Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-39
Completion of the Proof of the Arzela-Ascoli Theorem and Introduction to Montels Theorem

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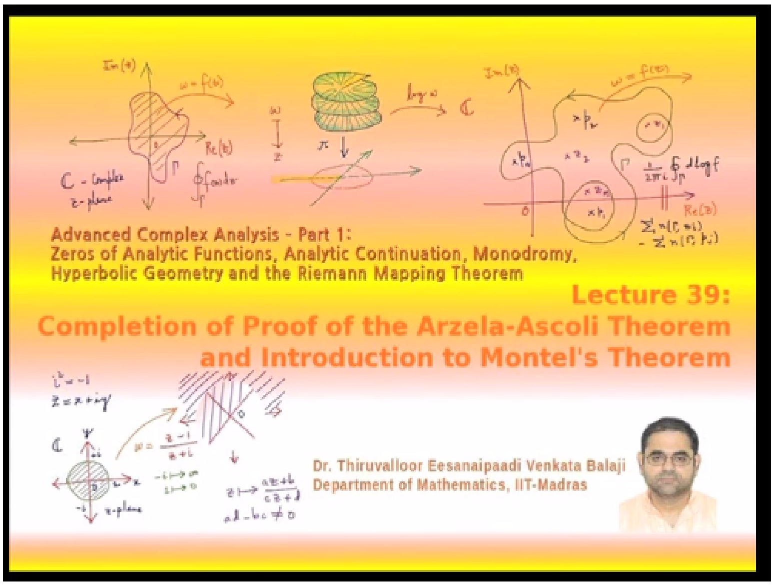

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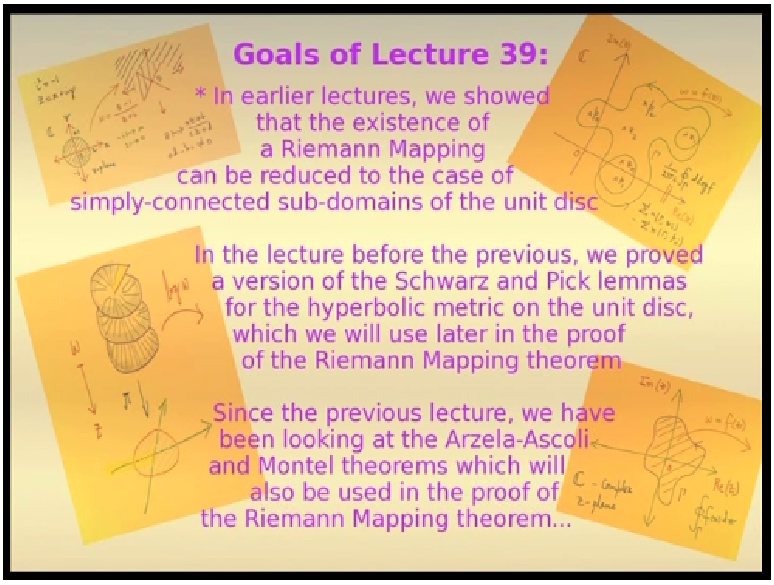

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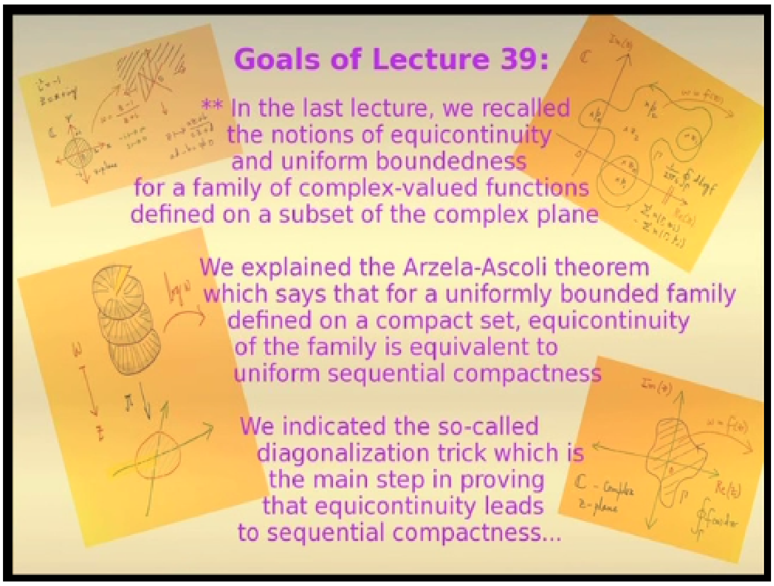

