Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-33
Reducing Existence of Riemann Mappings to Hyperbolic Geometry of Sub-domains of the Unit Disc

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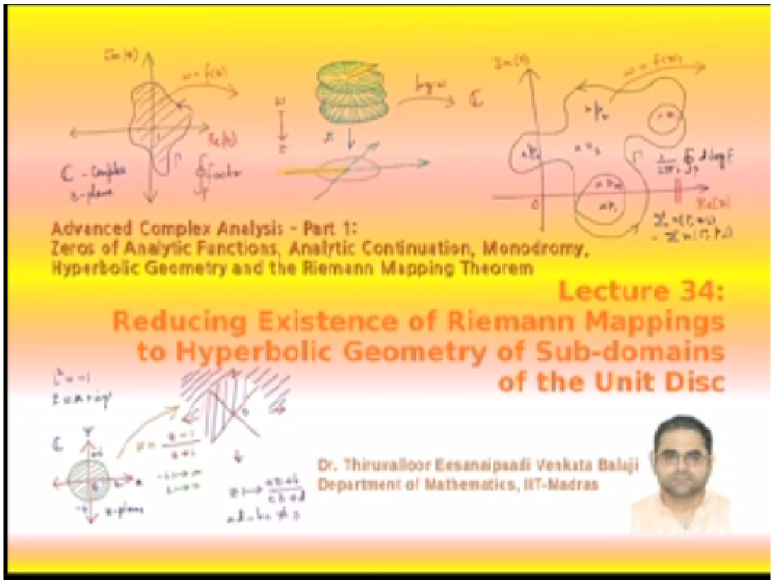

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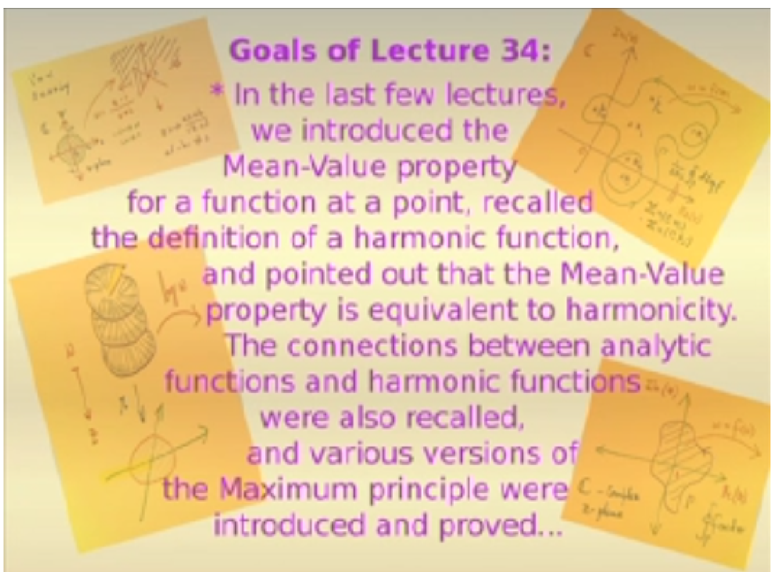

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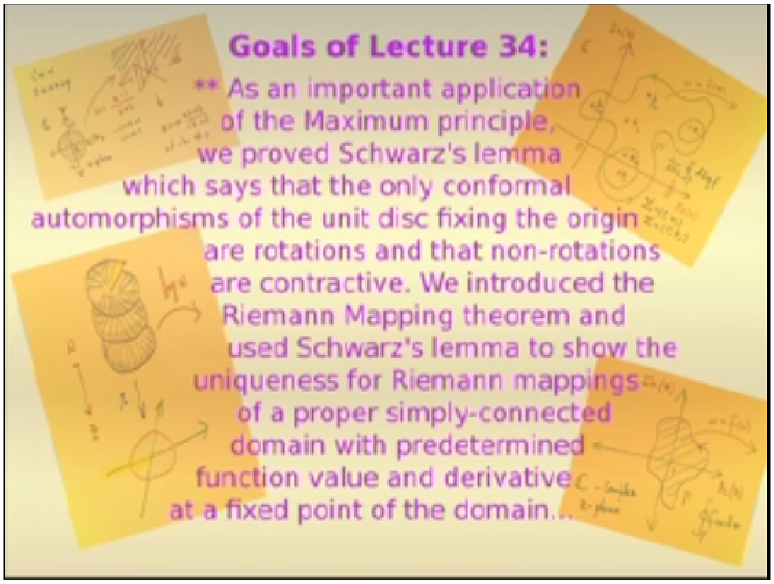

