Discrete Mathematics

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Lecture - 12
Proof by Contraposition

Welcome to the third video lecture in the third week. This week we are looking at the proof techniques. Till now, we have looked at proof by contradiction and in the last week we have looked at constructive proof namely direct proof and case studies. In this particular video, we will look at a variant of the proof by contradiction namely contrapositive.

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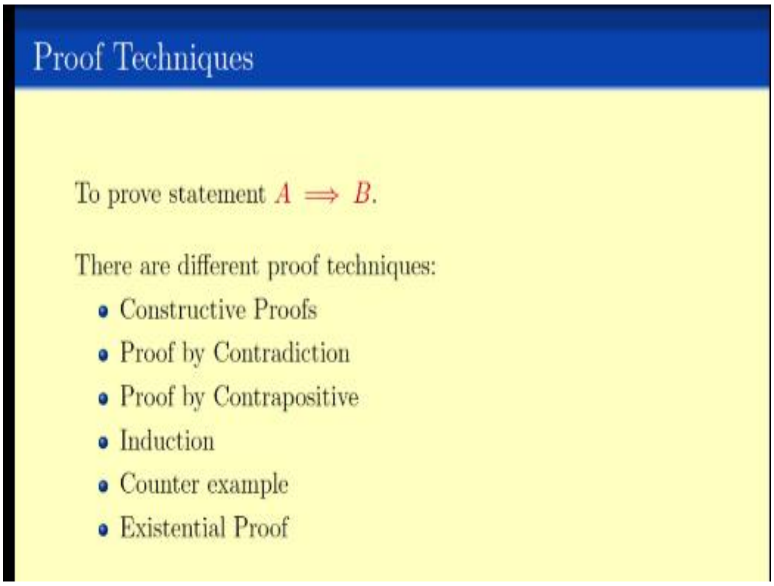

So to recap, to proof a statement A implies B there are various proof technique that we are studying and we will be studying for the next couple of weeks also.

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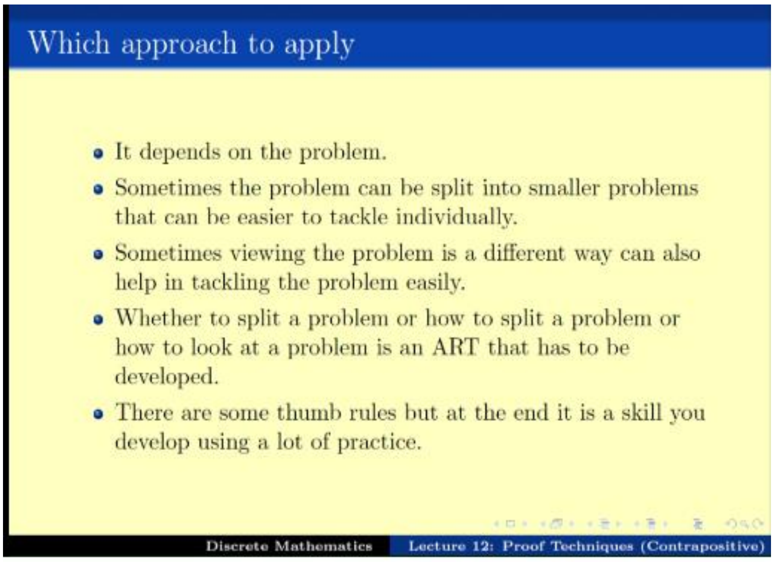

This is a slide I have told read out almost every time I have spoken about two techniques namely which proof technique to apply for which problem is something that you have to develop. We will be introducing you to various proof techniques and give you some thumb rules about which proof technique should be used or can be used for which problem. But at the end, it is you who have to develop an art of understanding which proof technique to be applied.

Now we started with some simple observations as to how to split the problem into smaller problems.

