Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-42
Completion of Proof of The Riemann Mapping Theorem

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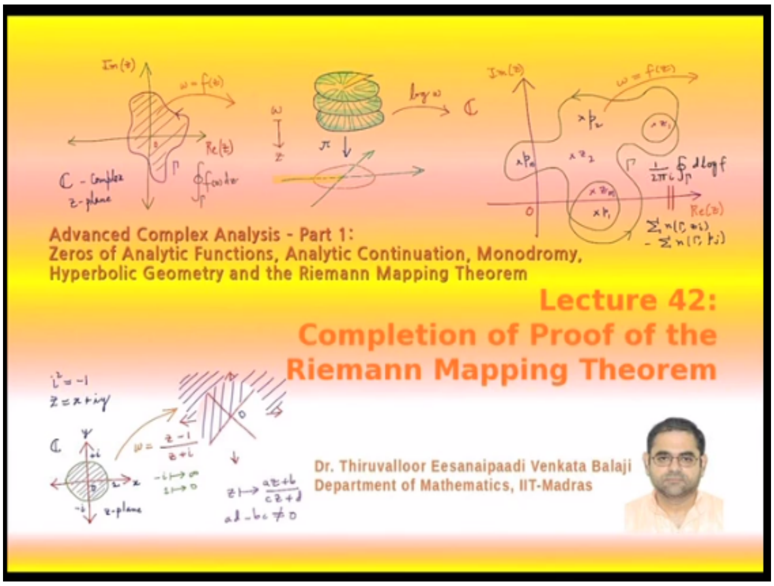

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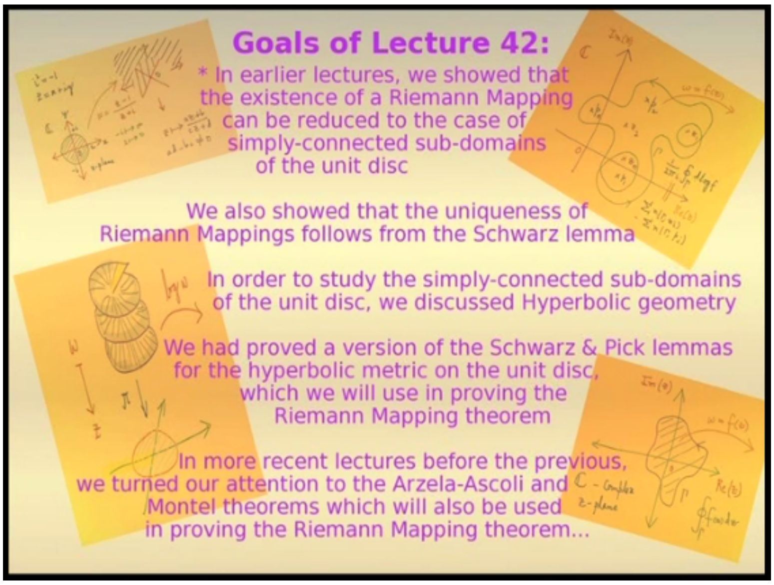

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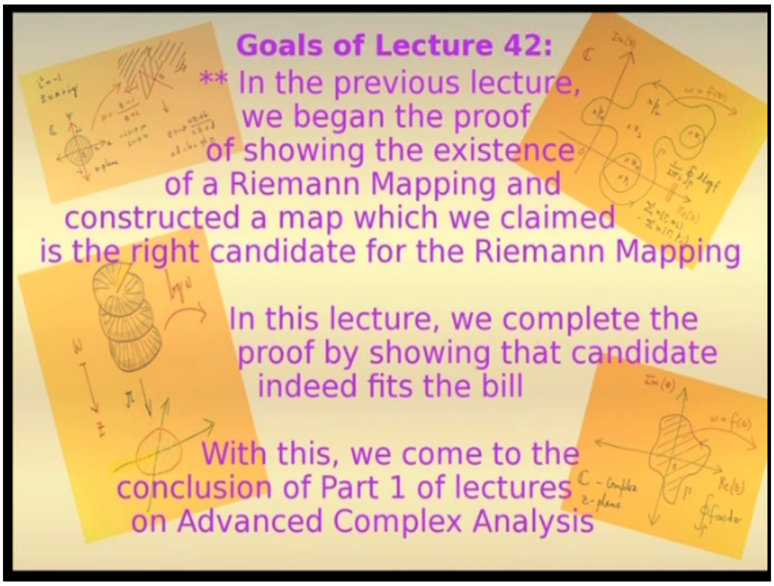