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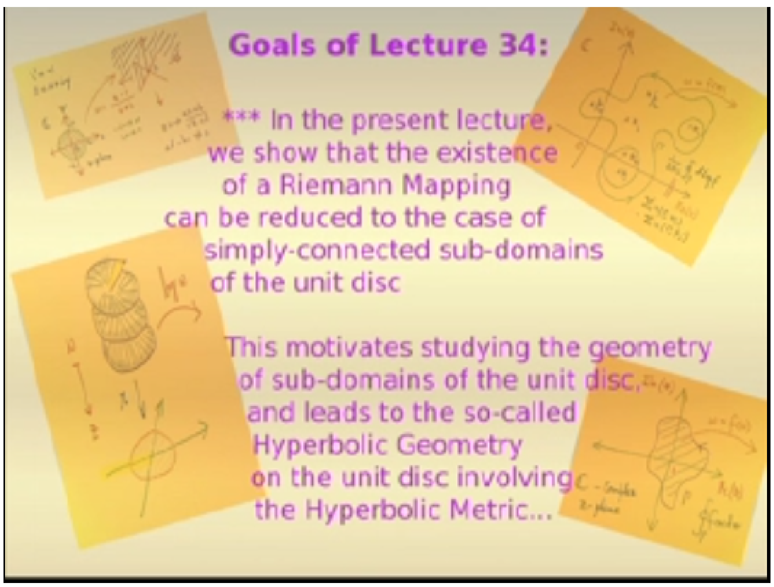

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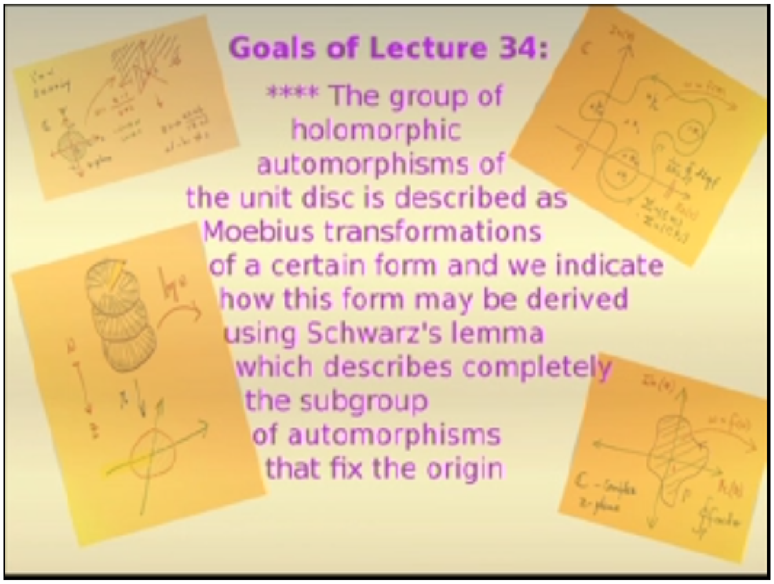

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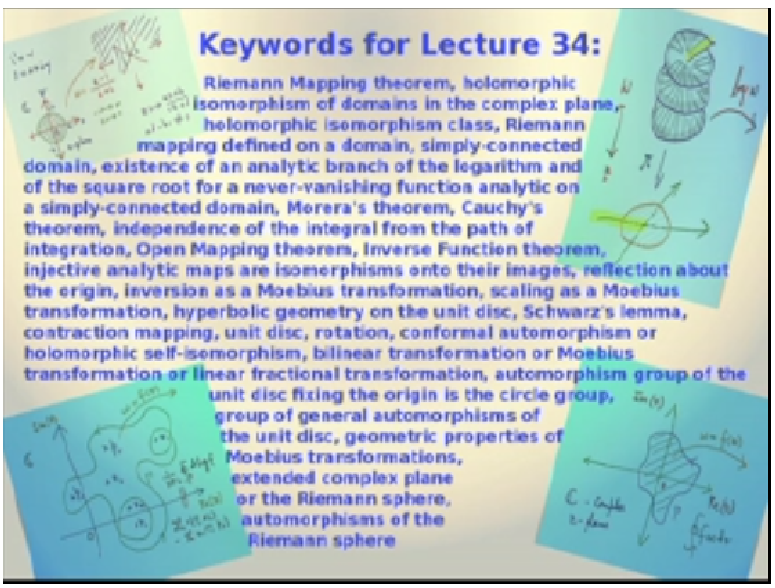

So, just to recall what we saw in the last lecture.

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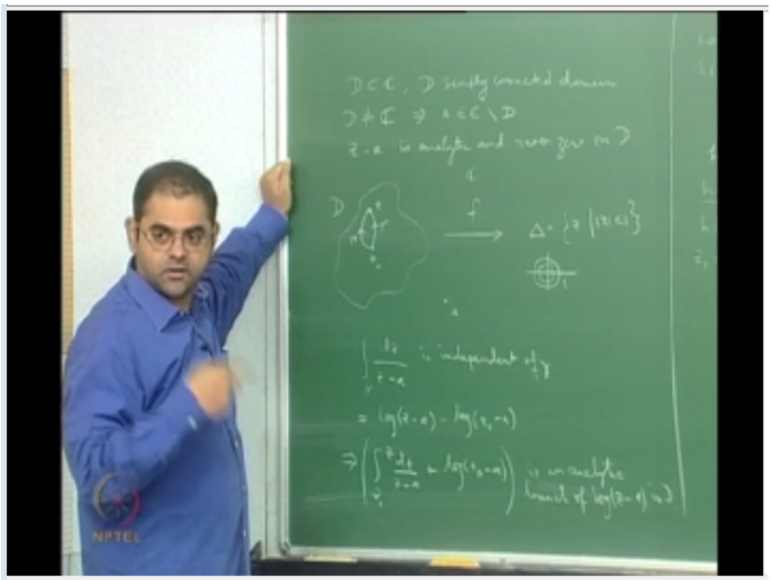

We have D inside the complex plane D is simply connected domain and of course d is not the complex plane okay. So, you know there is appoint a in the complex plane which is outside D okay. And if you look at the function z-a this is analytic and never 0 on D because a point a is outside D alright. And you know if you have a non-zero analytic function on a simply connected domain.