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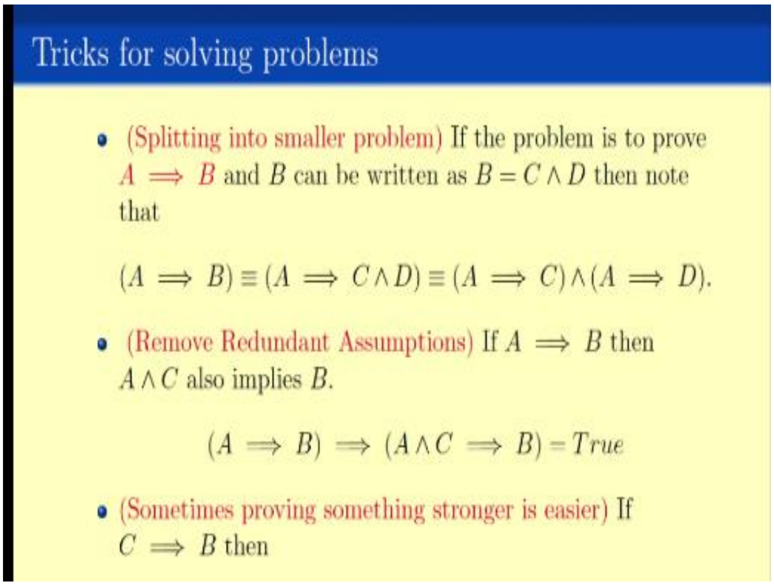

The first two issues think that we say was how to split them into smaller problems if when proving A implies B, B can be written as C and D. In that case A implies B it is same as proving A implies C and A implies E. The second one that we saw was how to reduce or remove redundancies in the assumption. So if there is some assumption that can be removed then removing it to simplify the problem help us get a greater trick on the problem.

And hence you help us always. Now which assumptions to be removed and which cannot be removed depends a lot on your own experience of handling proof technique. The third one was that to observe that sometimes proving something harder and the easier. For example, if C implies B it might very well-happen that moving A implies C is easier than proving A implies C. In that case getting or making a problem harder makes it easier to handle.

We solve such examples in last week. Other than this, 3 small observations we have seen a few proof techniques.

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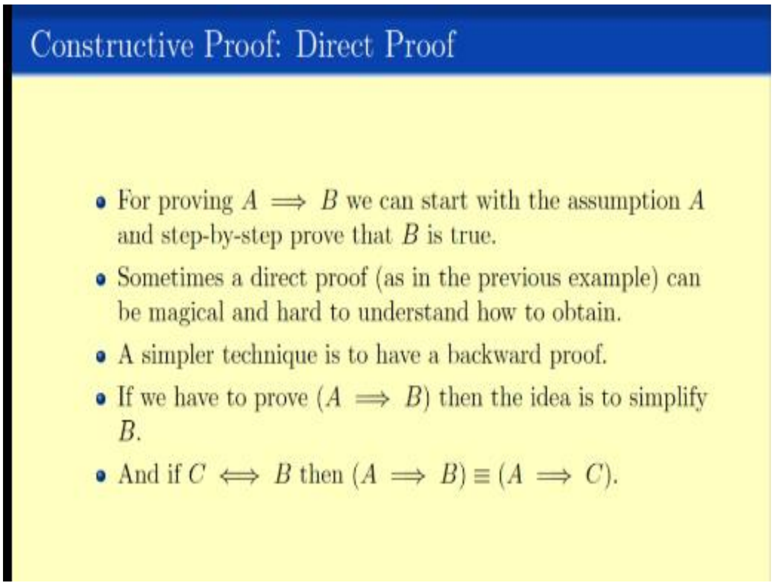

The first one was the constructive proof, but where we will apply the direct proof technique namely you start with A and you prove B. There are two ways of doing it. Number one, of course you can start with A and step by step proof B, but sometimes that can be magical. The direct proof can be very magical and it is not very clear how to obtain such a proof. But it takes way to get around it is by doing a backward proof.

Namely to prove A implies B, first start with B and simplify what we have to prove and if B is simplify then you simplify B to C then A implies B is same as proving A implies C and since C is simplified moving A implies C will be an easier job. So this is backward proof, but in either case so this proof gives us the proof technique called direct proof.

