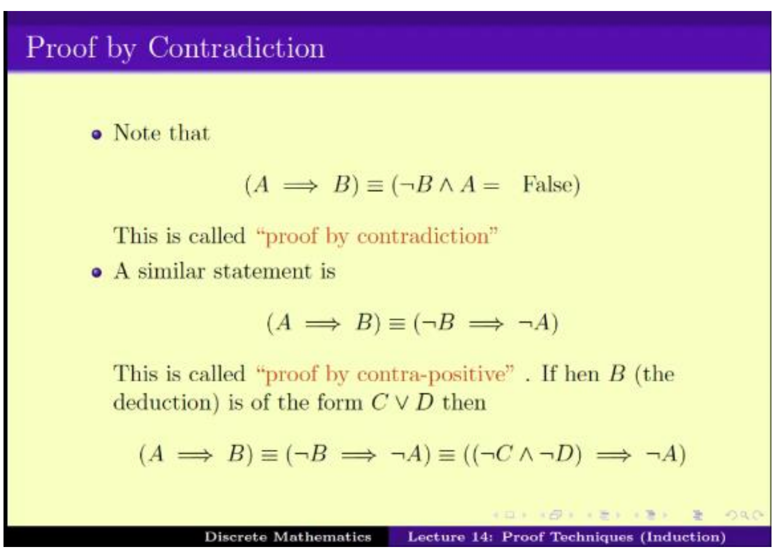

The third technique that we have seen is what we called the Proof by Contradiction. The main idea is that to prove A implies B one can also prove not B and A is false. A very similar statement is also not B implies not A. So this is called the Proof by Contradiction. Here if B that is the deduction can be written in the form of C or D then A implies B one can apply the proof by contra-positive.

And get a statement of the form of not C and not D implies not A, this would an easier technique to prove or easier problem to prove. A Proof by Contradiction and Proof by contra-positive, we have spent again a week on this for particular two techniques and these are very powerful techniques for problem solving. So this helps us to view the problem in a slightly different way which is possibly easier to adapt.

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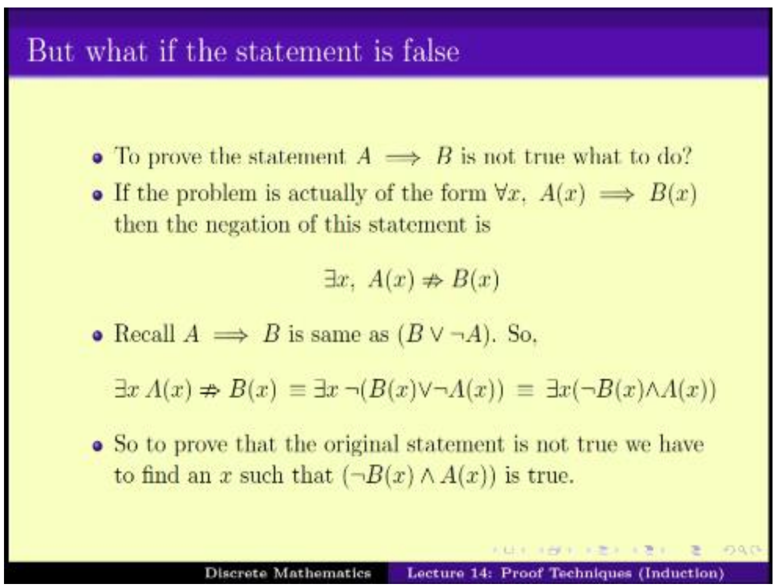

So one more thing that we have looked at is what we call Proof by Counter example namely if we have to check whether the statement A implies B is true or not and particularly if – the problem is of the form for all x A(x) implies B(x) where the A(x) and B (x) have two predicates. Then to prove that that is not true we have to prove that the negation of this one is true. Or in other words we have to prove that THERE EXISTS x where A(x) does not imply on B(x).

And by usual assumption like usual deductions namely A implies B is same as B or not A, we can replace this THERE EXIST A not implies B(x) as THERE EXIST x not B(x) and A(x). So thus to prove a statement A implies B is not true I have to find an x which does not satisfied B(x) but satisfied A(x) and this is called the Proof by contra example.

