Discrete Mathematics
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Lecture - 11
Proof by Contradiction (Part 2)

Welcome everybody to the third week, second video lecture. In this video lecture, we will continue with our understanding of various proof techniques, particularly understanding of the proof by contradiction. We have started this particular technique in the last video lecture and we continue to study this by looking at more problems.

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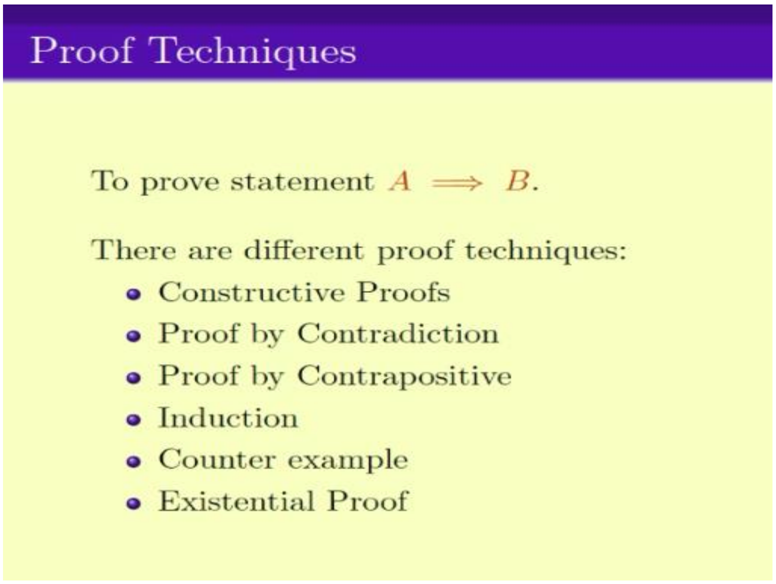

So to recap, to prove a statement like A implies B, there are various different proof techniques. We have already seen proof techniques namely constructive proof, proof by contradiction and so on. We will be seeing much more other proof techniques also in the next one or two weeks. So one thing that I have always mentioned and I repeat here again, if you ask which proof technique to apply to which problem, it depends on the problem.

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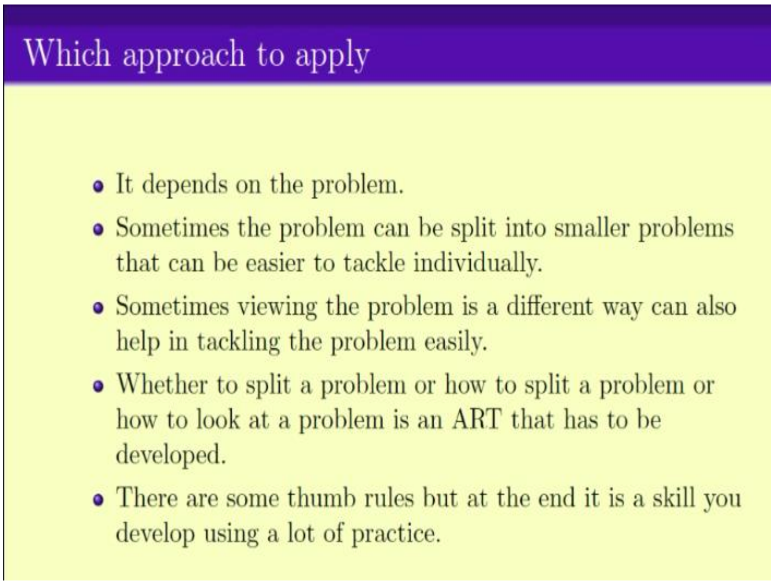

The answer of which proof technique should to be applied depends on the problem. While for some of the problems it can be split up into smaller problems, that can be easily tractable. Why for the many other when one can view the problem in different way and that make it easier. But which problem to split and how to split it and how to view it depend on yourself. This is an art that one have to be developed, you have to develop.

In this course, we will be giving you all the various tool that are there. We will give you thumb rules, but at the end it is your skill that you have to develop, that you have to apply to find out which problem should we solve by which technique.