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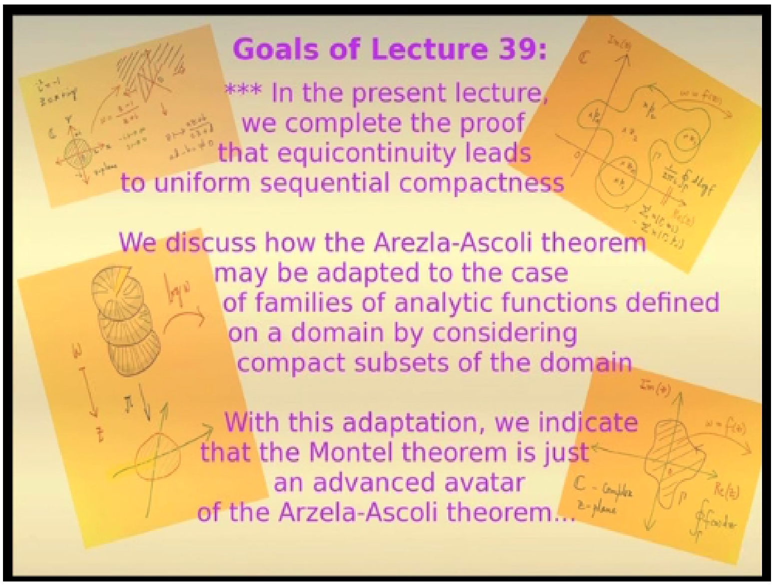

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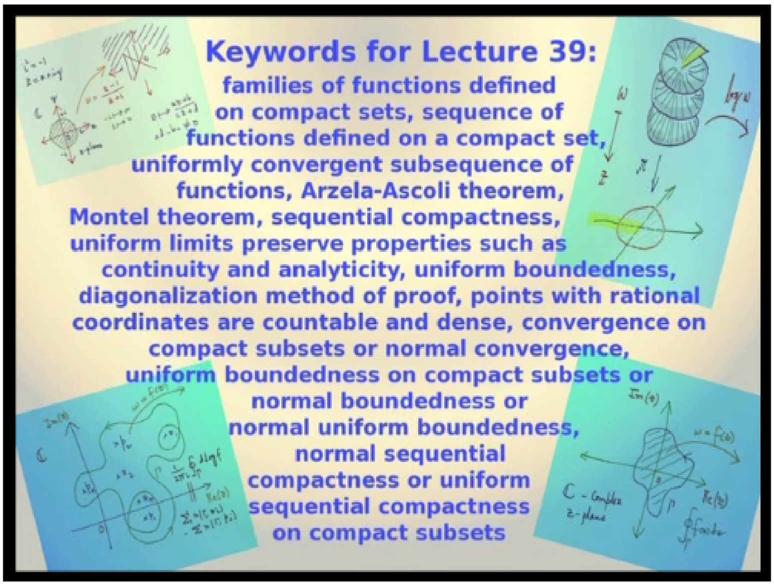

So **so** we are continuing with the proof of the Arzela-Ascoli theorem alright, what you have done is we have a set E which is a compact set in the plane alright and we have this family script f given to us, so family of continuous complex valued functions . We assume that this family is uniformly bounded right, and you further assume that these families it will continues at every point.