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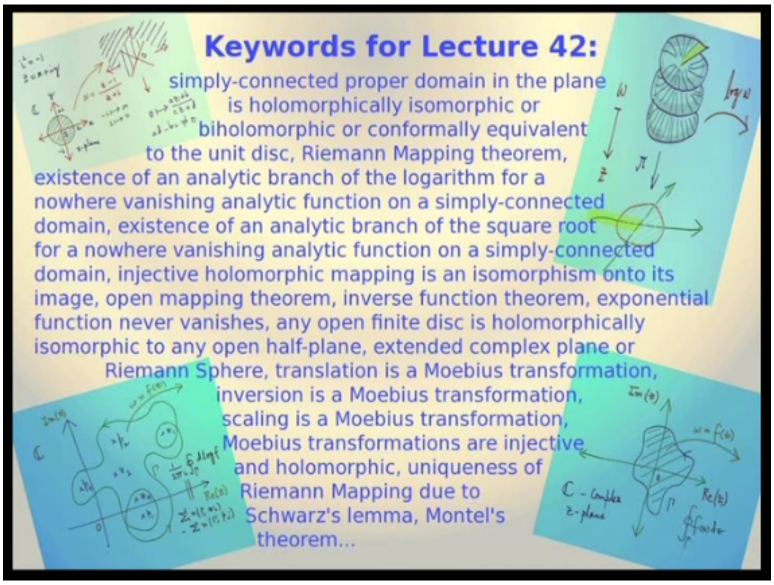

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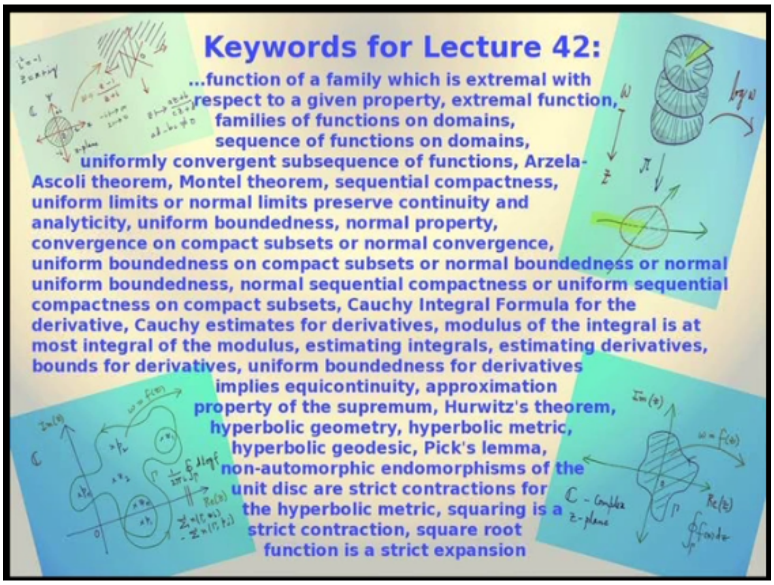

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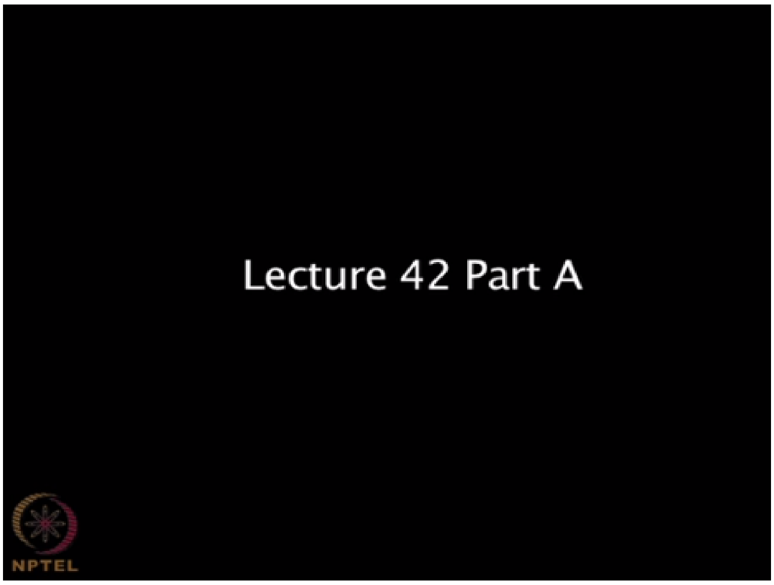

Ok, so let us let us continue with this proof of the Riemann mapping theorem.