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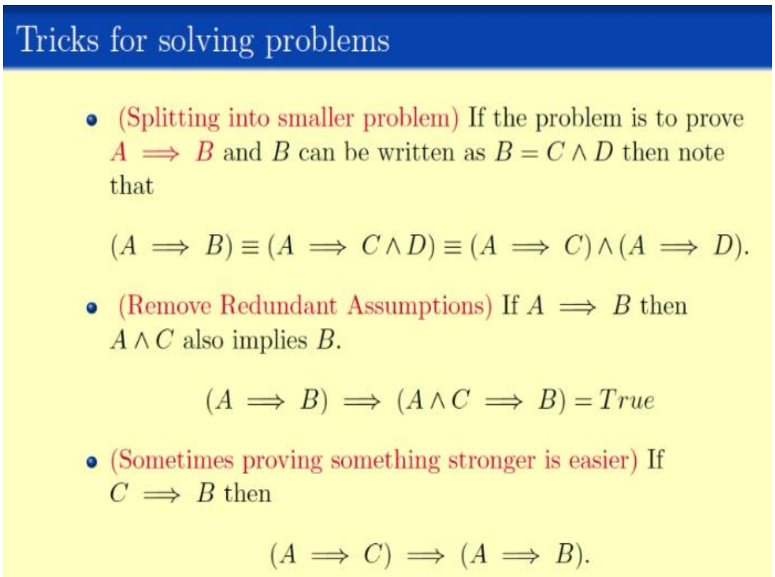

To start with, we looked at some of the tricks that can be applied to solve the problems. The first trick was splitting it into smaller problems, namely when to prove A implies B, B can be written as C and D. In that case A implies B, is same as proving A implies C and A implies D. We also saw that we can remove redundant assumptions and by doing so we can make our problem simpler which can be easier to handle.

The third technique we saw is that sometimes proving something stronger is easier. In a sense that, if C implies B, it may be possible that A implies C is easier to prove that A implies B. But since C implies B, so A implies C, is good enough for proving A implies B. We saw three applications of these three techniques in the video lecture last week.

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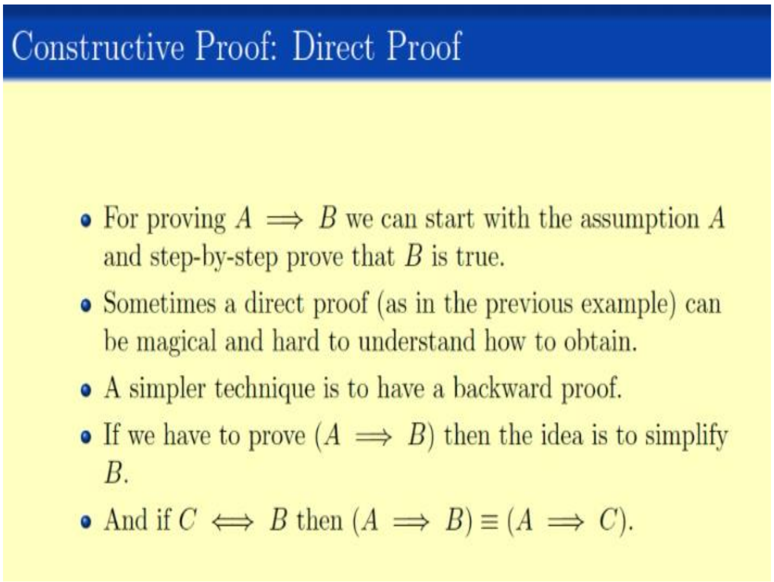

Other than these three small tricks we also started looking at constructive proofs, or what we call as direct proofs. So there we have two different kind of constructive proofs, first of it is direct proof. The idea was to prove A implies B, you work with A and step by step prove B. Sometime we saw that proving it in this way can be hard, so can be magical. So instead one can come up with something called a backward proof.

So that was a simpler technique the idea is that instead of starting of A and slowly massaging it to get B, you start from B, you start simplifying B and if you can simplify B to something like C, then proving A implies B is same as proving A implies C and proving A implies C can be an easier job, because you have simplified B to get C. So this is the main thing of the direct proof.

There was another kind of the constructive proof that we looked at, namely what we call as the case study.