Now all these various proof techniques that we have done are actually proof technique that can be applied to any field of math not particularly discrete math, meaning this can be applied to discrete math, continuous maths or any logical set of statement. But the next proof technique that we are going to see is something that can be applied to only discrete objects and hence a very unique proof technique and (()) (09:08) one of the most powerful proof technique that we have.

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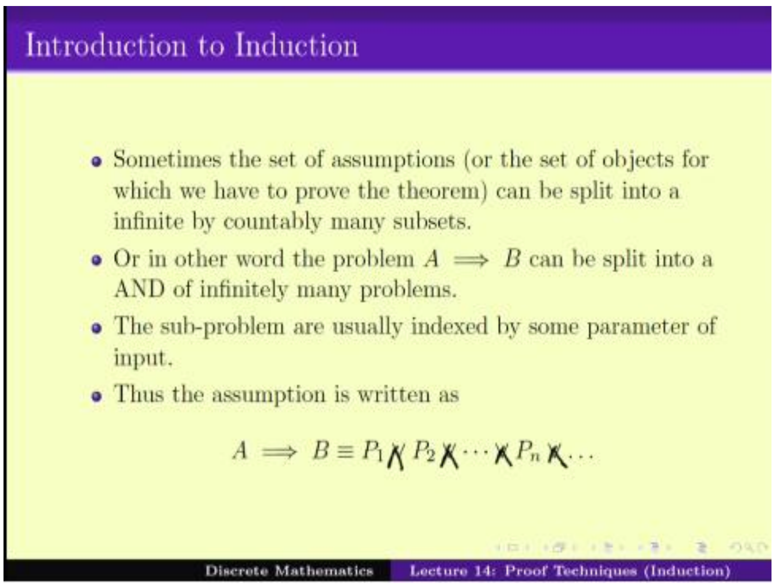

We call it as Induction. So the main idea is that sometimes the set of assumptions or condition for which it should be proved or the object for which we have to prove the theorem they can be split up into infinite by countably many subsets. So in other words, let us prove it A implies B we can split A as or this whole problem as AND of infinitely many problems. So these sub-problems are indexed by some parameter of the input.

In other words, I would like to write A implies B as P1 and P2 and so on till Pn and so. So I have made the mistake in the slide here-- these are all should be AND. So P1 AND P2 AND P3 AND… till Pn, okay.

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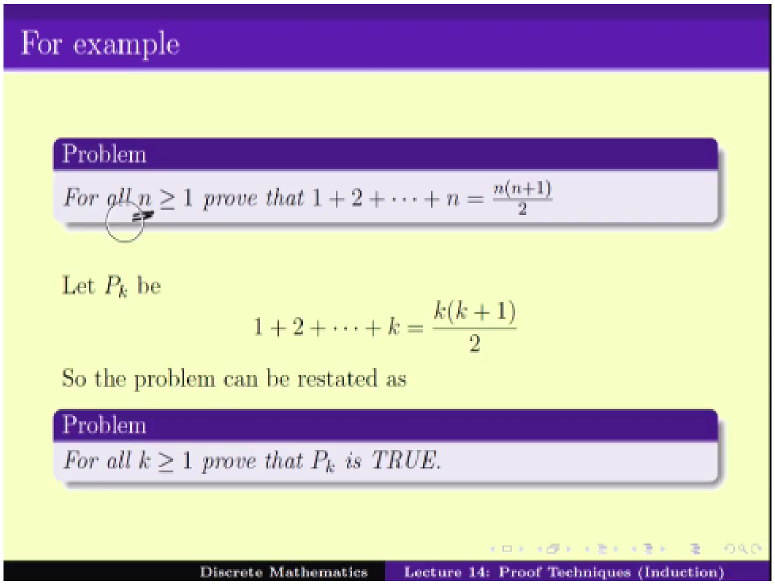

Let us look at the example. So consider this problem right, for all n greater equal to 1 we want to prove that 1 + 2 + ...till end n into n+1 by 2. Now how do you prove it and what are these smaller statements? Now let me define Pk to be this particular statement 1+2+ up to k equal to k * k + 1. So the problem statement can be restated as for all k greater than 1 prove that Pk is true. So here the Pk that are sub-problems. Right?