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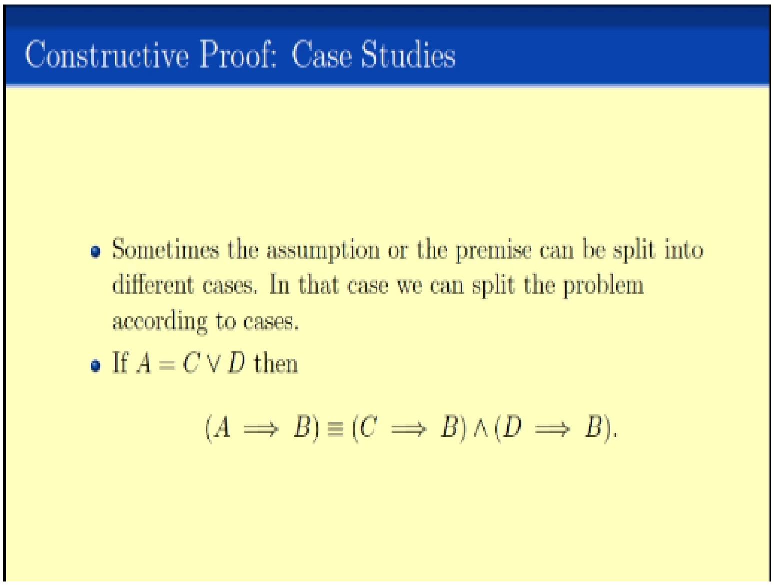

The another constructive proof technique that we saw was what we called as case study. When we case study the idea is that you can sometimes split the assumptions in to cases. Namely, if you can write A as C or D that A implies B is same as proving C implies B and D implies B. One can split the problem into smaller parts. Now how to split the problem or assumption into smaller parts is something that has to be practised or understood by yourself.

We have seen a few examples of applying the case study proof.

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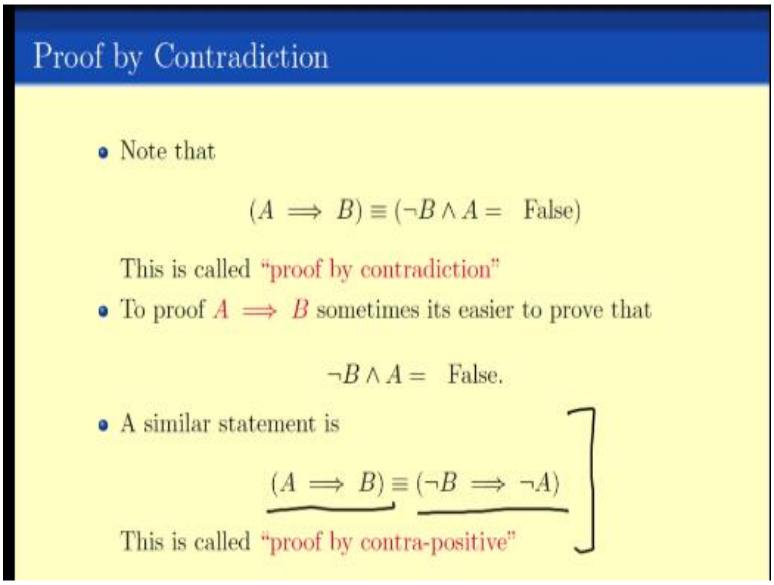

Now in the last couple of videos, we have been looking at the proof by contradiction. The idea is that to prove A implies B it is sometimes easier to prove not B and A implies false. So in other words, if we have to prove A implies B instead of looking at A implies B once can in fact not B and A and prove that they are contradiction. In this particular video, we will be focussing on another kind of proof by contradiction namely proof by contrapositive.

Here, what we use is that A implies B is same as not B implies not A. This is something that can be very useful at times. These are very similar to the proof by contradiction, but the formulation gives the slightly different and for certain kind of problems this is very helpful you will see some examples.