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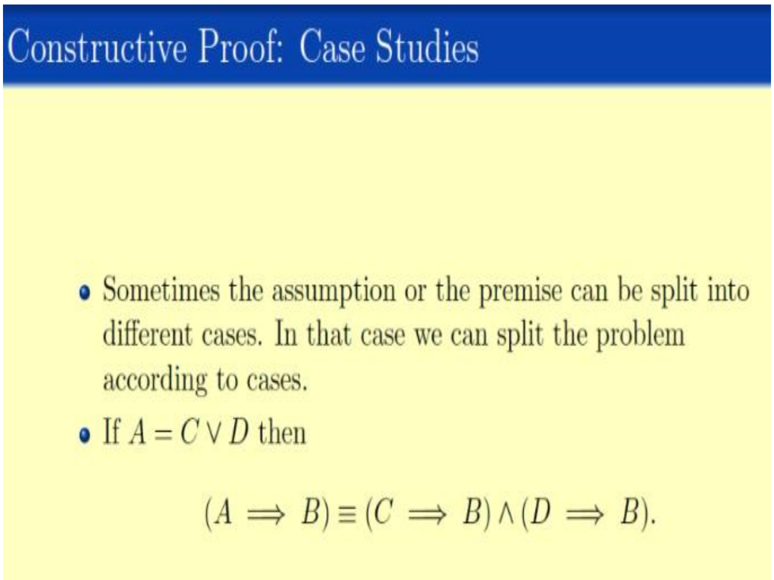

So in this case, we split the assumptions into parts. In other word, if A is written as C or D, then A implies B is same as C implies B and D implies B. And the main feature is to split A into two cases or three cases or whatever number of cases of C and D, such that C implies B and D implies B are easier to prove. So we saw examples of all these various techniques in our past video lectures.

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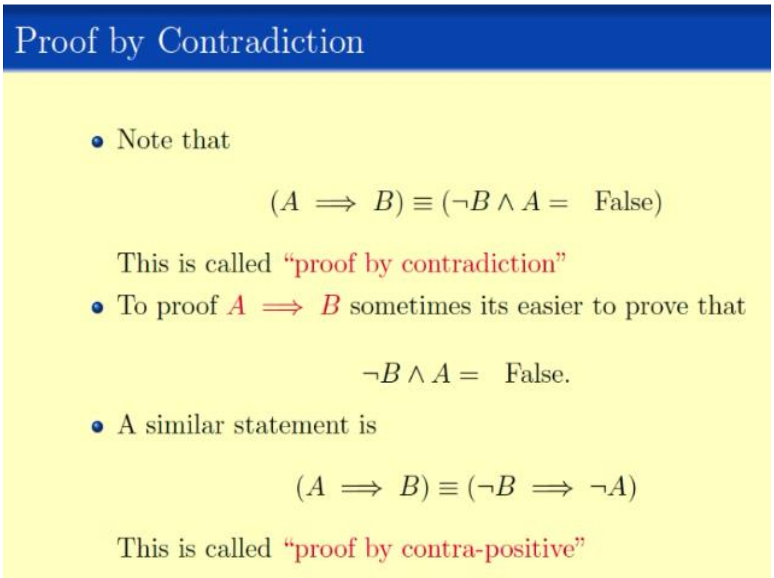

In the last video lecture, we started with this new proof technique called the proof by contradiction. So the idea here is that to prove A implies B, it is the same as proving not A and B is false. So, sometime instead of proving A implies B, one can end up proving not B and A is false. This technique is called proof by contradiction, or in other word if you view the problem in a different way.

Instead of viewing the problem as A implies B, you view the problem as not B and A is false. As seen the statement is what you call proof by contra-positive, which we will be doing in the next video lecture. So in the last class we saw application of this proof technique to solve the problem, namely to prove that there are infinite primes. In this video lecture we will also apply this particular proof technique of proof by contradiction to a new problem.

But, before we start on the problem, let us try to understand again what is the proof of contradiction all about.

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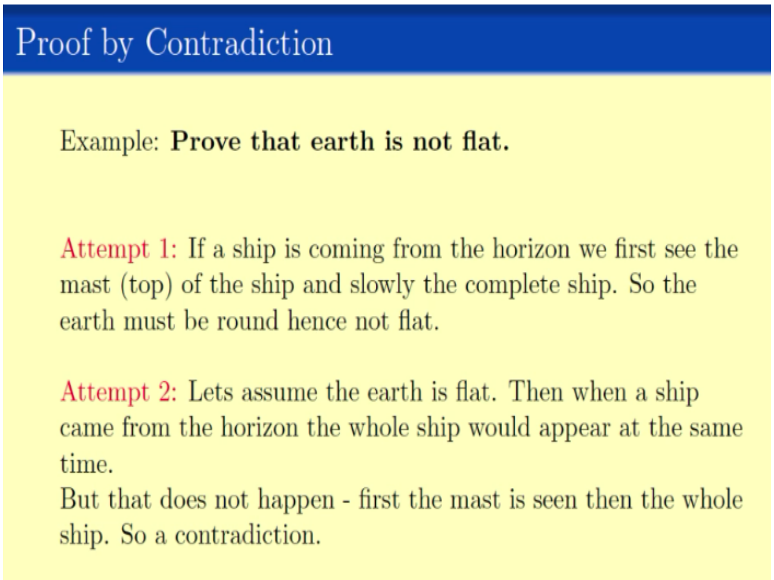

So it is like, there are two different ways of proving that earth is flat or earth is not flat. The first approach is a direct approach, namely, a ship is coming from the horizon, when it comes we first see the top of the ship, and slowly the whole complete ship. So the earth must be round, hence not flat. The other technique is to say that let us assume that the earth is flat. In that case, when the ship came from the horizon the whole ship would have appeared at the same time.