So the original problem have this broken up an AND of infinitely many problems mainly the Pk. Note that is usually this is how we end up doing there is a natural parameter or something here we called it n so using which we split up the problem into smaller problems. When we can split up the problem in a smaller problem, we usually say we are inducting on this particular parameter so in this small problem you maybe indicting on n.

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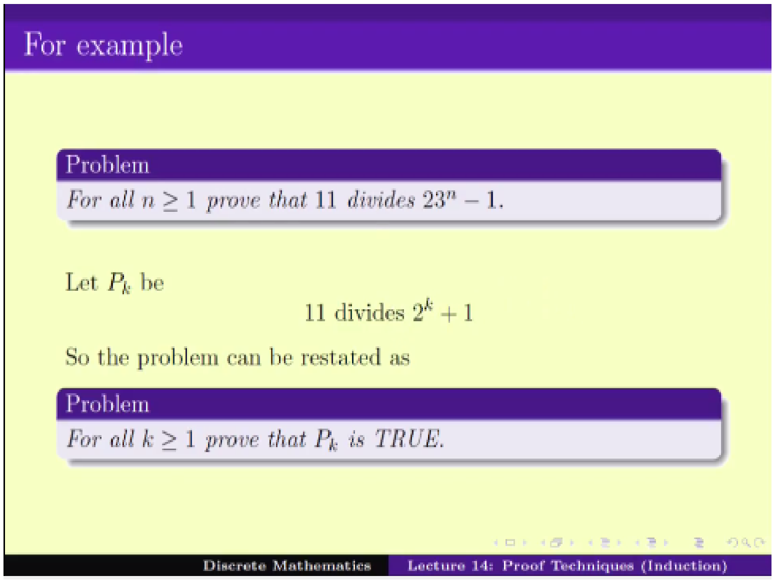

Let us look at the second example, say for all n greater than equal to 1 prove that 11 divides 23n - 1. Now can you guess what are the Pi's? So let us define it by again use the n as the parameter so in other word Pk is 11 divides 23k + 1. So 11 divides 23k+1. So this problem can be restated as for all k prove that Pk is true. So again I broke this original problem into infinitely many problems is parameterized by an integer namely here k. No let us try to understand how to bigger one more problem.

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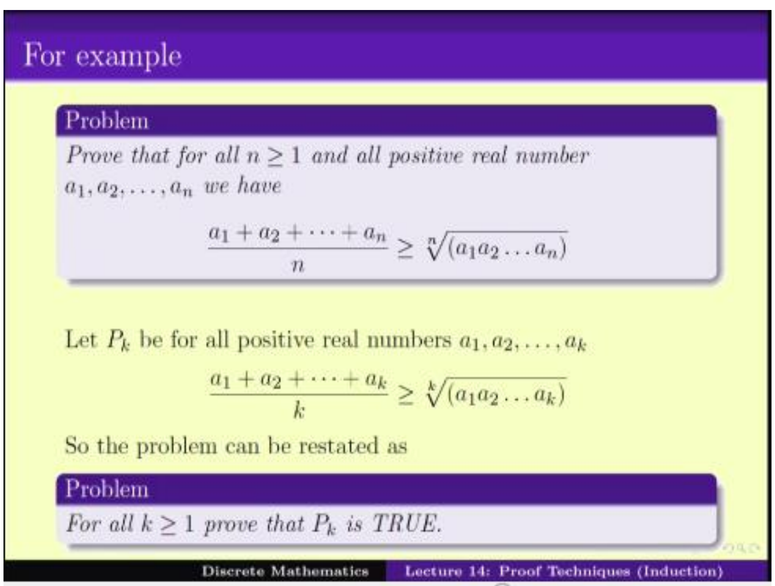

So here is the slightly more complicated problem. You have seen this problem maybe earlier this is called 'an', 'pn' equality which says that for all n and for all positive real numbers a1 to 'an' we have a1 + a2 + … till 'an' by n that means average of a1 to 'an' is greater than equal to the nth root of the product of ‘ai’ it is n root a1 and a2 times ..till ‘an’. Now again here what are the Pk. From here, there positive main parameters right, there are ‘ai’s, there is the n. So there is lots of parameters. There can be multiple ways of breaking up a problem into smaller problems and almost all these techniques can meet to a solution to some of them might easier or hard.