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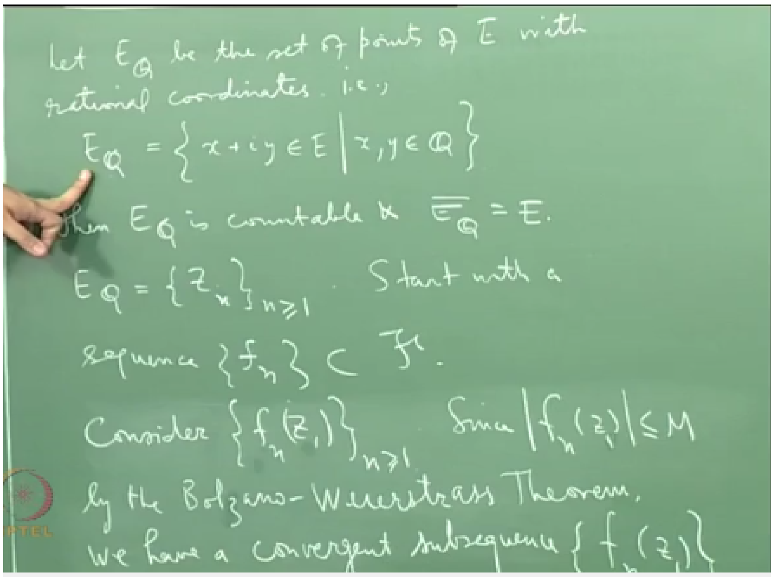

And we are trying to show that given any sequence in this family there is a uniformly convergent success ok. So we took the points the rational points of E ok the points of E with rational coordinates ok namely point of E which are complex numbers whose both real and imaginary parts or rational numbers. That is called the building subsidiary and we have been able to extract using diagonal ration trick.

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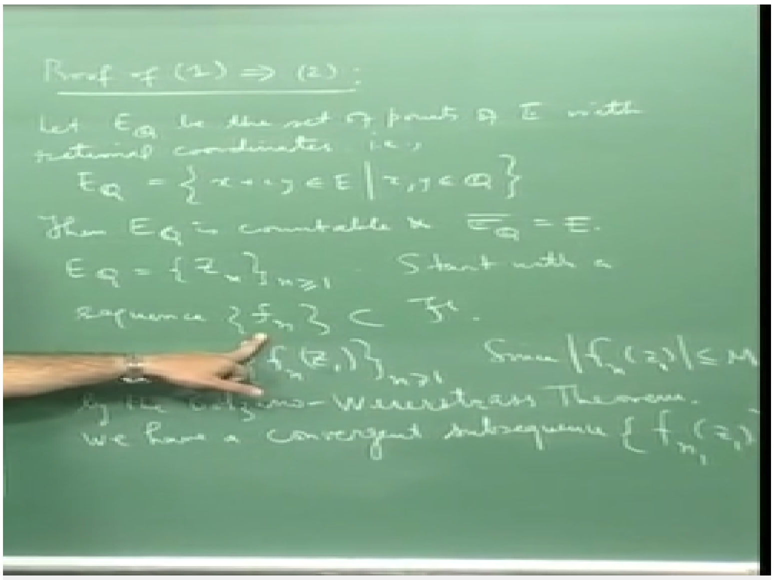

We are able to extract the sequence F, subsequence of the given sequence fn ok with the property that this sequence the subsequence converges point wise and every at every rational point of view ok. That is what we have proceeded up to so far alright. Now what I am going to do, now is the time I will have to use see so far what I have used is only the I just use the fact that the family is uniformly bounded.

I just use the fact the families uniformly bounded and I have not use anything else whereas I still have to uses the equal continuity of the family and I will have to use the fact that everything is happening on the set E which is compact, I have never use the compact itself right, you are going to use that. So how do you proceed, so we do the following thing, so note that so we have a quick continuity of the family script F.

And therefore this holds for any collection of functions in the family. So it also holds for this Fs ok. So let me write that down since F is equity continues on E so is f sub shell ok, after all this is just this is also the sequence in script m, in fact this is a subsequence of the given sequence a Fn let me start with right. So so given z0E ok. So you have this definition of equity continuity.

What it tells you is that suppose if an epsilon start with start with an Epsilon given to me ok, go given is not E and epsilon greater than 0 ok, there is delta greater than zero with the delta depending only on z0 and epsilon such that 12 the distance between you know so I should I guess you let me write it very generally distance between F of z and z0 is less than epsilon.

Whenever the z and z0 is less than delta ok, and here I can put for smallest any number here in fact I can put it for any member in script ok for all smallest in script in particular I can take for smallest any of the capital Fl also. This is just equicontinuity of the family script that I was seen and I am using that. Now what you do is you know for every for specific epsilon fix is Epsilon then you know for every point is iz0 .