But that did not happen, we first see the mast and then the whole ship. And that is the contradiction. A contradiction to the assumption that you have made namely, the earth is flat. As you can see that both the proof almost the statement are very similar. It is just in terms of saying with different wording. But that is what the proof of contradiction is, proof by contradiction is just restate the problem in a different way.

And in fact a proof of contradiction can variously be converted into a direct proof. But sometimes, getting a direct proof directly can be more complicated and that is why we go through the thing called proof by contradiction.

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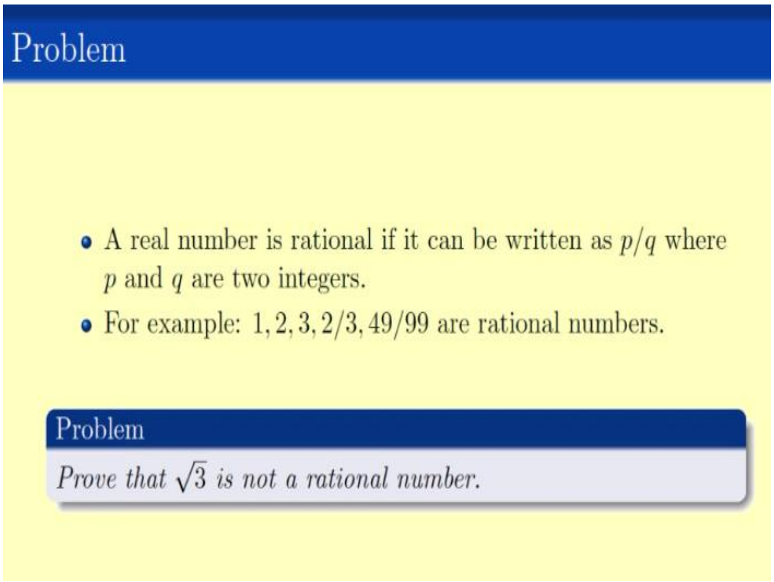

So now, for today's video lecture, we will be working with numbers. Here, you see a real number is rational if it can be written as the ratio of two integers, namely p by q. For example one, two, three, all this can be written as one by one, two by one, three by one and two by three, forty nine by ninety nine and so on it goes. The question is that can you prove that square root three is not a rational number.

So namely, square root cannot be written as p by q or a ratio of two integers. Now, we prove it using contradiction. Let us see.

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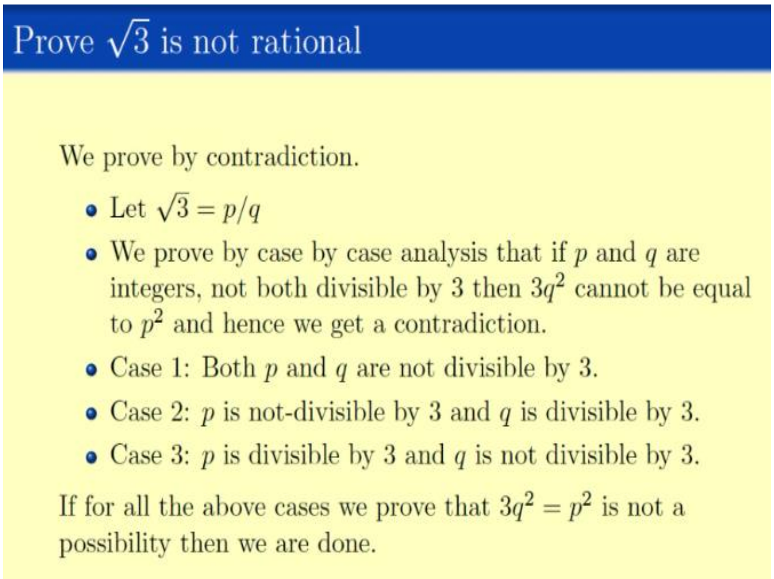

So let us assume that square root 3 is rational, or in other words square root 3 is equal to p by q. So that is the assumption, this is contradiction part, right, this is not B part. And now, we have to prove that not B and A is false, and what is A here, A is everything else. All the thing that we know of, so if square root 3 is equals to p by q, then do I get something weird statements.