Then you can get a analytic branch of the log of that function okay so, you know so basically if you want now your domain I am just drawing a picture all the time know about the domain is that it is is simply connected and it is not the whole complex plane okay. But what I am drawing here is something that is bounded need not be like this right for example it could be the upper half plane right which is unbounded.

But I am drawing a I am drawing this diagram so, that you just for some motivation so, the point a is in the complex plane – we said D so, you know if I take the point z then I am looking at z-a which is translation by –a alright z-a f of z equal to z-a is just translation by –a. So, it is a moebius transformation actually okay. So, I translates this whole disc by –a alright .

But there is no what I want , what I want to say is that z-a as an analytic branch that is a analytic branch of the log of z-a here. And how do you I do that you see I take a point a fix a point z0 in D alright. And you know and you give me any other point z in D alright take any path gamma alright. Then you define you know if you integrate along gamma I am trying to find an analytic branch of log of z-a okay what is it is derivative, derivative of log of z-a is 1 by z-a alright.

So, the integral of that should give me the log right therefore I integrate over gamma okay dz by z-a okay when I do this. So, gamma is from z0 to z then what I will get is a logarithm of z-a okay and you know if I , I will get log see in fact what I will get is log z-a final point –log z-a initial point okay. So, in fact so, you know this is independent of gamma this integral is independent of the path okay.

And that is because of simple connectedness because you know if I instead of gamma if I put another take another path gamma prime which is also inside D alright. Then you know these simply connected then gamma can be the integral over gamma forward by –gamma prime which is a loop will be 0 for the function 1 by z-z. Because 1 by z-a will be analytic in D that is because z-a never vanishes in D.

And by Cauchy's theorem this integral will be 0 okay so, this is independent of gamma and this is what this is just going to be log of z-a –log of z0-a this is up the indefinite integral is log of z-a the upper limit is z the lower limit is z0 this is what you are going to get and therefore you know if I take this + this I will get an analytic branch of log of z-a. So, **so** in other words you know integral over gamma.

So, I will just write this integral over gamma as z0 to z dz by z-a+log of z0-a is an analytic branch of log of z-a in d okay is an analytic branch of log z-a in d. So, you see I have used you know I used Cauchy's theorem I have used if you want I have used morreres theorem alright. I have used everything here okay to say that the integral is independent of the path. I need integral over a closed loop is 0.

And integral over closed loop is 0 because the integrand is 1 by z-a is analytic z and that is because z-a never vanishes okay. And therefore the Cauchy's theorem integral over gamma is same as a integral over gamma prime and therefore the integral is independent of the path alright.

And then of course the derivative of this which follows from you know the proof morreres theorem that if you differentiate this as an indefinite integral.

You will get log of z z-a alright and therefore this log of z-a when I write log of z-a it is some branch of the logarithm which is not just continuous. But it is actually even analytic the analyticity comes from observing the proof of morreres theorem okay. So, what will happen is that this expression here will be an analytic branch of the logarithm. And that is the reason why I am writing log.

But I am not it is some branch you do not know what branch it is some branch right. But it is analytic that is the most important thing so, I have a analytic branch of the logarithm and once I a have analytic branch of the logarithm what I have is I can so, let me again tell you I used the fact that D the domain D is not the complex plane.

Because I have chosen a point outside the okay if the domain D is a whole complex plane I do not have a point outside it okay. I cannot do this so, I use the fact that the domain D is smaller than the complex plane. Then I use the fact that domain D is simply connected because if a domain is simply connected then any loop if you take any loop inside the domain. The region inside that loop will also be inside the domain.

Because the domain cannot have any holes okay and I need all the region inside this loop found by gamma and the reverse of gamma prime to be inside the domain. Because only then I can apply Cauchy's theorem to say that the integral over gamma is equal to the integral over gamma prime aright. Therefore the moral of that story is that I have use simply connectedness of the domain.

I use the fact that the domain I snot the whole complex plane I use both alright okay. So, now this is so, I have yeah I have an analytic branch of log z-a here alright.