So I am varying the z0, then I get this disk centre at z0 radius delta, so that 2 aspect first thing as you vary the z0 ok, then of course the epsilon is fixed alright, but you vary the z0s, then you get these discs, the deltas will also vary because the delta depend on z0 also. But then you will get 1 disc surrounding each Z0 E ok and that will give you an open cover of ok.

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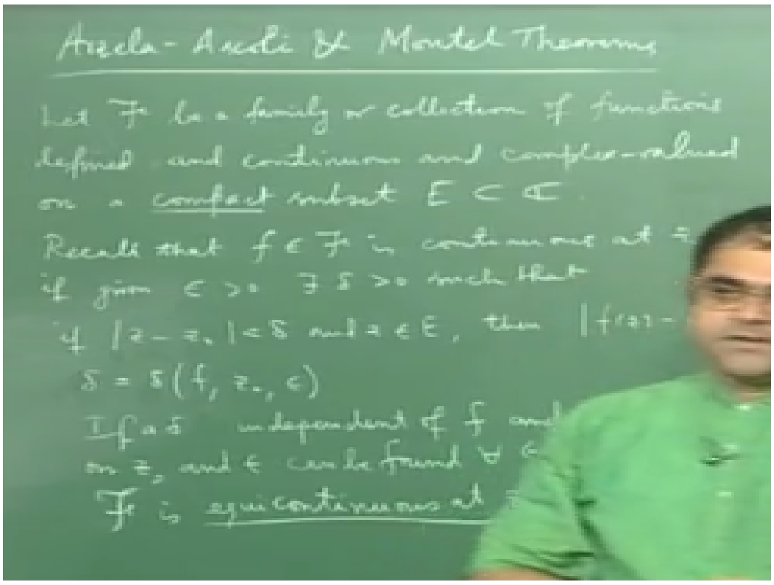

And now use the fact that E is compact to extract the finites of cover. So that is one part of the argument. So let me write that down the collection modz-z0 lesser than delta, epsilon as z0 varies over E is an open cover of E of E which is compact. So her e I am using the compact as a E and I am using the E I am also using the equicontinuity of them ok, so admits a finites of.

So you know you will get so the sub cover will begin by modz-z1 less than delta z1, epsilon or z-z2 is less than delta of z2, epsilon and so on modz-zk less than delta zn, epsilon. So I get this set right and of course you know I do not know all I know about zt is that is compact, I do not know about anything about its interior and so on so for, therefore each of this zs and each of this discs you need not .

I mean you need not all this contains z ok, but it could bigger than alright, so here also whenever I say modz-z0 is less than delta I am only looking at those z for which lie in E ok, I am not looking at z in this disc which does not which do not lie in E because then z does not lying in E F(z) does not make sense ok because all the functions in a family define E on E ok.

So whenever I write things like this I am assuming that the variable is also any right, so you have this word and let me call it zeta1 because I could get into trouble zetak, so it get such, so you know I have so I use z I use different symbol, so that I do not confuse them with zi which are supposed to be all the points E ok. So you have this cover and alright and then you see in each of in each of these things in each of these discs alright.

Of course I will be able to find point of view I can find one point of Equation ok, I can find one point of Equation in each of this disc ok. So choose zi1 in this disc 1z-zeta1 is less than delta theta1, epsilon similarly zi2 in z-theta2 less than delta of theta epsilon and so on. This Eqzi that any of them, so choose one point in each of this discs with zetak ok.

Where of course you know the zij or in Equation each has been eliminated as a zi, zn n ok. So you choose one rational point from each of this discs ok, I have finitely many discs it cover E ok and in each of this disc you choose a rational alright choose a rational. Now what we do is that see what you should now know to is it if you give me any point of E that point has to lie in one of these ok.