Now, to prove that, you get some true statements, or you may get some false statements, we will do a kind of a case analysis, we will apply the case analysis technique here. But to understand it, let us first simplify. When we write square root 3 as p by q, and if both p and q are divisible by 3, then I can just strike out 3 or divide both p and q by three, to get a smaller p and q, right.

If by chance square root 3 is equals to 18 by 36, I could divide by three and get down to 6 by 12, or something like that. Of course, that is not true, because this number is equals to half. But, the idea is that I can always write if square root 3 is a rational number, I can always write square root 3 as p by q, where both p and q are not divisible by three. That means one of them can be divisible by three, but not both of them, both of them cannot be divisible by three at the same time.

Now, if that is the case, main case, when I say square root 3 equals to p by q, this translates to, of course if I square both sides, three equals to p square by q square, or in other words 3 q square is equal to p square. And that is where we will be drawing our contradiction. Let us see, let us do case analysis. In case one, that both p and q are not divisible by three, that is, neither p nor q divisible by three.

Then can 3 q square equals to p square, right, that is the first case. The second case will be if p is not divisible by 3 and q is divisible by 3. And third case will be is p is divisible by 2 and q is not divisible by 3. Note that, there is only three cases, because the fourth case where both p and q are both divisible by 3 and in that case that we would have got a smaller p and q, as we just now argued.

So if you want to prove that all the three cases, we can get a contradiction, maybe we can prove that 3 q square cannot be equal to p square, then we are fine. With all the three statement if we prove that 3 q square is equal to square is not a possibility, then we get a contradiction.

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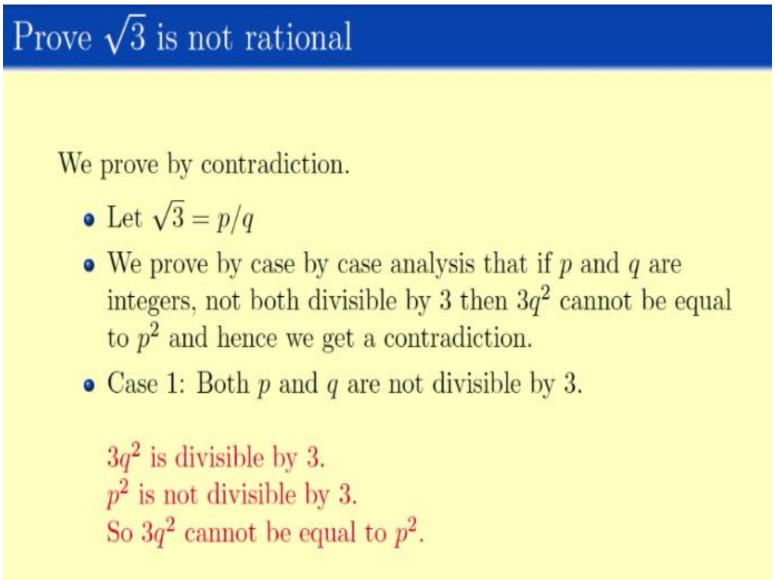

Now let us do it by case by case, let us consider the first case. First case, when both p and q are not divisible by 3. So in that case 3 q square is of course divisible by 3, as you know, that 3 q square has a 3 in it. But p square is not divisible by 3, because p is not divisible by 3. So if p square is not divisible by 3, but 3 q square is divisible by 3, then can p square be equal to 3 q square, of course not. So 3 q square cannot be equal to p square in this case one.

Now let us go to case two, this is the case where p is not divisible by 3, but q is divisible by 3.