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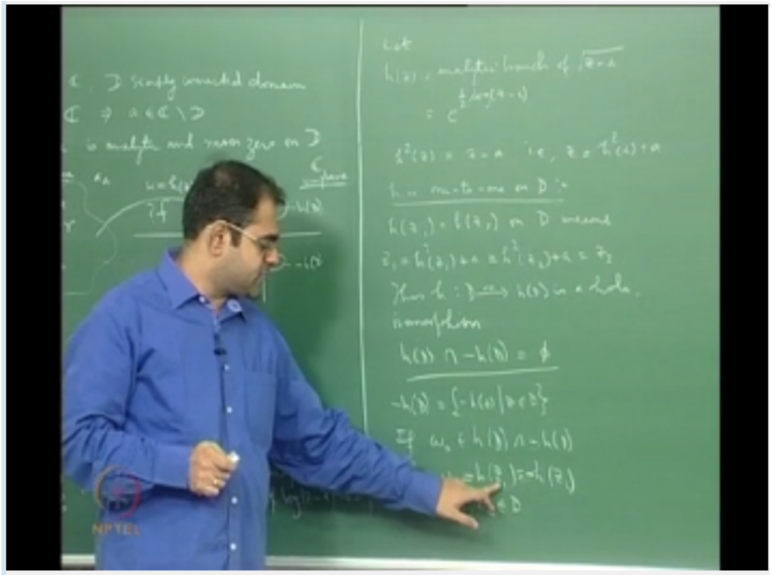

Now I can defined h of z to be an analytic branch of root of z-a okay analytic branch of root of z-a. And how do I defined it is very simple this is just exponential of half log z-a okay so, I already I have an analytic branch o flog z-a I multiplied by half n take e to the raise it it with that power and I will get an analytic function. And this analytic function is just root of z-a okay.

Because you know this if you think of it this half will go to the power here and the en log will constant I will get z-a to the half okay. So, this is an analytic branch of this square root function okay and now I want to I want you to understand that this has properties. So, first thing I want to says that you see so, you see since it is an analytic branch of z-a you will have zh square of z will be z-a.

So, in other words you will have z equal to h square of z+a I will get this alright. And from this I want to derive two properties of h namely the first property is that h is one to one. And the second property is that the image of h the image of D under h and – of h the images are disjoint okay**.** So, but before I continue let me try to say what I am trying to do I see what is a Riemann mapping theorem.

Riemann mapping theorem is you know give me a domain D which is simply connected and which is not the whole complex plane. Then it is conformably equivalent to the unit disc that is a Riemann mapping theorem so, I have to find from the domain D I have to find the Riemann map f okay. I have to find the Riemann map f which goes into the unit disc delta this is set of all complex numbers are the modulus less than 1 okay.

So, I have to find a map into this oaky from this domain into this I have to find a map f which is analytic which is one to one okay. And whose for which you also have an inverse okay which is also analytic in other words I want to find an analytic isomorphism of this domain D into the onto the unit disc that is the Riemann mapping theorem that is what I want to do.

But what I am trying to do now in the first step is try to at least map this disc into a smaller possibly smaller domain inside the unit disc that is what I am going to do in the first step first I will show that you can map this D into a smaller domain inside the unit disc okay. And then I am going to use that and some more techniques to show that you know you can the from that and using some analysis.

You can get a map which maps this not just inside the unit disc but onto the unit disc okay. So, you can use some analytic techniques to show that you can this instead of this map you can find other map for which I mean which covers the whole unit disc okay. So, the first step is you are just mapping this into a smaller domain inside the unit disc like this okay.

And what is that map, and that map is going to be hooked up using h okay that is the importance of h so, let me make this claim zi set some time ago h is one to one on D why is it one to one D because you know h of z1 is equal to h of z2 on D means well you put it here you will get z1 is equal to h square z1+a but h1z is hz2. So, it is h square z2+a but that is equal to z2 okay.

So, hz1 equal to hz2 means z1 equal to z2 which means it is one to one okay so, your map h is analytic and one to one okay. But what you know about an one to one analytic map it is an isomorphism okay we have seen this you have this seen in inverse function theorem if you have one to one analytic mapping it is already an isomorphism onto the image okay an analytic an non-constant analytic mapping is always an open map mind you.

If you take a one to one an analytic map then it is image is open and on the image you can defined the inverse you can defined the inverse because it is a one to one function. But the inverse function theorem will tell you that the inverse will also be analytic. Therefore one to one analytic map is an analytic isomorphism it is a holomorphic isomorphism or conformal isomorphism.

Therefore what this will tell you zh from D to h of D h from D to h of D will be a holomorphic isomorphism okay. So, thus h from D to h of D is a holomorphic isomorphism of course the other words that are used are well the literature instead of saying holomorphic isomorphism people say analytic isomorphism or they use conformal isomorphism as a people sometimes is use the word conformably equivalent okay.

So, this is because of the open mapping theorem and inverse function theorem okay now so, you know so, the point is well here is my D and then I have this h so, of course whether there exist an f like this is a question that is a Riemann mapping theorem. But I am looking at D and here is hD this is some other domain it is another domain okay.

And then I can also look at –hD okay –hD is consists of a all those complex numbers whose negatives are in hD okay. So, well in other words you know –hD will be just reflection of hD so, if I draw a diagram well I need to draw let me do this.