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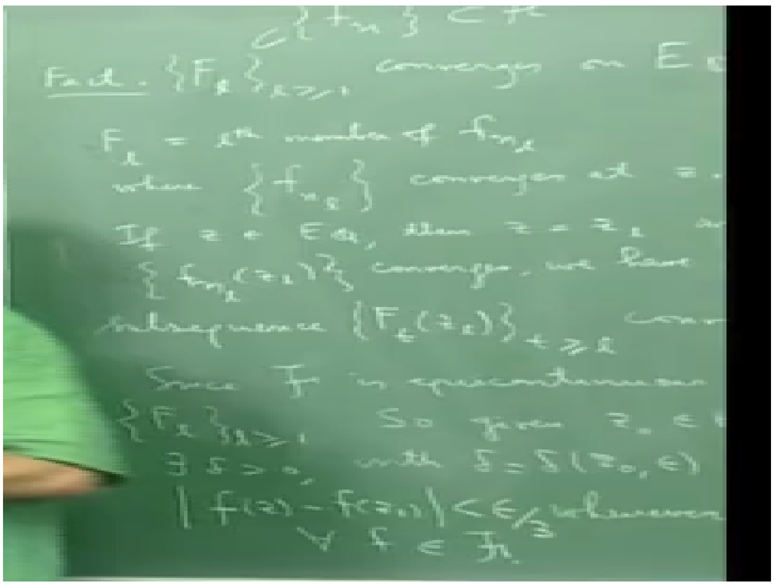

And corresponding to that you have a certain zij ok, so you know so that is how I actually extend the all my z to for example convergence to any point of E, ok so you know you are if you want if you think of so so if z belongs E then z is then z-zeta I let me use some quotation then not z-theta is less than delta zeta some l, epsilon for some l with 1 less than l less than k.

This will happen ok and zil is also is also in modz-z zetal less than delta etal, epsilon ok. So I have right now you know basically what I am trying to show I am trying to show that I am my aim is you know starting with a sequence fn which I have pick in this family F for which I extracted this subsequence with converges on the rational points.

I am trying to actually show that this subsequence converge is uniformly not just converges but converge is uniformly and that to not only at the rational points, but and all of I am trying to achieve both in one step. So here what I have is that this subsequence convert only on the rational points ok and convergence on the rational points means point rational it does not uniform alright.

But what I am trying to show is that not only does it converge at the rational points I am trying to say converges everywhere and not only I am trying this converges everywhere I am trying to say converges everywhere uniform ok, that is what I am trying to do. So how do I check that it converges uniformly. So you know I will have to just check that you know I just have to check that at any by substitute any z here in this sequence of functions.

I get the sequence of complex numbers complex values I have to show that sequences uniformly cause ok I have to show that that is uniformly cause right. If I show that then I will get uniform convergence of that sequence. So what I will have to do is it I will have to compare fm of z-Fn(f) for mn sufficient, this what I have to compare right. Now what you use is you use the fact that you see this z is in this which contains the point zl alright.

And which also contains zil, so what I do is I add and subtract the function value set these 2 point, so what I do is I write as Fmz-Fm of you know if you want zeta l then +Fm(zl) and do the effect of adding it then I put -fn of zl –Fn, now introducing the point zil then I end you by adding Fn of zil and then I write the final Fn(z) then I use triangle inequalities and group these use training to get modFm(z)-Fm(zl)+Fm(zl)-Fn(Zil)+ModFm(zil)-Fn ok I use alright.

So you know I am so basically the so the basically the diagram is like this, so here is if you want this is E ok and there is this point zeta l and also that is this point zeta l which is the point of E ok and there is this disc centre at zeta l radius delta ok and this intersect E where this intersects E, there is that is where I have chosen my point is z ok, my chosen point is there plus this also contains one point of the set of rational coordinates which have taken as zi.

So this is zil ok, this is my situation right, so so this is and this is nothing zeta, so this is my right now I mean basically if I show that for M and then sufficiently large if I can make this less than Epsilon independent of Z ok. Then that is enough to show that the sequence F of a sequence of capital (FL), they converge uniform alright. So that is my aim, my aim is you want to find above beyond for which for all value of Mn.

And beyond which for all value of z ok, I can make this less than epsilon ok, so what do I, how do I do that. So you know the so the first thing that you must know z that the first expression in the last expression are the difference in the function values the same function involving the same function. So this involves Fm and this involve fm alright and of course all these Fs are all part of the family script F which has equicontinuity alright.

So I will have to make an adjustments, so that you know this term works out lesser than epsilon by 3 this terms works out less than epsilon 3 and this term also works with less than epsilon by 3. So that when I add all 3 it works out less than epsilon is what I want, so how do I make this and this epsilon by 3 it is very simple because I will use I use equicontinuity in the neighbourhood of zeta l.