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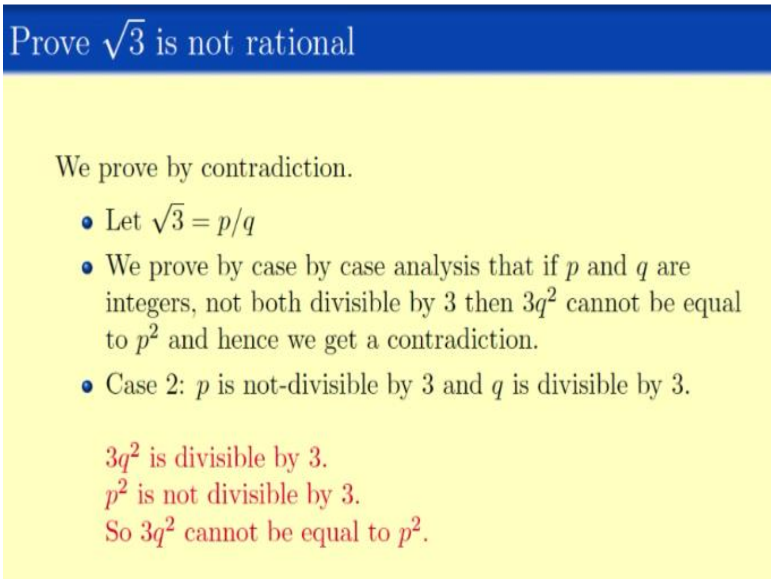

Again the same argument, 3 pq square is divisible by 3 and p is not divisible by 3, therefore p square is not divisible by 3. And hence, 3 pq square cannot be equal to p square. Now let us see the third case, namely if p is divisible by 3 and q is not divisible by 3. Now, p is divisible by 3, that means p square is divisible by 3 and on the other hand 3 q square is also divisible by 3. So that argument will not work, so what shall we do, let us see.

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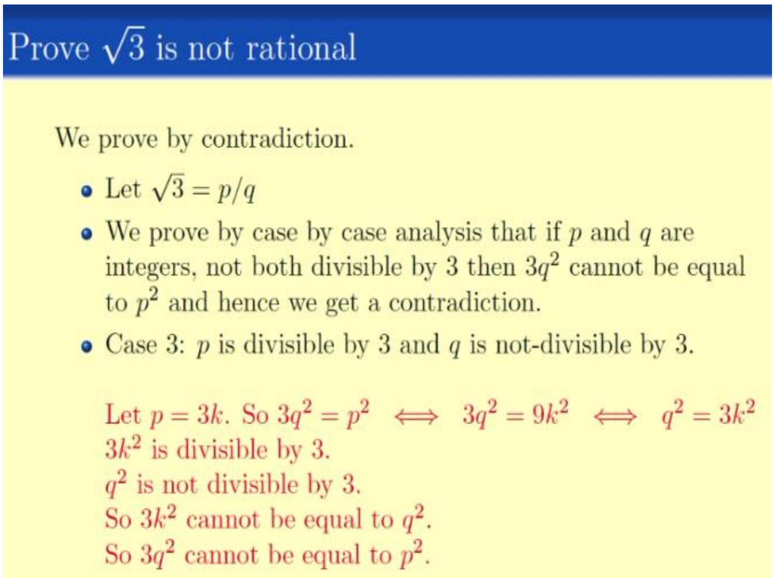

So let p is equals to 3 times k, because p is divisible by 3, right. So 3 q square equals to p square is same as 3 q square equals to 3 times k whole square, because I assumed that p equals to k. So 3 q square equals to 9 k square, which is same as q square equals to 3 k square, right. Now 3 k square is also divisible by 3, because 3 k square has a 3 in it. But, q is not divisible by 3, so q square is not divisible by 3.

And hence, by almost the same way of argument 3 k square cannot be equals to q square. Thus, even in this case square root 3 cannot be equals to p by q, or in other words 3 q square cannot be equals to p square.

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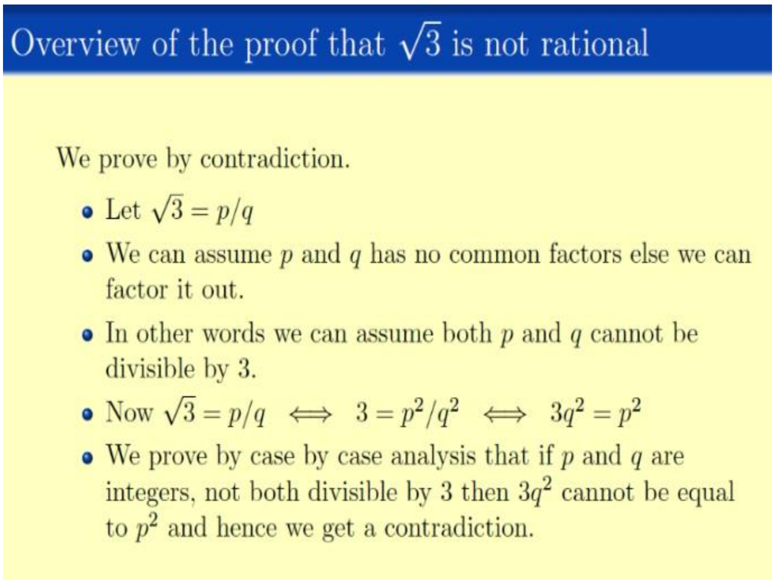