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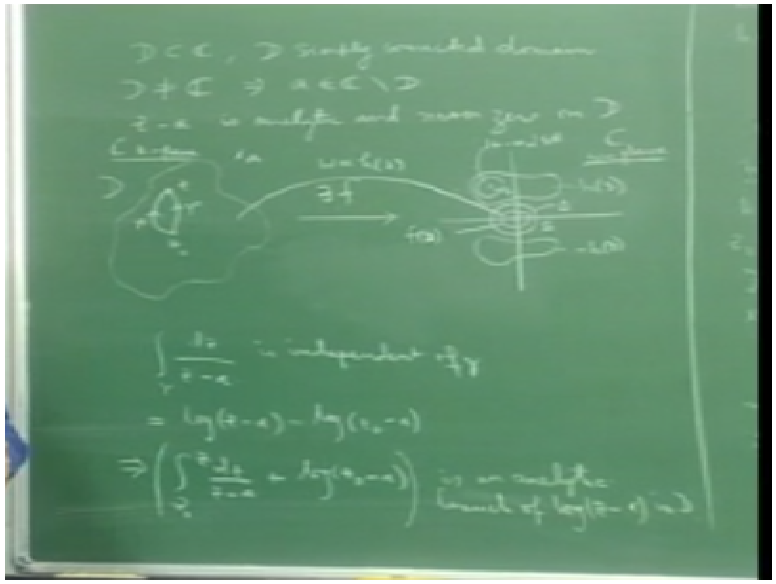

So, here is my point somewhere outside let me saves some space here and here I will draw the unit disc and well so, if I draw this little bigger my situation is like this here is a complex target complex plane. And well this complex plane here is the w plane is the z plane and here it is a w plane. And I am using the map w is equal to h of z where h is a square analytic branch of the root of z-a.

And you know well the image of D under h is going to be something well so, let me draw something here. So, this is hD okay it some domain in the complex plane okay mind you hD is isomorphic to D. Therefore it will also have the properties of D namely it will be a domain and it will be simply connected. So, h of D will also be simply connected domain alright.

And what is –hD this is the reflection of hD so, I will have this – of hD alright. I have the map –h see if **if** h is analytic –h is also analytic and in fact –h will give you the other branch okay if h is one branch of root z-a –h will give you the other branch. Because you will always get two branches for the square root you will have two branches for the nth root you will have n branches.

So, if you take –h it will be the other branch analytic branch of the square root and hn-h will be mirror reflections of each other under the real axis. And the claim is that hn-h cannot intersect okay no point of h can belong to –h. I need this fact so, here is one more claim h of D intersection –h of D is the null set there is no point in h an d-h okay where –h means the reflection of h.

So, -h if you want it write it as sets –h of D is a set of all – of h of z where z belongs to D okay. You just apply –h instead of h and you know if h is analytic on D –h is also analytic on D an analytic function defined by I mean an analytic function multiplied by a constant is also analytic function after all –h is h multiplied by -1, -1 is a constant. Therefore –h is also an analytic function right.

Now you see what I want to say is there is no point common to both of this okay. So, you know if there is a point common to both of this. Let me try to get a contradiction then it means the w0 is h of z z0 and it is also equal to h of well let me use z1 and is also equal to z2 –h of z2 for z1 and z2 in D this is what it means to say that there is a point common to h of D and –h of D.

So, this there this so, if you call that point is w0 then there is a z1 in D which is mapped by h into w0 and there is a z2 in D which is mapped by –h into w0 okay. And well this cannot happen why this cannot happen is just a simple calculation because you know now I can use **use** the same kind of calculation here.

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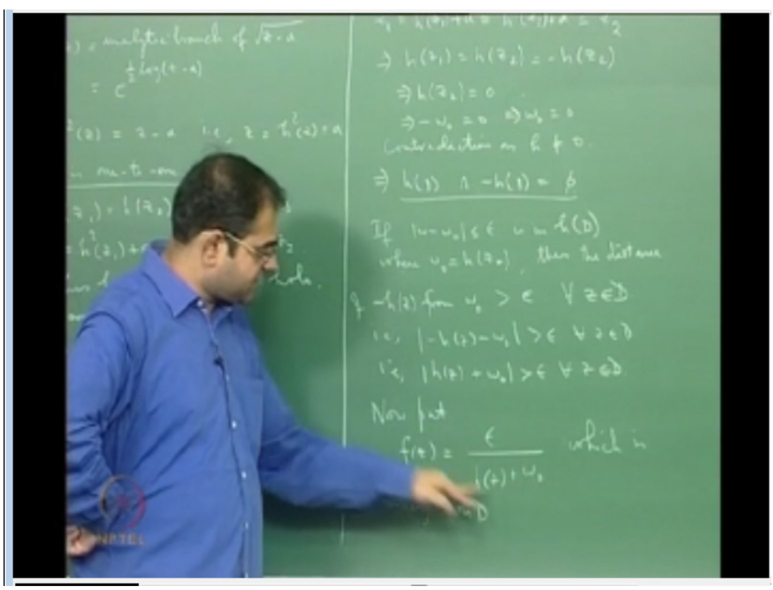

I will get z1 is equal to h square z1+a and but you see h z1 is –hz2 therefore h squarez1 is the same as h squarez2. So, this is equal to h square z2+a and that is equal to and that is equal to z2 okay see whether you take h or –h h square is always z-a. Because both h and –h are two branches of the square root both h and –h are branches of root of z-h so, h square is always z-a okay.