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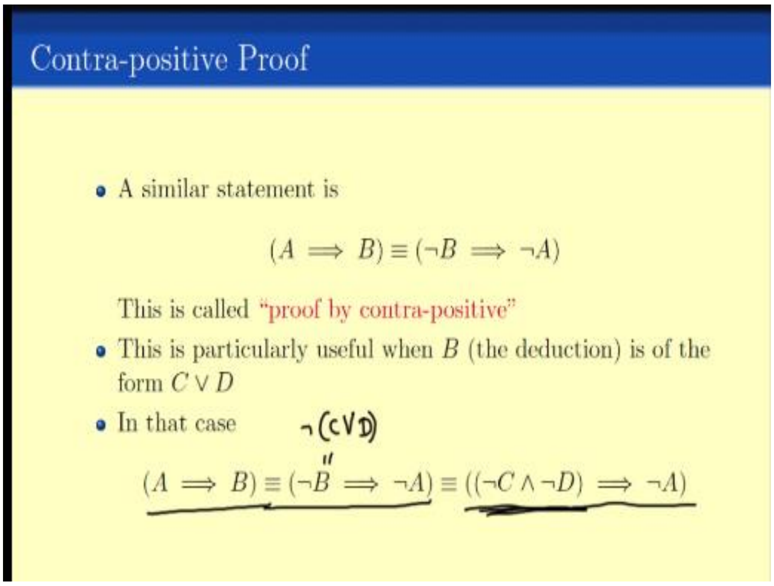

Now, well here is the contrapositive statement. A implies B is same as not B implies not A. It is particularly useful when B the deduction is of the form C or D. In other words, A implies C or B how to solve that? So here A implies C by the contrapositive statement same as not B implies not A. Now what is not B? Not B is not of B or D and here I can apply De Morgan formula a not a C or D is same as not of C and not of D.

So this problem becomes same as not of C and not of D implies not of A. So this is a very simple way as we have seen in many times having more and more conviction of the assumption is not bad. So this is basically I am saying that let assume a not of C and not of D that carry to not of A. We will see examples or problems where this can be handled.

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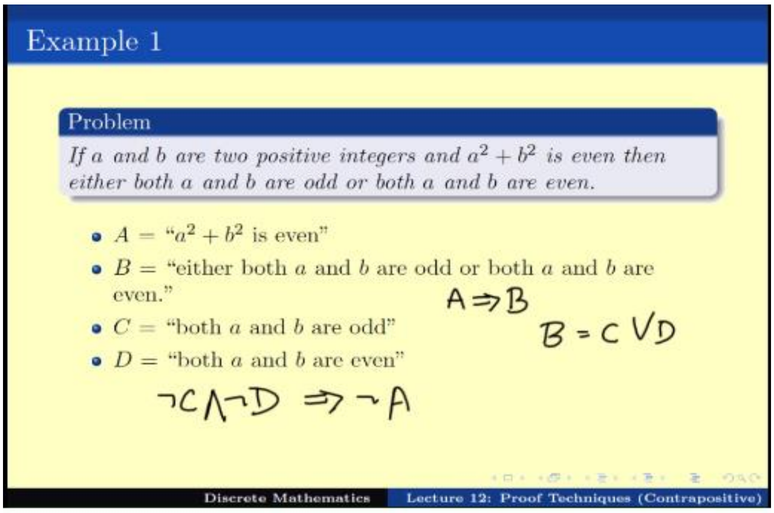

Here is the first example it says that if A and B are two positive integers and A square +B square is even if either A and B are odd or both A and B are even. Now just try to split up into that A, B, C, D formula. So A is A square + B square is even. B is either A either both A and B are odd or both A and B are even. Of course, we have to prove A implies B, but then it is B I have written as some B can be written as C or D, but C is first of all, both A and B are odd and D is both A and B are even.

Thus if you not apply this contrapositive of this statement here what do we get? We should get not of C and a lot of D we call not of C and not of D implies not of A or in other words both A and B are not odd and both A and B are not even then A square + B square is not even. Now let us reformulate the statement again for not C and not D what is it saying that A and B are not odd and both A and B are not even. Another way of writing it is of course one of A and B is odd the other is even and not A is A square+ B square is not even.

In other words, A square+ B square is odd. So the formulation is of course that if one of A and one of B is even and odd then to A square + B square is odd.