Thus to wrapping up the whole proof, with proof by contradiction, we assume that square root three is equal to p by q, which is the opposite of what we have to proved. And then we assume that p and q has no common factors else we can factor it out. So in other words, we can assume that both p and q cannot be divisible by 3. And now since square root 3 equals to p by q, or in other words 3 q square equals to p square.

We say that, prove by case by case analysis and prove that, this cannot be true, 3 q square cannot be equal to p square for any integer p and q, and hence, we get a contradiction. So this is an example where we not only apply the proof of contradiction, but also the case study proof into it. And there are lots of similar problem that can be asked.

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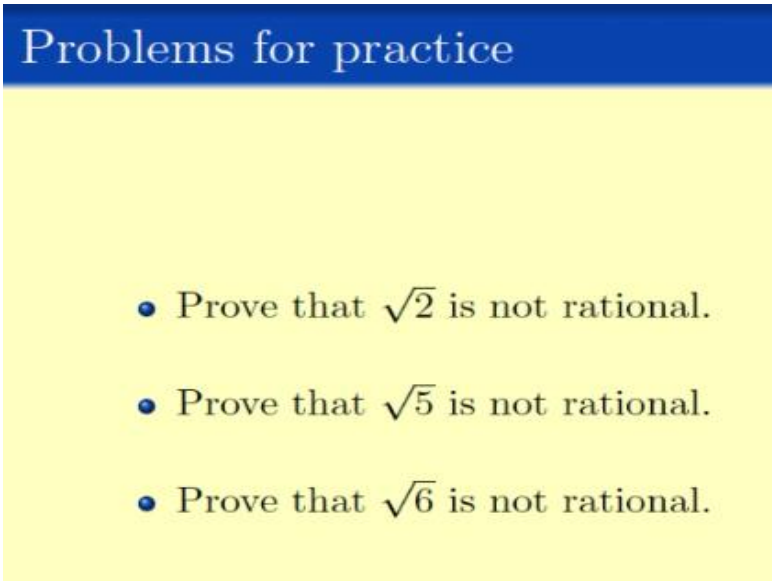

Particularly, the need to solving the problem that are the practice problem, namely prove that square root two, square root five and square root six are not rational numbers. So none of these are rational. Now I would also like you guys to solve some more statements or observations about rational numbers.

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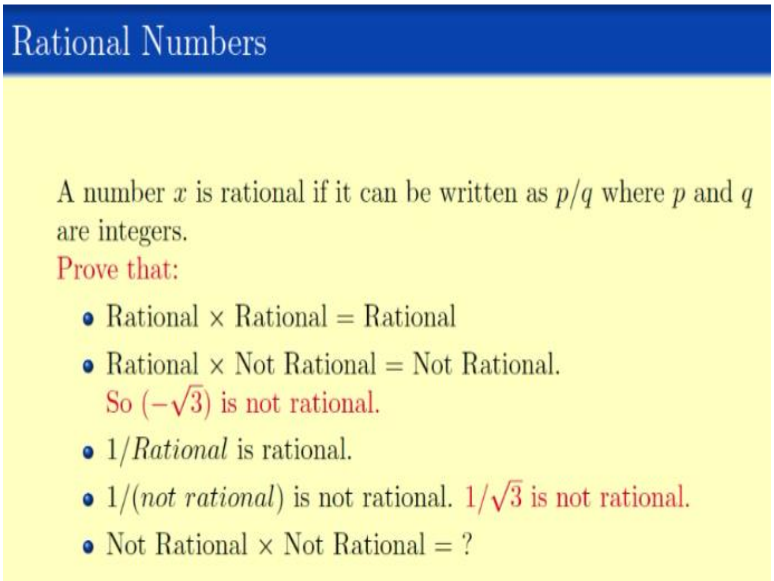

Namely, here are they prove that a rational times of rational is a rational number. And rational times of non-rational is non-rational number, for example, the square root three is not rational and minus one is rational. So minus square root three is not rational. One by rational number is rational, one by a non-rational number is not rational. Therefore, even one by square root three is also not rational. And what about non-rational times non-rational.