So, I am using that here so, I will get z1 equal to z2 okay but if z1 is equal to z2 then it means h of z1 is equal to h of z2 okay. So, this implies h of z1 is equal to h of z2 but h of z1 is also equal to – of h of z2 so, what this will tells you is that it tells you that h of z2 is 0 okay. But what is h of z2 after all **is** h of z2 is –w0. So, -w0 is 0 is will you that w0 is 0 okay. But then that will mean that h of c but this cannot happen.

You know because see w0 cannot be 0 that is a contradiction because w0 is a value of h or –a and also where it is also value of h and it is also value of –h. But just use the fact that is the value of h w0 is a value of h but h is what h is exponential, exponential function can never take the values 0. Therefore it is a contradiction so, contradiction as h is 0 not equal to 0 that is because h is exponential the exponential function can never take the values 0 okay.

Therefore if you assume that there is a point in the intersection of hD and –hD you get a contradiction okay. So, **so** this implies that hD and –hD they do not intersect their intersection is a null set okay. So, the diagram is the hD and it is reflection they do not meet each other they do not intersect. And now why do I need this, I need this for the following reason see if I take a point w0 let me choose a point w0 in the image.

And choose a small disc centred at w0 which is in the image so, this is a disc mod w-w0 lesser than or equal to epsilon. I take a small disc closed disc in the image okay then let us see what happens if mod w-w0 lesser than or equal to epsilon is in hD okay. Then you see where of course you know w0 is h of z0 okay. So, I mean I take this point z0 and it goes through a point w0.

And I take a small enough disc closed disc which is in hD okay centred at w0 radiu sepsilon okay then the distance of h of z then the distance of –hz from w0 I sgreater than epsilon. You see for all z in D see try to understand the statement see you take any z in D okay. Then hz is going to lie in hD okay but if I take –hz it is going to lie in –hD okay. So, you know hz is going to lie here –hz is going to lie here.

Therefore the distance of –hz from w0 has to be certainly greater than epsilon okay. Because if the distance of –hz for some z the distance of –hz to w0 is lesser than epsilon. Then that point will be an intersection of hD and –hD which is not possible okay. Therefore you get this fact okay. And this that is in other words what will you get is if you write it the distance between two complex numbers is given by taking the modulus of the disc difference.

So, it wil give you that –hz –w0 is greater than epsilon for all z in D so, you get modulus of hz+w0 is gretaer than epsilon for all z in D. You get this okay now you know now you put f of z to be epsilon by hz+w0 okay put f of z equal to epsilon by xz this w0. If you do this then this is analytic which is analytic on D see this is analytic on D because you know hz+w0 this always greater than epsilon okay.

Therefore it never vanishes so, 1 by hz+ sorry hz+w0 is always greater than epsilon and epsilon is positive so, show it is modulus of hz+w0 which is greater than epsilon which is positive and does not vanish. So, let me hz+w0 never vanishes so, 1 by hz+w0 is an analytic function and I have multiplied by an epsilon the numerator so, this an analytic function what is more this is actually this is even one to one analytic function.

Because after all what I have done is a I have taken the one to one function h I have translated by w0 translation is also a bilinear transformation. It is analytic and it is one to one and the I would then I have inverted it then I have multiplied by a epsilon so, I have applied a serious of moebius transformation to h okay to get from how do I get f from h first I take h and translate by w0 that is the translation oaky.

So, I translate by w0 then I invert okay I apply the transformation w going to 1 by w an inversion which is also a moebius transformation okay. Then I multiplied by epsilon and multiplying by a complex number is also moebius transformation so, I get h f from h by applying 3 by composing with 3 moebius transformations therefore since h is one to one f will also be one to one.

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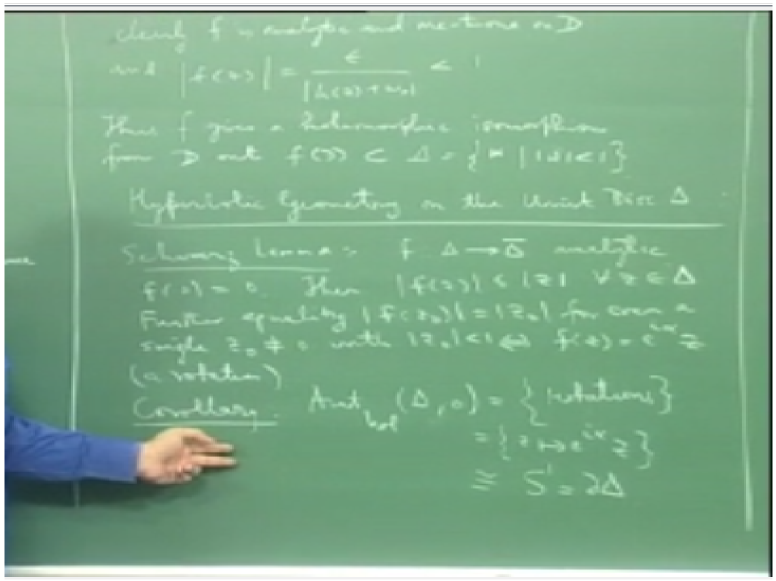

So, f will also be one to one analytic map on D but the beautiful thing is now f will land inside the unit disc okay clear clearly f is analytic and one one , one to one on the unit disc I mean on D and what is mod fz, mod fz is going to be epsilon by mod of hz by w0 which is less than 1. Because of this in equality this inequality tells you that epsilon by mod of hz+w0 is equal to 1 **is bis** strictly less than 1.