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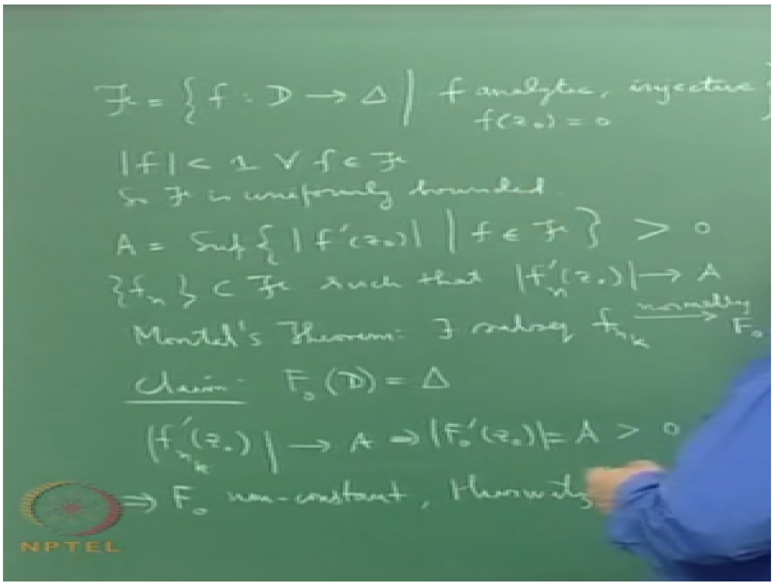

So what we have is see we have we have this family script F consisting of analytic functions from the given simply connected domain d which is not the whole complex plane ok and taking values in the unit disc delta right, see this is unique disc centre at the origin radius one ok, open unit disc, such that of course is analytic injective and taking ff z0 taking z0 is 0, so z0 so z0 is a point of t.

And so we have this family script f, then what we did was we had taken so the fact that each f takes values in the unit disc, that told you that modf is always take less than one for all f the family. So this tells you that the family is uniformly bounded ok, so so script f is uniformly bounded and of course that we already seen that this family is non empty ok.

Because you can always find a **a** holomorphic isomorphism of d on to a sub domain of unit disc ok that was the first step of the Riemann mapping theorem, that we started to prove ok. So this this family is non empty ok and the family is uniformly bounded alright and therefore now you apply, so what you do is you take so Montel theorem will apply to say that if you have any, if you take any sequence in this family.

Then there will be a uniformly there will be a sub sequence you should converge uniformly on compact substation namely you are always find normally convergence of sequence. Normal convergence means uniform convergence or compacts surface ok. So Montel theorem will apply, but to which sequence do we apply to so what we do is we look at the we look at the supermom.

The supermom of derivative of these functions at z0 models for derivative ok, the fact that this is a finite because of it was basically because of cautious estimates ok, therefore the supermom is a finite number he called it as a alright and we prove that since all the this a is of course not negative in fact, so the fact that all these f are all injective means that all these derivatives can never vanish ok.

Therefore a is a positive quantity alright and then what we did is we took sequence fn in a family as f such that the if you take the corresponding sequence of derivatives and take the mod like that converges to a and this is because a is a supermom, a hassle has a approximation property supermom of set of real numbers can be gotten as the limit of a sequence from that set ok.

So you that is how you get the sequence ok, the sequence of functions in f such that they are derivative that Z not the modulus of the derivatives are z0 text A ok, and then by using Montel theorem tells you that there exis a sub sequence fn k which converges normally to a function f0 ok.

So this is the application Montel theorem which says that whenever you have uniformly bounded family of analytic functions then you take any sequence in that family the sequence will admit a subsequence which converges normally ok. So normally on d which means uniformly on compact subsets of t alright. Now what we did then of course the big deal was that then the claim was that the image of f0 under d.

The image of d under f0 is actually the whole unit ok, so this is f0 succeed in mapping the given simply connected domain which is not the whole complex plane, isomorphic on teo the unit test ok which is the main aim of the Riemann mapping theorem. So the first thing is that so there are few there are few facts that one has to understand the first thing is as I told you first of all if you the fnk of z0.