Is it rational or not? I leave this for you guys to check it out. Is the product of two non-rational numbers, non-rational or rational. Now moving on, let me prove one more thing, namely what about if we I add two non- rational numbers, or something like that.

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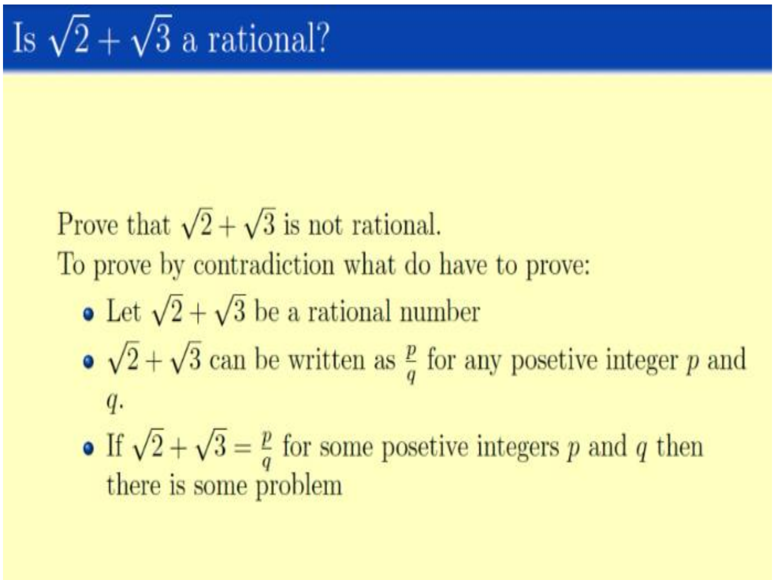

So is square root two plus square root three rational? We have proved that square root three is not rational, and I have asked you to prove that square root two is not rational. But what about square root three plus square root two. Let us prove they are non-rational, right. So the proof is again by contradiction, and there it is. So let us, square root two plus square root three be a rational number.

That is square root two plus square root three is p by q for some positive integers p and q. So therefore, square root two plus square root three equals p by q, which means that square root three is p by q minus square root two. If I square both sides, what will I get, I get three is equals to p square by q square minus twice square root two p by q plus two, which means if I guess, move these things around we get twice square root two p by q is equals to p square minus q square by q square.

And therefore, square root two is equals to p square minus q square by two p q. Now, since p and q are integers, so p square minus q square is also integer is let it be p prime, and two p q is also integer we call is q prime. In other words, if square root two plus square root three is a rational number, I end up proving that square root two is a rational number, which is something that I know to be false, right.

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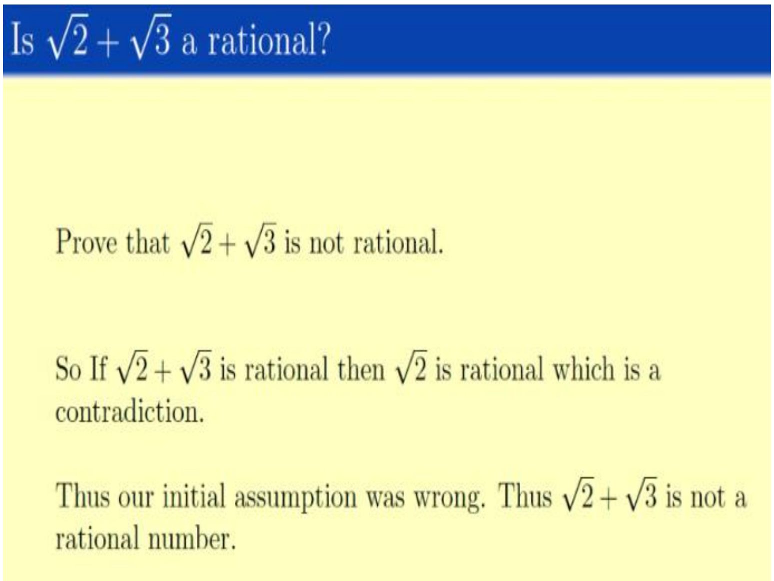

So in other words if square root two plus square root three is rational number, then square root two is a rational number, which is a contradiction. So our initial assumption was wrong that is square root two plus square root three is non-rational number. I may ask you guys to repeat the same proof, where instead of assuming that you know the proof of square root two is non-rational, you really use the fact that square root three is not rational, which is what we have proved here.

So, this brings us to the end of the video lecture, in the next video lecture we will be talking about proof by contra-positive. Thank you.