So, I will get mod fz less than 1 what does it mean it means that f takes values in the unit disc okay. Thus f gives a holomorphic isomorphism from D onto fD which is conained in unit disc delta which is set of all setz mod z less than 1 okay. If you want again put w because the okay. So, what we have done so for is you have just use the existence of square root of z-a to derive that you can find an analytic map which goes which maps the domain D isomorphically onto a sub domain of the unit disc.

So, it lands here this is unit disc okay so, there exist an analytic isomorphism of the domain D onto a sub-domain of the unit disc, the sub-domain is fd, this is the unit disc delta and this contains this sub-domain which is f of d. And f from d f of D is an analytic isomorphism because it is and 1 to 1 alright. So, this is the first step that using which you can map any simply connected domain which is not the whole complex plane onto a smaller domain in the unit disc what we want is we want a map that will fill the whole unit disc okay, that is the main thing.

So, **so** the moral of the story is that somehow you have to show that if you give me a map if you give me simply connected domain inside the unit disc okay, you have to somehow show that is also equivalent to the unit disc okay, I have to somehow show that a simply connected sub-domain of the unit disc is also conformally or holomorphically equivalent to the unit disc , that is what I have to do.

So, this so what we have done is we have translated the problem for an arbitrary simply connected domain which is not the whole complex plane to a problem of looking at a simply connected sub-domain of the unit disc which is not the whole unit disc okay. So, the this step reduces to studying everything the inside the unit disc okay, so what it tells you is that you have to study the unit disc carefully.

And that is what we are going to do in the following discussion, what we are going to do is we are going to study geometry on the unit disc a very special geometry called the hyperbolic geometry on the unit disc okay. So, it if this leads us to study the hyperbolic geometry on the unit disc. So, let me write that down hyperbolic geometry on the unit disc, so you know we already know sub facts about the unit disc.

So, what are the couple of facts that we know about the unit disc, we know 1 fact that we know is a Schwarz's lemma okay and the other thing is the corollary of the Schwarz's lemma which says that any automorphism of the unit disc a holomorphic automorphism the unit disc that fixes the origin ahs to be a rotation okay. so, let us recall that Schwarz's lemma. So, what is Schwarz's lemma f from delta to delta bar analytic f is defined on the unit disc.

And takes values in the inside the closed unit disc f takes 0 to 0 okay, then Schwarz's lemma says that the length of fz cannot exceed the length of z or all z in the unit disc which is Schwarz's lemma and it tells you that you get equality for a single non 0 member of the unit disc if and only if it is a rotation okay further equality mod f z0=mod z0 for even a single z0 not equal to 0 with mod z0 less than 1 implies f of z is a rotation okay.

And it is not only that it implies it is if and only okay. So, you have equality even for a single vector if the image of that vector I mean if you think of the point as a vector in the complex plane the vector joining with the origin to that point, the question vector of that point okay. Then if the length of the vector is equal to the length of it is image even for 1 point, then your analytic map has to be a rotation there is no other choice okay.

So, this is the Schwarz's lemma and here is a corollary and the corollary is the step of automorphism the holomorphic automorphism of delta, so this is the set of all maps from delta to delta which are holomorphic isomorphism they are in other words they are injective holomorphic maps a they are bijective holomorphic maps from delta to delta and you know bijective holomorphic map is automatically a holomorphic isomorphism because of the inverse function theorem and the open mapping theorem.

Therefore this is a set of automorphism of the unit disc and what are they, they are just rotations okay. So, in fact the point is I come back to it soon but here we are only looking at auotmorphism which fix the origin okay. So, let me put let me write delta, 0 this is the set of all these are all the rotations, so this is just the set of all z going to e to the i alpha z okay. And so and that is that can be identified with the each rotation therefore is given by this angle alpha okay.

So, you can identify it with just the unit circle S1 which is the boundary of the unit disc every point of S1 has a unique angle alpha namely the angle joining that point to the origin made with the x axis the real axis. And therefore the set of holomorphic automorphism is just S1 and this so what is the map you send any holomorphic automorphism is are the form z going to e to the i alpha z and you send z going to e you send this map z going to e to the i alpha z to the element e to the i alpha which is on the unit circle which is the boundary of the unit disc okay.

That gives you a bijection and this is not just a bijection okay, it is actually a group isomorphism. Because if you compose to rotations the angles will get added okay and therefore on the circle also the you have the set of all complex numbers of modulus 1 is also a group, e power i alpha and e power i beta ewhen you multiply you will get e power i alpha+beta.