In modulus the derivatives tends to a that will tell you that f0 derivative of f0 at z0 is also A, the modules of that is also A. the modulus of that is also A ok, because fnk tends to f0 because it is normal convergence that derivatives will also tend to the derivative of f0 not and if you apply z0 and take modulus you will get this fact. So this limit function okay mind you normal limit of analytic function is analytic.

Therefore is f0 is certain analytic alright and its derivative at z0 if you take the model of it that will be A alright. That is because of this convergence and the definition of A ok and the sub and the sequence we have pick and this so the derivative is at z0 ah is certainly the derivative at z0 for the function f0 is not zero. So it is a not non constant function.

So f0 is not is non constant ok, because had being concentrate the derivative would be 0 alright everywhere, so it is a non constant function, but then you now apply Hurvitz's theorem, it tells you that whenever you have a sequence of injective analytic functions that converge is normally to a function, then the limit function is either constant or it is also injective.

But since the limit function is not constant Hurvitz's theorem will tell you that f0 is also injective that is 1-1 or univalent in other words ok. So that is because of Huruvitz's theorem. Then the other thing is that you know since modf is always a less than 1 for all f in script f , so that will tell you the you know of course the subsequence is also from there.

So modfnk is always less than 1, so that will tell you that mod f will also be less than 1 ok, so here I mean modf0 will also be less than 1 ok. and why is this true because you know of course if all these functions are bounded by 1 therefore the limit functional also be bonded by one ok, but I want to see it strictly bounded by 1, I want to say the limit function cannot take the model of limit function cannot be equal to 1.

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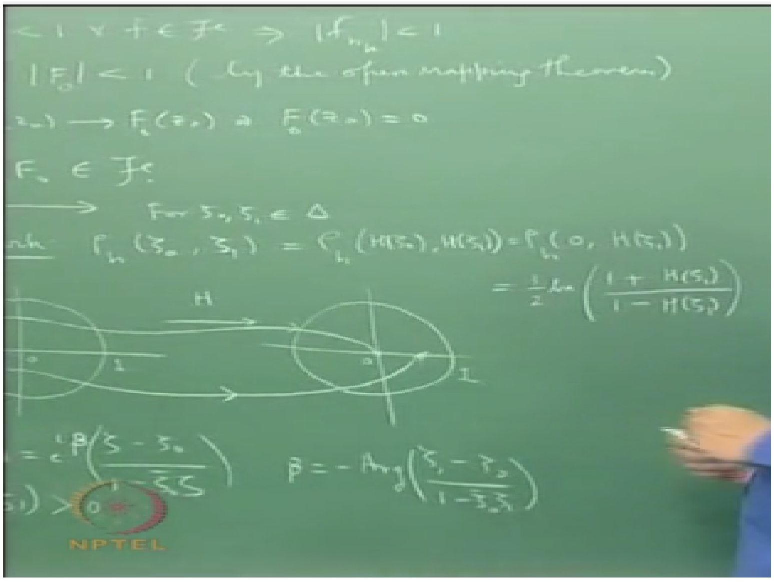

And why is that true because if it is equal to 1 that means there is some point in d where f0 take a value which lies on the unit circle ok, but then there is this open mapping theorem which tells you that the non constant analytic functions always the image of any open set with the image of any open set under any non-constant analytic function is an open set.

That means what is another way to say the other way to say it as it an analytic the values at analytic function take, there are interior values, the analytic function cannot take a boundary value, see what is open mapping theorem says, it says you take you apply you take an open set and apply the analytic function to get an image set. That image set is open that means each of the values it takes is surrounded by a disc full of which are adjusting the values of a function.

So it can take a boundary value, so if if you can also think of the open mapping theorem saying that an analytic function can take a boundary value. So this f0 cannot take the boundary value 1 because you know if it take boundary values 1 ok then it will also take values outside, little outside the unit disc ok, if it takes the boundary value one ok, if it takes a value on the unit circle ok.

The boundary which is the boundary of the unit disk, then there it will also take all values in a neighbourhood surrounding that point on the unit circle and that neighbourhood that part of that neighbourhood will be outside the unit disc ok and therefore it will take some values outside unit is ok. So that you will get points you will get points in d where the limit function f F0 is taking values outside the unit disc.