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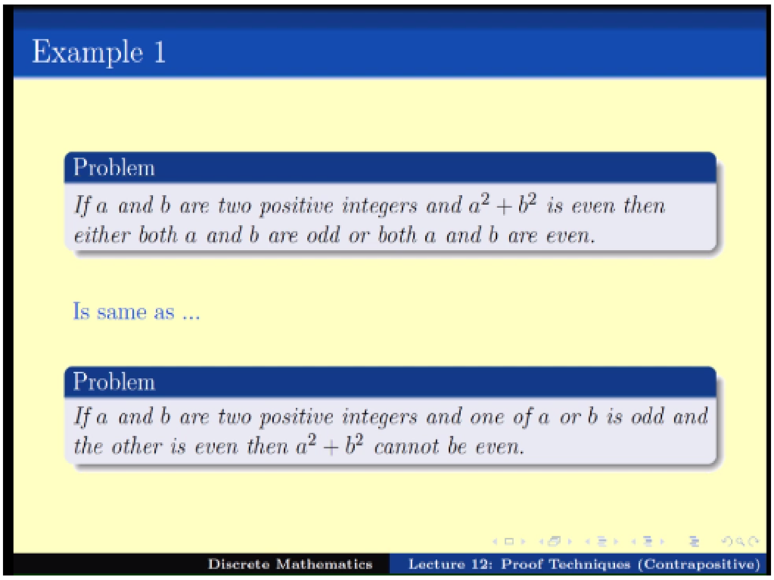

This is the formulation that we are applying and we get that this problem becomes same as if A and B are two integers and one of A or B is odd and the other is even then A square+ B square cannot be even first of all.

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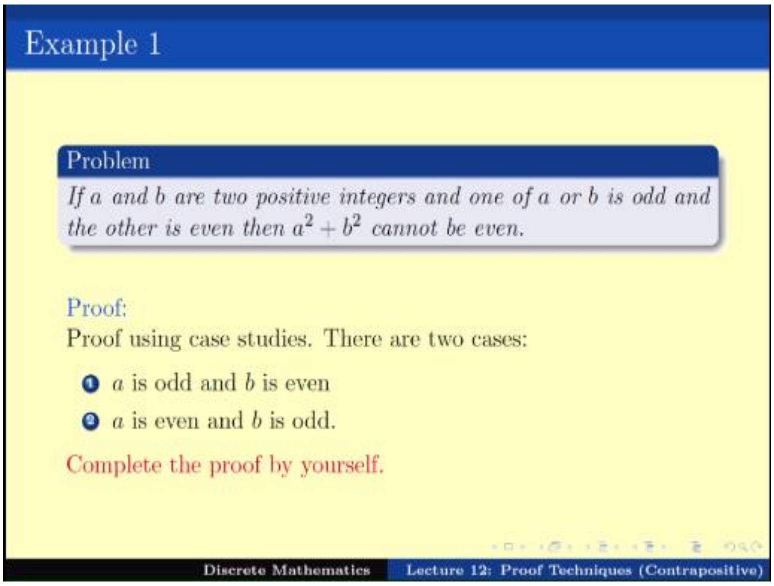

Now to prove of this one okay we have to do case studies and it is very simple. First case is A is odd and B is even and second case is A is even B is odd. It is clear that these are the two cases you can apply this two case study and prove the following statement. So rest of the two for you to prove in your leisure time to completely proof by yourself. The main thing to notice is that by applying the contrapositive rules.

We could easily convert this problem into something quite same. Sometimes some of the statements of the examples can be much more complicated than what you all think-and that is when contrapositive statement helps a lot.