In this problem let me break it up into this following ways again it says that let Pk be has that for all positive real number a1 to ak the average of a1 to ak is greater than k root of the product. And since the original problem says that we have to prove it for all n so this statement if burst out proving that for all k greater than equal to 1 prove Pk is true. So this is another example of particular of the whole problem into smaller problems.

This is something extremely important as the first step of induction and I told the induction starts with breaking up the problem or the assumption into infinitely different affecting any sub-problems parameterized by some integer which is some kind of a property of the input.

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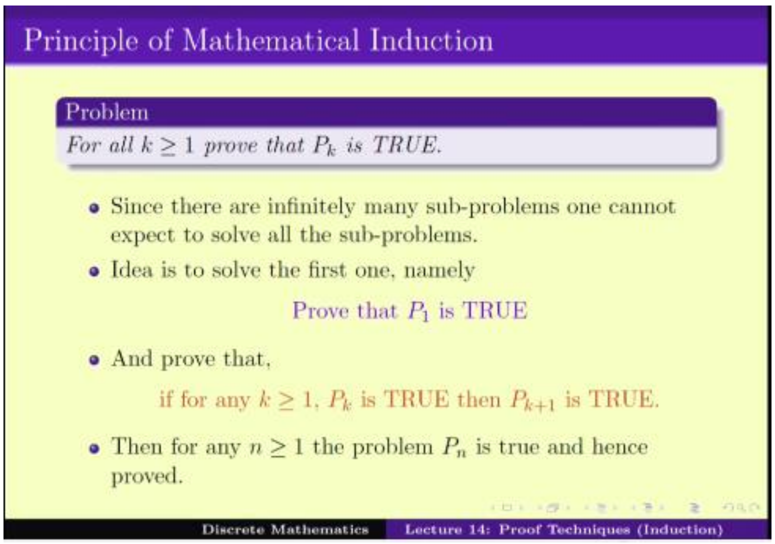

Now what do I do to accept that. I cannot apply things like proving for all the different ‘Pi’. So for example if I have proved that for all k greater than equal to I prove that Pk is true, there are infinitely many sub-problems and one cannot expect to solve all the sub-problems. So how do we do that? So the idea is that first prove P1 is true that is something you have to prove and next assuming that I have managed to prove Pk is true for some Pk for some k prove that Pk + 1 is true.

And by doing so, I should be able to has convinced you that for all n this number Pn is true, hence proved. There is a quick remark, we have already seen in the beginning of this set of proof technique that if the A can be broken up or particularly in the case study problem if the A can be broken up into finite number of part the problem breaks up in finite number of a constant number of sub- problems and we solve each of together one-by-one.

But here, since there infinitely many sub-problems we cannot do such things. So this seems to be a pretty natural level of doing it. So I will just tell you that whether this actually ends up proving all -for the all n or not have some bit weirdity. Namely assuming the, I cannot move using propositional logic or predicate logic that this statement will end up proving the whole problem.

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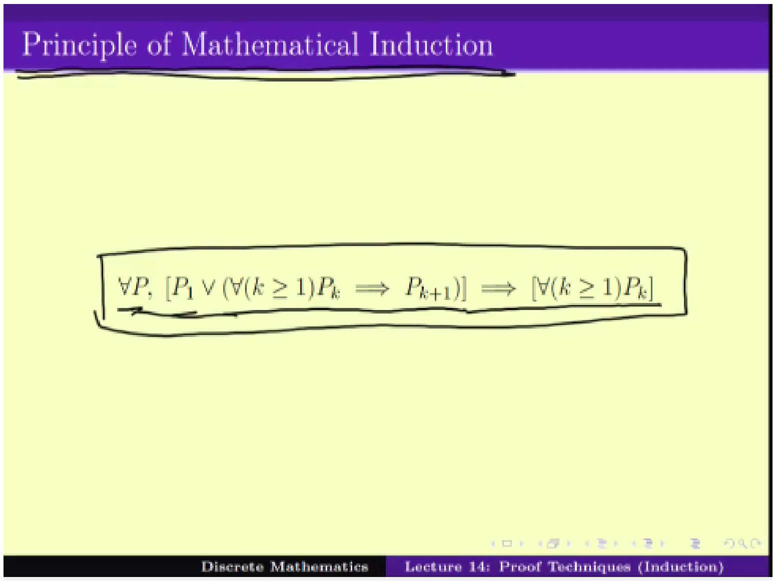

So usually what we do is that we call the 1 the principle of mathematical induction, it is an axiom in maths which says that if for any predicate, if for any problem, if I can first prove P1 and then for all k if I can prove Pk implies Pk + 1 then this imply that for all k I have proved Pk. So this is an axiom in the mathematical framework and one might tend to argue whether this axiom is right or wrong but there are lot of mathematicians who accept this one as a reasonable axiom, meaning this statement is true.