But it is the limit of functions all of which are taking values only inside the unit disc and that you give your quantity ok, the limit of a sequence of values which line in the unit disc cannot converts to a values which is outside the unit disc. Ok so that contradiction will tell you that f0 also is strictly bounded by one in modulus ok and so used by open mapping theorem.

And so and what so, so the moral of the story and of course you know if you take fnk of z0 is zero by definition and a distance to f0 of z0, if you tell you that f0 of z0 is also zero. So all these observations tell you that this f0 is actually an element of the family script, so f0 is an inductive analytic function from the given simply connected domain d which is not the whole complex plain.

Taking values in the unit disc and taking z0 to 0 ok, so f0 so in other words you see you have this supermom A, the supermom A is attained by a function, that function is also in this family , that is what we have to do, ok the supermom A is attained by a function which is also in the same family alright, on which the supermom is defined right. So this f0 is in f, so this is the fact that we need ok. That is an observation that we need to make.

Then now comes now I will have to tell you that you know f0 takes d on to unit. So I mean the moment I say this it means that you know f0 is already injective, so it is an injective and you know an injective holomorphic or analytic map is an isomorphic on to its image. Therefore this once you prove this you have already prove that f0 is an isomorphism of D on to the unit disc.

Ok which is the purpose of the Riemann mapping theorem alright, so now you know so the fact that f0 the image of d0 fills out the whole unit disc that is the fact which requires the use of hyperbolic geometry ok, which is what we are going to look at, so what I am going to do is so what I am going to do is I am going to break up at this point and say few things ok. The first thing I want to say is that you know.

So we go to we change our focus for a little little amount of time and look at go back to (()) (14:34) ok, so the first thing I want to say is so here is that, so here is a remark, so the remark is you know that if you take a point if you take 2 points to the unit disc, so I think I maybe I should not use z0, because z0 already use, let me use something else zeta 0 and zeta1 ok.

What is rho os h, roh up h is the hyperbolic distance, this distance between these two points in the unit disc for the hyperbolic metric ok and you know that is the arc length of the arc join these two points on the circle passing through these two points and orthogonal to the unit set ok. Because that is the geodesic for the hyperbolic geometry, the path of shortest length ok and of course if one of these points lies .

If one of these points is the origin or if both of them lie on a diameter then that geodesic turns out to be just a straight line segment ok and otherwise it is not a straight line it is always a curve path, it will be an arc of a circle which it passes through these two points and which is perpendicular to the unit circle which is the boundary of the open unit disc right. Now what is the formula for this.

The formula for this is you see you know you know given any two points in the unit disc you know you can always find a mobiles transformation and automation of some disc that maps one of them to 0 and maps the other one onto if you want a point on the real axis ok. You can always do that ok, so you know you know the situations like this I have.

This is my unit disc and you have I have 2 points z0 and zeta 0 and let say zeta 1 ok, then you know I can write this mobius transformation if you want let me give it a name H ok, and you know I can find a mobius transformation which will map the init disc to back to the union disc, that is it will be automatic of some unit disc and it will map the point zeta 0 to 0.

I can make zeta 0 go to 0 and I can make zeta1 to a point on the real axis, I can do that ok, this can always be done how because you know if you are just define H of z H(zeta) to d you know if you put zeta-zeta0/1-zeta0/zeta ok. This will map zeta0 to zero and this is certainly of the form of the general form of an automorphism of the unit disc ok. And then the only thing is when I put zeta1 I will get Hz1 zeta1-zeta0/1-zeta0/zeta1, that may not.

That is the point in the unit disc it may not lie on the positive real axis, but you know if I whatever is argument is if I multiply by you know e power i beta ok, so that beta is the negative of the argument of this when I substitute zeta=z1, then h (z1) will lie on the real axis. So you know so this is something that this is an adjustment we can always do, so what we will end with is take this so you know H(z)=e power i beta, beta times.

This you put this ok, if you multiply by e power i beta it is still an atom of the unit disc because e power i beta is just rotation about the earth ok and you know general automobile disc looks like this, that was something that we have seen already ok, where you put beta to be negative of principal argument of you substitute zeta1 without the e power i beta term ok.