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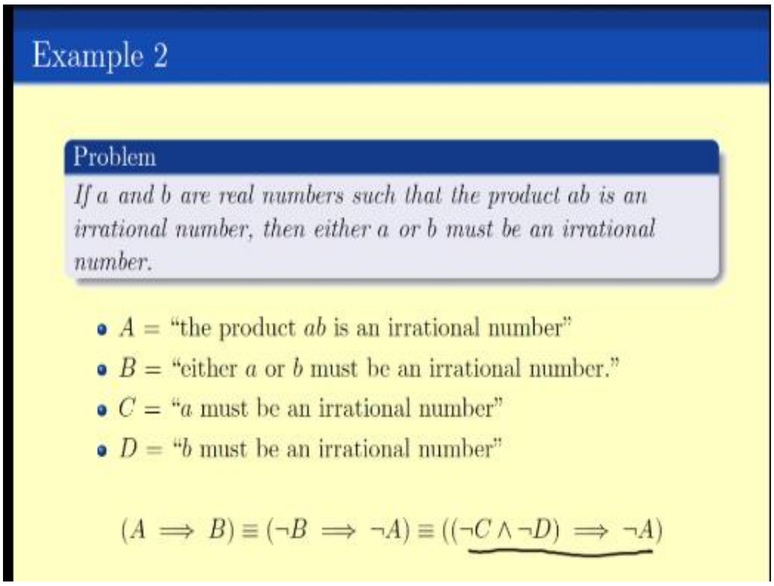

So let us look at our second example it states that if A and B are real numbers such that the product of A and B is an irrational number then either A or B must be irrational. So this is very similar to some problem that was assigned in the last video. So how do you prove it? Again let start to break it up. The first assumption is both A and B is an irrational number the product of A and B is an irrational number. B which is induction that you have to do either A or B must be an irrational number.

If you want to split up into the C and D way C becomes C is A must be an irrational number and D becomes B must be an irrational number. I think let us say to what is using this rule of contra positiveness, we get that NOT of C now what is NOT of C? That means A must be an irrational number. NOT of C means A is a rational number. So if A is a rational number and B is a rational number then AB is a rational number.

So that is what we are getting. This problem which was tend to be a bit more complicated at least initially (()) (14:08) is same as proving that if A is rational number and B is rational number then prove that AB is a rational number. This is one of the problems so the problem is same as this particular statement if A and B are rational numbers and AB is a rational number.

This is one of the problems that was assigned to me last week and hence you should be able to now solve this problem yourself completely.

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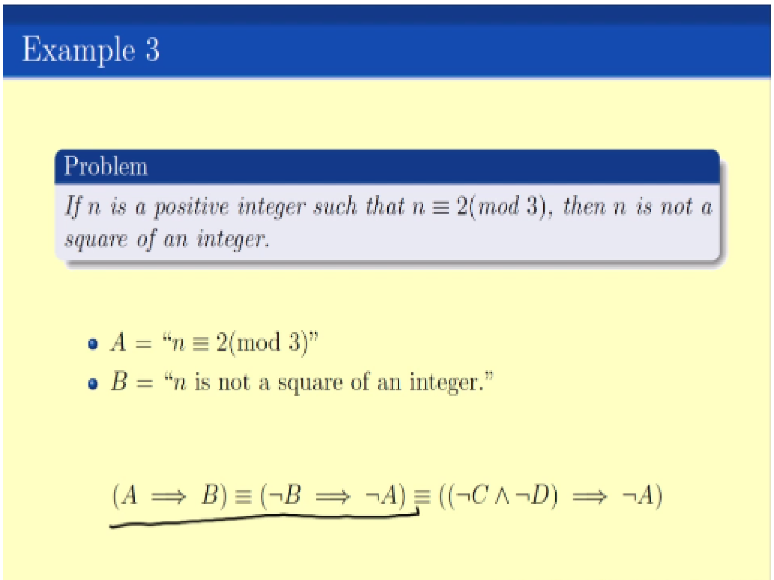

Moving on to the third example in this example, we say that if N is a positive integer such that N is congruent to 2 mod 3 then A is not a square of an integer. Now here let see what is A? A of course is saying that N is congruent to 2 mod 3. What is B? B says that N is a not a square of an integer. Now in this particular problem the clarity that is not more economical way of splitting this B into C or D.

So we have to what write away with that, but let us write out the contrapositive statement first know this namely all the 3 up to 2 you will be first part because there is no C and D. So A implies B is same as not B implies not A. Now what is not B? Not B, not B is A square of an integer. So that in other words A is a square of an integer implies C then not A means A is not congruent to 2 mod 3.

If A is not congruent to 2 mod 3 what is the congruent 2 the other 2 namely 0 and 1. So in that case, we prove that it is congruent to 0 mod 3 or 1 mod 3. When we are converting on this statement to a problem where it do becomes A implies C or D, but it so happen that kind of B and C. Let us see.