So this is an axiom and this we call as a principle of mathematical induction. So using this principle of mathematical induction, we can now have a technique of proving this infinitely many collection of sub-problems.

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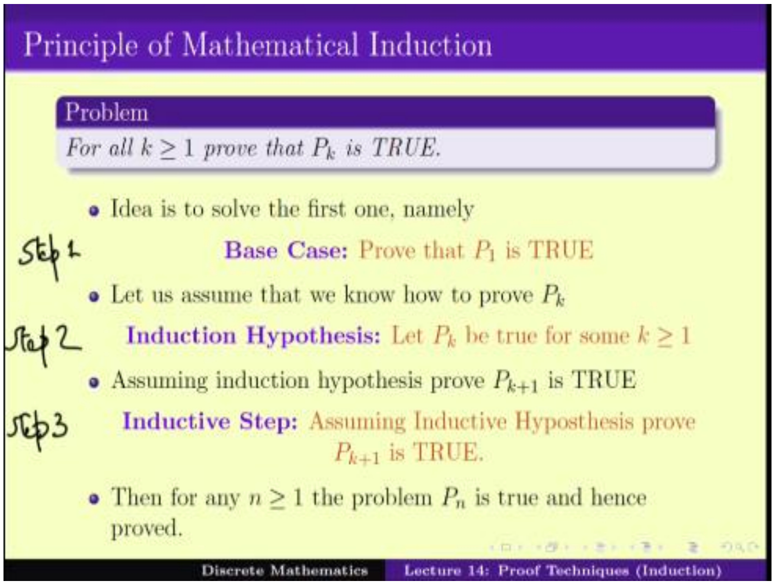

So to this prove statement that for all k is greater than equal to 1 prove that Pk is true. There are three parts to do it. First part is what is called the Best Case where you prove P1 is True. The second part is called Induction Hypothesis where you assume that we now Pk is true for some k greater than equal to 1. And Inductive Step is that assuming induction hypothesis can you prove direct statement namely any prove Pk + 1 is true.

So these are the three steps that are there, step one, step two and step three. And once you have the step three than it follows that I have the whole problem. Namely we have proved that for all n Pn is true. This is what you guaranteed by the principle of induction hypothesis and we will be using this one to two our problems.

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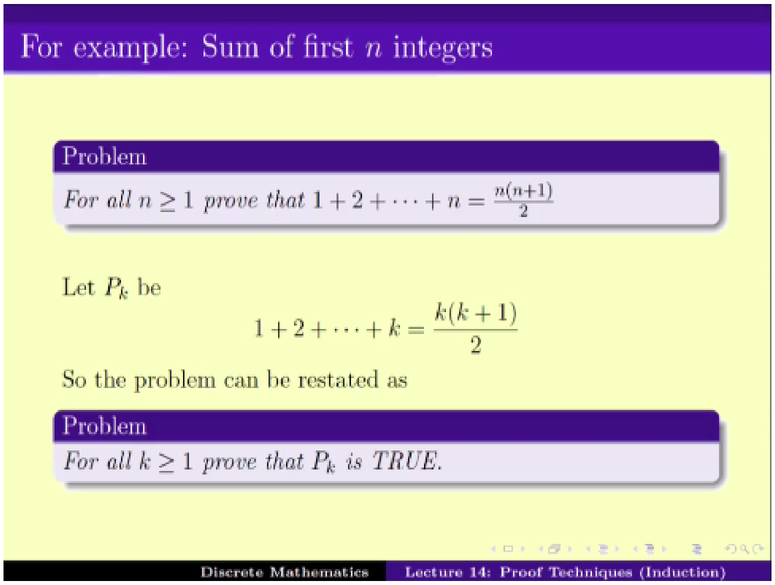

So let us us look at this first problem. So the problem was that for all n prove that the sum of the n number first n integer with n into n+1 by 2. Now as I split up this problem into sub-problems namely where Pk is sum of first k object k integer is k into k+1 by 2 and the main problem was down to proving that for all k this statement is true.