So now if you calculate H(zeta1) and you calculate the argument of a H(zeta) arguments of H(zeta1) will be argument of beta and argument of e power i beta which is beta+this ok. So you will get zero. So the fact that is principle argument is 0 tells you that it is on the real axis, so it is a positive real one, it is a fraction and you should say unit disc, so it is a fraction. So you can do this. Now but what is this you know what is advantage of doing this.

See the advantage of doing this is you know that the hyperbolic you know you know that the automation for the unit disc are isometries for the hyperbolic every for the distance given by the hyperbolic metric the every autumn of the unit disc is an isometric ok in fact in a way the you know the original version I mean the (()) (20:15) for example says that that you know **if** if you have an analytic function which maps unit disc in the unit disc.

If it is not an atom of them then it will be a strict contraction in terms of hyperbolic metric. Otherwise it will be it will preserve the hyperbolic metric and it will be an atom of disc ok. in this case this H is an automatic distance so it preserve the hyperbolic metric, so this si the same as hyperbolic distance between H(zeta0) and H(zeta1) and this is just the hyperbolic distance between 0 because H(zeta0 f0) and H zeta1 is whatever it is.

Of course it is a real number ok and you know we derive a formula for this is, this is half lawn of half lawn of 1-hzeta1/1+Hzeta I mean 1+Hzeta1/1-hzeta ok. So this is the formula for the hyperbolic distance between any two points a unit disc, because this is something we already calculated. If you had a real number R here ok, here R is a fraction you will get half lawn 1+R/1-R right.

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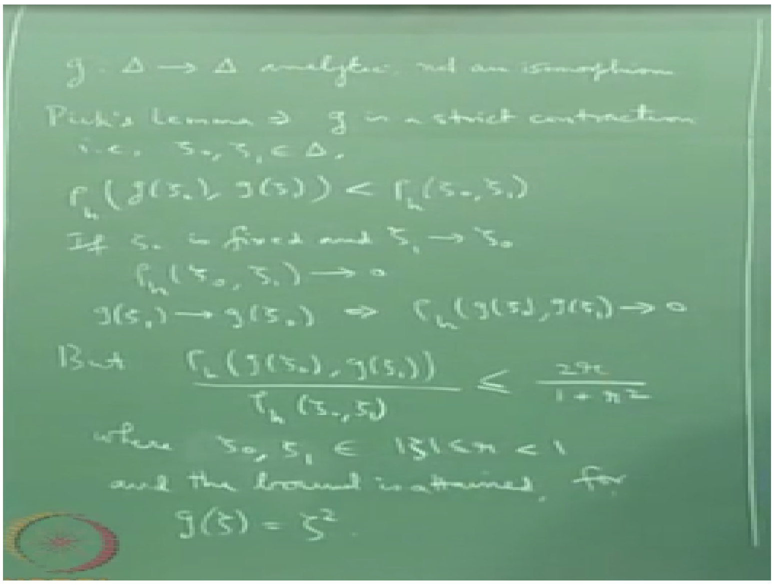

That is just the radial distance from the point from the origin to the point with point lying on the real axis with coordinator ok. So this is a calculation we have already done, and need you to remember this. Then should take analytic or not a nice not anything take a function take an antique map from user and display the disk is not an isomorphism ok, take a function take an analytic map from unit disc.

The unit disc which is not an isomorphism alright take such a map, now what this will tell you is that you know I mean what does pick's Lemma tells you that g is a strict contraction with respect to the hyperbolic metric ok. So that is what pick's Lemma tells you alright, what does it mean, it means that if you take 2 points z0 and z1 in the unit disc ok.

You apply the points 2g to take the other that is you take their images g and you apply, you calculate the hyperbolic distance ok, the distance between the image point will be strictly less than the distance between the original points. So it will be strictly less than the hyperbolic distance between the original points, so this will happen, this is pick's Lemma, that any analytic map self map of the unit disc which is not isomorphism, which is not an automorphism.

It has to contract ok you take 2 points and then you take their images the difference between the images and the hyperbolic metric is smaller than strictly smaller than the original distance this is the original points centre under the hyperbolic metric ok, if you know if z0 is fixed and z1xz0 ok what will happen you see zeta1 has a point comes closer and closer the obviously distance between the two points will come closer and closer and tend to 0.