This delta of zeta(l), epsilon such that Fz-F zeta l in modules can be made less than epsilon ok, but I can choose that epsilon to be less than I can make epsilon epsilon/by, so what I do is that you know here I do I make the I go back and make this change, so what I do given epsilon greater than 0 alright, you choose a delta alright and this depends only on z0 and epsilon and not on the function F in the family.

Such that this is not less than epsilon, but let macro environment make it as epsilon/3 ok. Suppose you do this right then would I be done, sense of taking the deltas you make the change of taking delta by 2 ok, but still continues to be an open cover, alright and so I will have to change delta/2 everywhere ok, so I get the finite sub cover and then I choose one rational point in each of these finitely meaning open disc which cover E right.

And then I note that z is a point of E then this is going to lie in one of the one of the sets because they cover E ok, so it lies in one of the one of disc I choose that one and call centre as interest of l alright and I also have and I can also get a rational point of E in that set ok right and now I write this, but when I write this expression what I do is I do not I **try** try not to bring in this centre ok.

But I just work with this rational this point with rational coordinate that I have chosen, so in subset I put zil here and also zil here ok. So then tell then I am in good z so see again zil here I get zil here and now I have the same zil it is the centre term. So you know if I put this z and zi l that will be less than that will be less than delta by triangle equality because you know z-zil is going to be less than or equal to modz-theta il+mod of eta il-ail.

And which is less than delta of theta l, epsilon each one is less than half delta right, and then I get this equality and I get this and of course this si this is zil ok and then if I look at the third one and comparing zil and z the function values at these points of the function fn. So again I do not mezup with zeta l but I compare z ok and again I have the same in same equality.

And that will tell me that that will tell me that this is still epsilon by 3 ok. So I am in good save, so I have to only worry about this fellow and here is where I use the fact that the F the subsequent of F will converts the every rational point ok. So since the (FL) converts so that could be Fn converts on EQ and you know that zil is the point of EQ for the given epsilon there exist n which will depend on you know epsilon and it depend of epsilon and it also depend on zil.

Such that the distance between the distance between F m at zil and Fn at zil can be made less than Epsilon for M and n greater than equal to this n of epsilon, zil ok. This is just the cos nature of this sequence Fm of zil because Fm of zil converge such cosin so, this si the question condition ok and but then I get all these ns alright and the you know now if I start with any Z ok this for that z this l may change ok.

I I start with if I start with any Ze this index L that comes in all this that will depends on Z ok but there are only finitely many ls, the L can only range from 1 to k ok and therefore there are only finitely many such numbers capital Fs ok and so if I take M and N to be larger than the maximum of all these, then the models of this is always less than its independent of z and that gives me uniform convergence of sequence of capitalist S ok.

And that finishes proof ok, so for N, M greater than equal to maximum of all these N epsilon, zil 1 less than or equal to l less than k we have modFm of z-FM of –z is less than epsilon ok independence of z of ZE and this independence of ZE tells you that the sequence of F converges uniformly one. So this implies FM converges uniformly with this. This is yeah yeah you are right.

I want you are right all the three should add up to Epsilon is already epsilon which is epsilon by 3. So this one has to be also adjusted to be epsilon by 3, so I put epsilon by 3, so you choose n, so that instead of getting this less than epsilon you get epsilon yeah thank you for pointing that out, only then you will get the, sum less than epsilon otherwise you will get something more.

So the moral of the story is so you basically the game of adjusting is Epsilon and delta carefully alright and even if you go down the little you can I you know how to know what to do to adjust ok it is some in something this is epsilon, delta adjustment is something that you always tell do in any analysis course alright. So the idea is therefore you know you use you use the uniform boundedness.

So you know let me some let us summarise what the proof did, you see to have this family scripted from that family you have taken a subsequence. And what do you want to show that this subsequent having from this family outing in a sequence and subsequence you can see and what you want to show is that it has a subsequence which converges uniform. So how do you get that subsequent what you do is you first get hold of subsequence which convergence on the national point ok. That uses uniform boundedness and the diagonolization trick ok.