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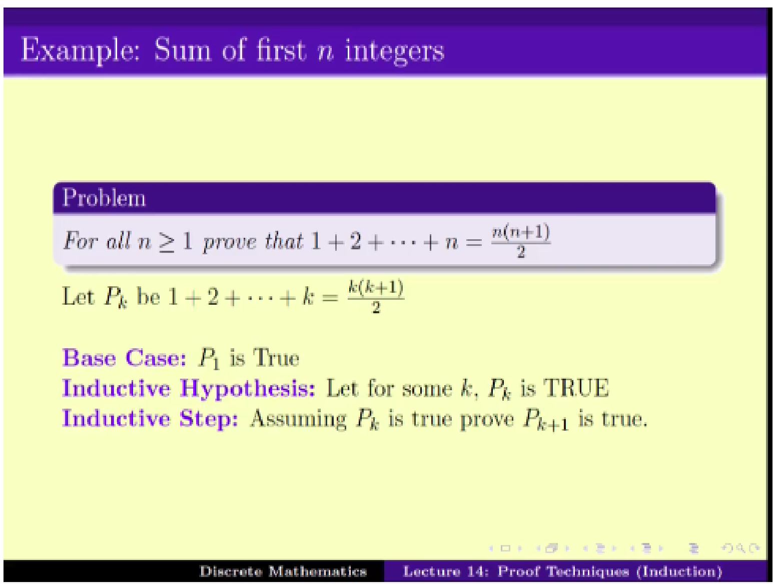

Now let us use the principle of mathematical induction. So what should be done? So first of all, so this is the Pk and the some of the first k element is k into k +1 by 2. When first prove Base Case which is that P1is true. We have to assume the Induction Hypothesis so that means for some k Pk is true and we have to prove that the Inductive Step that assuming Pk is true prove Pk+1 is true. This is a very simple kind of a step-by-step way of proving a problem.

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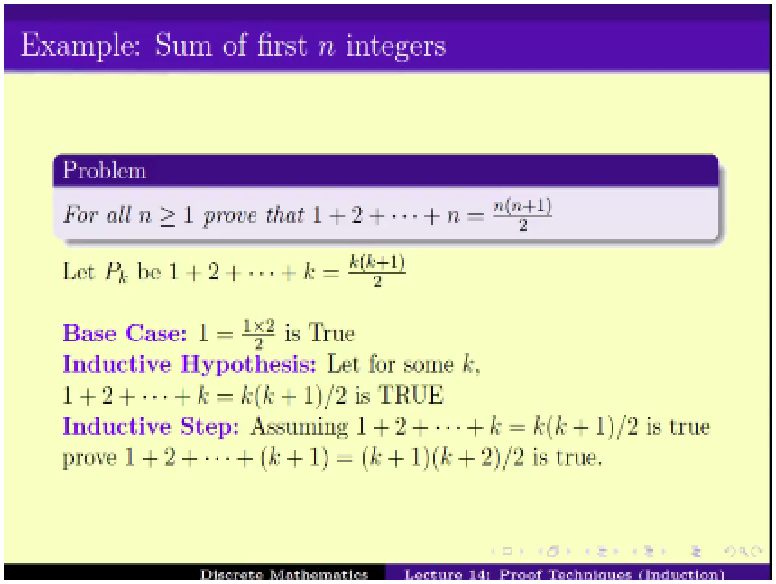

Now let us put these numbers back. So first thus, Base Case turn out to be so it was P1 is true or in other words, the first letter 1 is equal to 1 into 1 + 2 which is 2 by 2 and this is in the true. Now Inductive Hypothesis says that for all k I have to prove that Pk is true for some k we have assumed Pk is true, so let us us assume that 1+1 Pk first k element integers add up to k into k plus one by 2.

And the inductive step says that assuming inductive hypothesis prove that the first k+1 number is k+1 into k+2 by 2. As you can see this whole problem has been boiled down to some three basic steps.