So what will happen is that the distance the hyperbolic distance between zeta0 and zeta1 that tends to 0 ok and zeta1*zeta0 g(z1) will also tend to g(z0) because of continuity of g beta, g(zeta1) tends to g(zeta0) will imply that the hyperbolic distance been g(zeta0) and g(zeta1) will also tend to 0 ok. So I am just stating the obvious thing that as the points come closer their images also come closer ok and both sides will tend to 0 ok.

See if 2 quantities go to 0 ok it is not necessary that their ratio goes to 0 ok, it can happen that the ration can go to infinity or it will go to zero or it can go to some finite value. So these 2 go to 0, but the fact is that if you take the ratio rho should take the ratio of the distance between the images and the distance between the two given point ok, this quantity can be approximated by the following number 2r/1+rsquared ok.

Where zeta 0 zeta1 belong to modz less than or equal to small r which is strictly less than 1, so in fact I want to say in fact I do not you want to say is I want to say that this is always less than or equal to ok, this is a bounded quantity and this is the bound ok and this bound is and in fact this bound is even attend ok, and the bound is attend ok, so there are you can find points zeta0 and zeta 1 in this disc for which the ratio of the distance is exactly equal to this.

So that the bound is also attained, so far so this is very important this is not for any g, this is for g zeta-zeta squared, this si for this, this is for this square function ok, so I mean if g is any analytic self map of from the unit disc to unit disc which is not an isomorphism it is a contraction ok and the point of both sides is both sides goes to zero as the points approach each other ok.

But if you take the ratio, the ratio is 0/0 form ok and a 0/0 form can behave in any way that you want, but it will go to z because the numerator is 50 less than the denominator alright, this is sticky less than this ok. So you can expect it to go to 0 alright, but then you can get away question for what this ratio is a for zeta 1 very close to zeta0 ok and in fact if you take both zeta0 and zeta1 in this close disc ok.

Then this is bounded by this number 2r/1+R squared where than that is for the particular case of the function g which is zeta square, ok the square function. The square function is of course a analytic map from the unit disc to unit disc, but you know it is not an isomorphism because it is not injective, because you have 2 square roots going to the same it is 2-1 in a deleted I mean in the deleted disc.

I mean from the disc you remove the origin then both z and –z will go to the same value, so it 2-1 map alright, so this is not an injective map, so it si not a holomorphic isomorphism, therefore by pick's Lemma it will be a contraction, but the point is when you calculate this ration for zeta1 and zeta0 inside this close disc, this is the bound for this ration ok, but the question is how do you get this number ok.

For that you know you have this point zeta0 and zeta1 you can move you can move zeta 1 to the origin, you move zeta1 to this arc by using map H like this ok and then you calculate now you calculate this ration, you calculate this ration this ratio will be roh h of you know this ratio it will be roh h of 0r square/rho h of 0r ok and divided by Rohit zero Hour I am calculating it for I am simply calculating it for the for these two points.

And there the effect of the function is zeta going to Szeta squared for those two points ok instead of calculating for 2 general point zeta1 and zeta0 because I can move them to these points alright. So you know for example of you do this calculation ok you will get half lawn you will get half lawn of 1+rsquared/1-rsquare/half lawn 1+R/1-R ok. This is what it will be alright and then you know if you try to apply.

If you try to study it is R tends to zero ok this is a you know R tends to 0 this is lawn this is lawn 1, this is the 0/0 form, so you can apply (()) (30:16) will tell you that you can differentiate both the numerator and denominator ok, it is a zero by zero form, if you take limit as R tends to 1 I mean R tends to 0 ok, it is a it is an infinitesimal mean it is 0/0 indeterminate form.

So if you apply (()) (30:39) this can be what you will get is you know if you calculate it you will get 2R/1+R squared just apply rule for R very very small ok, and that is very simple you just differentiate the numerator, and then you differentiate the denominator and divided you will get this equation. So so the moral of the story is that I need to know how this quantity looks like you know when the points zeta 1 comes very close to the point zeta 0.

I need that fact alright and in particular of course you know you must remember that this is less than 1 resetting a less than one because if you cross multiply and move things to other side I will get 1-r the whole square which is strictly greater than 0 ok. So this I mean the fact that this is less than 1 is this statement that I mean it again reinforces a fact that it is a contraction is a strict contraction alright.

Now anyway actually probably even this expression is not so important for me I want to say that there is a constant ok this ratio is less than equal to a constant which is less than 1 ok I need that fact for the square function right. Now what you do is you now we do this very nice Lemma so here is the lemma which is kind of and student call Lemma in fact is that it is more than a theorem because it uses lot of stuff ok and which is it is a critical Lemma that we need to complete the proof Riemann mapping theorem.