Once you have a subsequence which converges point wise on the rational points then you use the compactness of E and equicontinuity to show that it converges uniform, not only on rational points but every point ok and what it means is therefore if you take the limit of this subsequence you will get a continuous function, you will certainly get a continuous function. So in particular you know this this applies to for example .

If you say say functional analytic functions in the interior of E ok then you will end up with analytic function in the interior of E, the limit function will also be analytical in the interior right. So so that is the so that is one way of the other theorem which is what we need ok there is the other statement in Arzeral theorem which says that if you have this property that given any sequence in the family F.

This is uniformly bounded, you are able to extract a uniformly convergent subsequence then you have to the other statement d the other statement with theorem will tell you that you have to show that this family is equal to and that is proof and I am going to ok it is a proof by contradiction similar to proof that could have seen in real analysis ok. So I am not going to going to do that.

But what I need to do is to Montel's theorem ok, so let me explain Montel's theorem. So let me explain Montel's theorem, so this is this is the theorem that we actually need alright and what is the theorem well so now you see now you want you want a statement for analytic function, so you must understand how to modify statement, you see when you are looking at analytic function.

You know analytic function that depend on any open set, they are not depend on close set. In particular you cannot think of them as being define as compact ok because analyticity at a point means it means that the whole neighbourhood as a point where the function is defined. So it has to be an interior point unless the point is an interior point of a domain of a function you cannot talk about this analytic either.

So I cannot simple talk about you know analytic function on a compact that does not make you any sense right. Therefore but therefore you know how will you adopt the theorem to the complex analysis that you know analytic function. So the idea is what you do is whatever conditions you put they should be connected in the theorem all conditions are conditions of functions on a compact right.

What are conditions you have they all conditional on a compact set, the hypothesis are all conditions of function on a compact set and the conclusion is also about functions on a compact set ok and how would you adopt it to complex analysis setting where you want to apply to analytic functions what you do is you do take analytic function on a domain but put all the conditions on compact subsets of the domain ok.

So you are working with analytic function but whatever continuity I mean whatever conditions you put tried to put them on compact subsets of the domain. So this is the so for normal conditions ok. For example if a sequence of functions converges on compact subsets is called normal convergence ok, if a sequence of function is uniformly bounded on compact subsets.

It said to be normal bounded ok, then the natural version of theorem for analytic function that you can expect is that if you have a families analytic function on a domain ok suppose they are normally bounded ok. That is they are bounded on compact set right. Then the equicontinuity is equivalent to normal convergence namely normal convergence of a subsequence of any given sequence ok.

So the statement will be that you have you know the family of analytic function on a domain ok and instead of assuming them to be uniformly bounded on the domain you will assume them only to uniformly bounded on compact subsets of the domain which will anyway follow actually of the because of the cost estimates just because of analysis. so you do not have to assume that.

But it is already there ok, then the Montel's theorem will tell you that if these functions are equal continuous if the family is equally continues then given any sequence in this family you can extract the subsequent which converges normally namely which converges on compact subsets ok and this version of the theorem this version of theorem is called Montel's theorem ok.

So you must understand that this is going from you know Azela Ascoli theorem to Montel's theorem in the Azela Ascoli theorem all conditions all hypothesis, all conclusions of functions defined on compact sets ok, but when you come to Montel's theorem you working with analytic function and an analytic function are not defending complex here. They are define basically they define only on open sets.

Therefore whatever conditions you put you do not put on the functions themselves on the whole domain which is open, but you put it on compact subsets so you changed everything to fit into compact subset ok, so instead of uniform bounded you will get normally bounded. So instead of saying they are uniformly bounded on the whole domain you will assume that they are uniformly bonded on compact subsets.

And instead of getting uniform convergence on the domain which is too strong which usually do not get, you will get uniform convergence on compact substation domain which is called normal domain and if you adapt it to this situation in this way when you get Montel's theorem and the proof is surprisingly Azela Ascoli theorem plus another application of diagonolization.

So already the Azela Ascoli theorem has used one level of diagonolization as we saw in the proof, but when you adapt it to compact analysis as Montel's theorem you have another or you have to do one more diagonolization where at each stage you have to apply Azela Ascoli theorem and that will give you model ok. So we will discuss that in the next lecture.