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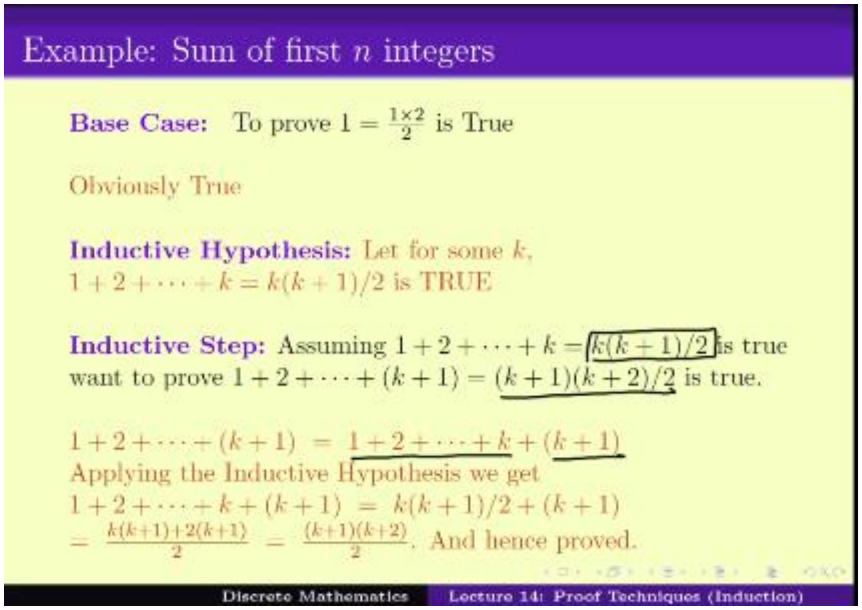

So if you see the first step to prove that Base Case is true, yes you can see this particular case is very obvious here. 1 into 2 by 2 is actually is 1 is obviously true. Okay. So the Induction Hypothesis is let for some k, the some of the k in length integer is k into k+1 by 2. And the Inductive step we know what to do. So, now let say prove it and we have to prove that some of the first k +1 making something.

So first of all note that the some of the k +1 integer is sum of the first k integer plus k+1. Now the Induction Hypothesis we know that some of the first k integer is k into k + 1 by 2 so you can plan begin and you get that so the sum of the first k+1 into k into k+1 by 2. And if now just do the calculation here you get that this is actually equals to k+1 plus k+2 by 2 which is what we wanted to prove. So thus we have proved the Inductive Step.

So why is this original problem might have looked a bit daunting namely how do you prove that in some of any first n number of integers n into n+1 by 2 – there is a principle of mathematical induction gives us a way to follow by following three basic steps Base Case, Inductive Hypothesis and Inductive Step.

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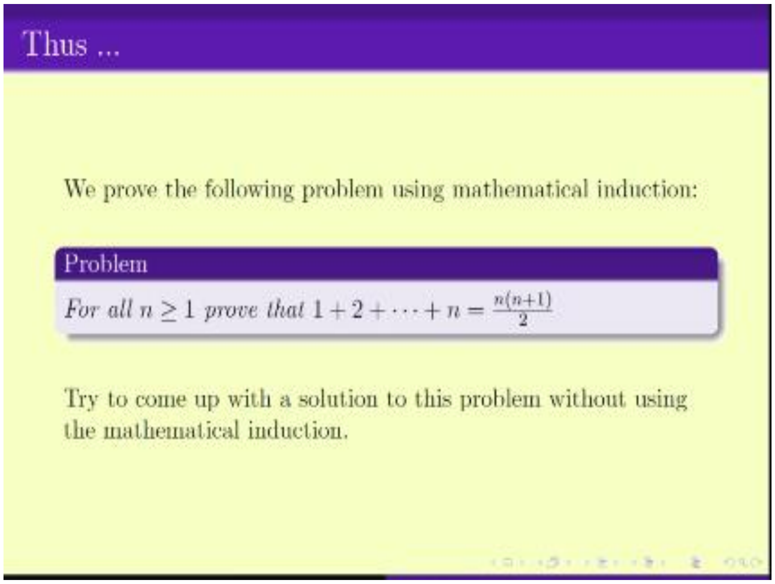

So we have proved the following problem, and I would ask you guys to try to come up with the solution to this problem without using mathematical induction. In this case, it is viable though bit fitty. In the coming week and next week, we will be looking at various variances of mathematical induction and how to use that to solve different kind of problems. Thank you.