So the Lemma as a following, the Lemma is if let me use d0 in delta is a simply connected domain such that domain such that D0 is not equal to whole unit disc, suppose you have simply connected domain into unit disc which is not the whole unit disc ok and if zero is a point of D0 then there exist there exist and an analytic map or holomorphic map size from D0 to delta.

Such that shy take 0-0 and the derivative of shy at the origin has smallest greater than 1 ok, so this is a other technical Lemma ok, it is a rather technical Lemma which uses hyperbolic geometry, but this is what we need for completing the proof the Riemann mapping theorem. So what this tells is you see the situation is like this situation is I have this I have this unit disc I will draw a bigger ones.

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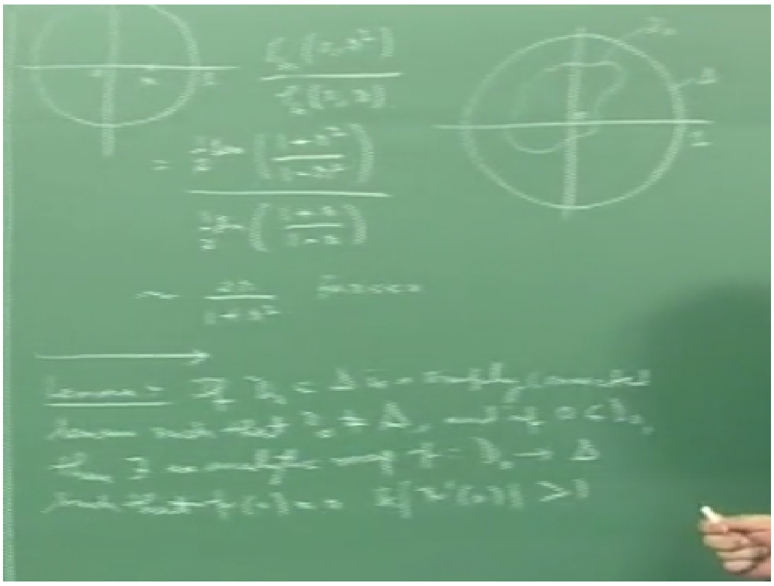

So that I can fill in with other things, so and I have a domain D0 which is not equal to the whole unit disk ok and such that it contains the origin ok, then what this lemma says is that I can find you know here holomorphic map that maps t0 again into another domain in the unit disc ok such that 0 will go to 0 alright, but the derivative at 0 can be made greater than 1 ok.

So this is like you know you see the point I want you, I mean the way I wanted to think about it is like this, see what is go back to our Schwarz Lemma ok, go back to Schwarz Lemma, go back to actually infinitesimal version of Schwarz Lemma, so what does it says from if you are having an analytic map from the unit disc to the unit disc ok, then the derivative at the origin in modulus has to be bounded above by 1.

The derivative at the origin cannot exist 1, it has to less than or equal to 1 and it will be equal to 1 exactly when this is an automorphism, if is strictly less than 1 it is certainly not an automorphism ok, now that that Lemma has this si like you know the effect of statements similar to that Lemma not for maps from the unit disc to the unit disc, but for map so much smaller simply connected sub domain of the unit disc to the unit disc.

So what that Lemma says is that if you are map is from the unit disc to the unit disc then the derivative at the origin modulus is bounded above by 1, but if you try to map a smaller domain a smaller simply connect domain to the unit disc by an analytic map ok, you can always succeed in exceeding you can always succeed in exceeding that bound for the derivative at the origin ok.

This will not happen if D0=delta, if D0=delta then Lemma the version of lemma will tell you that the derivative is less than equal to 1 and in fact if it is not a an automorphism it will be strictly less than 1, you can never exceed 1, and you will get 1 only when it an automorphism ok, but the movement you have smaller simply connected domain you can manage to find the analytic map which whose derivative at the origin modulus is greater than 1.

This is like you know this is this tells you what you can this tells you about analytic maps from a simply connected proper sub domain of the unit disc to the unit disc, it always tells you that you can get you can always find one thing that that you know that is opposite to what you will get for the short Lemma had D0 been in the whole units right, so you should see it in that you should see it in that point of view ok.

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Continued in Lecture 42 Part B