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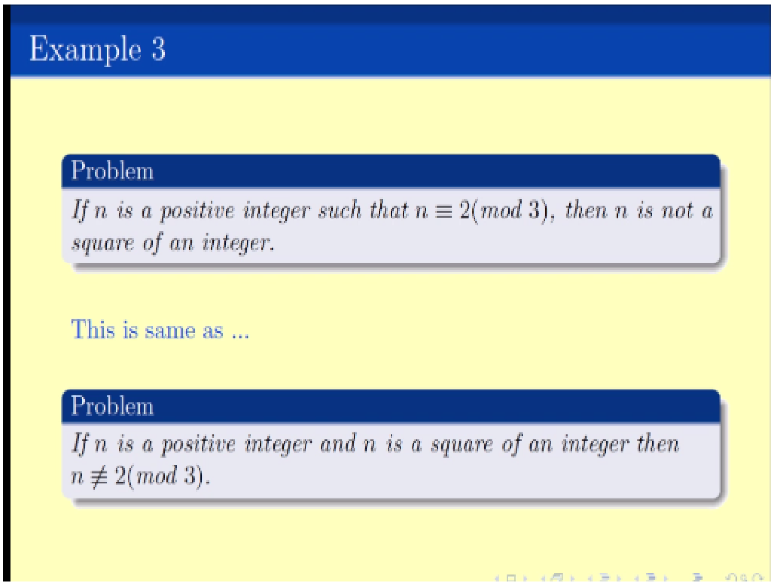

In other words, this problem is same as if N is a positive integer and N is a square of an integer then N is not congruent to 2 mod 3.

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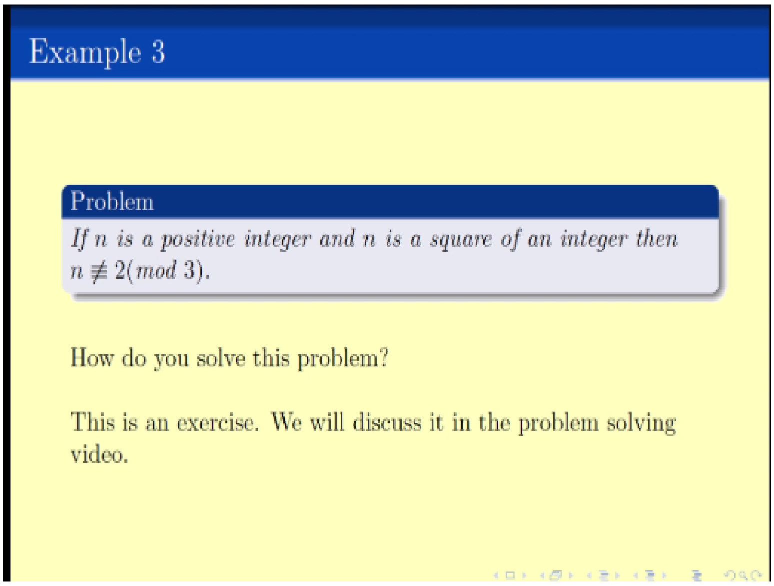

And how do you solve this problem or leave it to these guys to find out the way of solving it. This is a nice exercise it used to culminate many of the problem technique that we have done. It is a simpler application of one technique that we have studied till now. I advise you guys to look at all the examples that I did in this video and try to prove it directly and then you will only appreciate why to prove by contradiction is such a useful technique to have.

By doing to prove by contradiction we are able to simplify or view the problem in different ways which helps us to solve it and we are seeing many of the proof the last video as well as this video. The final proof of a problem was simply involve applying multiple proof technique in the same problem may proof by contradiction follows by a case study, approved by contrapositive followed by a direct proof and so on and so forth.

So this brings us to the end of this video lecture. In this video lecture, we saw proof by contra positiveness. It is a pretty powerful proof technique. In the next video lecture, we will be looking at a new proof technique or counter examples. It is very useful for certain formula. We will continue our study of proof techniques in the next week when we will move into some even more interesting proof technique mainly in that. Thank you.