So, this bijection is not just a bijection of sets, it is even a bijection of groups okay, so the group of what holomorphic automorphisms of the unit disc which fix the origin can be identified with the unit circle as a group under multiplication okay, this we have seen that this is the corollary of the Schwarz's lemma alright. So these are the first 2 facts, now the whole point about hyperbolic geometry comes from what is called hyperbolic the hyperbolic metric on the unit disc.

And the hyperbolic metric that you have a nice hyperbolic metric which is preserved by all these holomorphic automorphisms of the unit disc okay, now that itself comes from something that is called Pick's lemma which is a kind of nice version of Schwarz's lemma okay. So, let me make one more statement here I have looked at all the automorphisms holomorphic automorphisms of the unit disc with the origin fixed okay the origin going to the origin.

But what about any automorphism of the unit disc what is the general automorphism of the unit disc, so let me write that.

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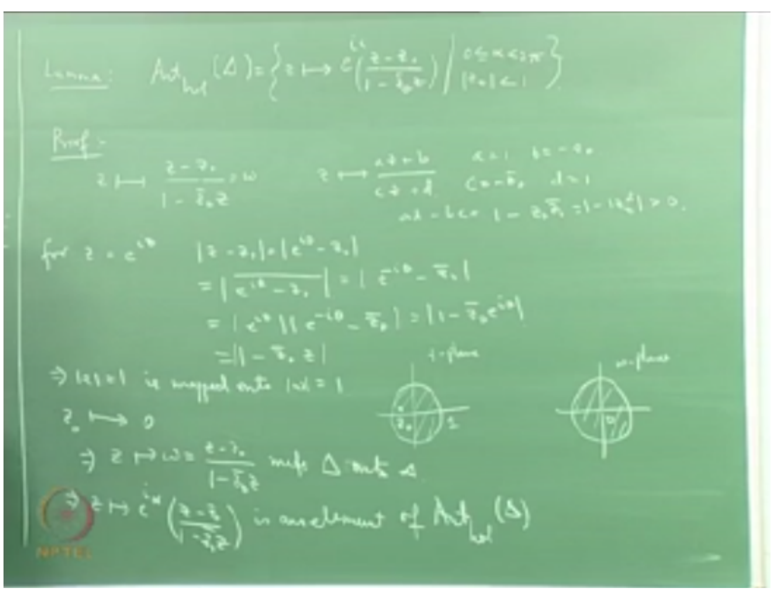

So, here is a lemma, so the lemma is automorphisms any holomorphic automorphism of the unit disc what is it, it is of the form z going to e power i alpha*z-z0 by 1-z0 bar z where of course alpha is a where is an angle between 0 to 2 pie and z0 is a complex number with modulus less than 1. So, here is a lemma, this lemma tells you what are all the holomorphic automorphisms of the unit disc, what are all the 1 to 1 analytic maps of the unit disc onto itself.

A general map of this type and you get this case when you put z=0, you take you put z=0 this expression will become z and the I will get the map **z=** z going to e to the i alpha z which is just a rotation about the origin okay, more generally if you are automorphism of the unit disc does not fix the origin it may map origin to something else then it has to look like this okay, this is the statement.

And what is the proof of this and the proof of this is that well see if you look at the map z going to z-z0 by 1-z0 bar z, if you look at this map okay. Then this map mind you this map is a Moebius transformation okay, this map is a Moebius transformation. Because you see it is of the form see it is of the form z going to az+b by cz+d where a is 1, b is –z0, c is 1, c is –z0 bar and d is 1.

And if you calculate ad-bc you are going to get 1-bc is going to be z0 z0 bar, so I am going to get 1-mod z0 square and that is in fact that is greater than 0 because mod z0 is less than 1. Since z0 is a less than 1, mod z0 square is also less than 1 and therefore this is a positive number. So, this is of the form z going to az+b by cz+d with ad-bc non 0, in fact ad-bc is positive, so it is a Moebius transformation.

And you know the Moebius transformation is a bijective map of the extended complex plane to the extended complex plane it will always map you know the properties of Moebius transformations namely it will map straight lines and circles onto straight lines in circles on the complex plane. But even a straight line of the complex plane can we thought of as a circle on the extended complex plane.

Therefore if you make the statement for the extended complex plane it will always maps circles to circles okay. and it will map the interior of the circle the interiors and exteriors it will preserve because of continuity okay. So, now what you must understand is that if I calculate if I take the image of if I take z=e power i theta okay and calculate mod z-z0 I will get mod e power i theta-z0.

But this is equal to it is the modulus of it is conjugate okay which is equal to e power –i theta-z0 bar alright and that is equal to if I multiply throughout by **e** e power i theta which I can do because of because the modulus of the e power i theta is always 1, what I will end up with is modulus of 1-z0 bar e power i theta okay right. And that is just 1-z0 bar z mod, so what this tells you is that mod z=1 is mapped onto mod omega=1.

So, you takes unit circle to the unit circle, see if you call this as a omega as z-z0 by 1-z0 bar z, if you put z=e to the i theta then mod omega turns out to be 1. So, in other words mood z=1 goes to mod omega=1, so this Moebius transformation maps unit circle to the unit circle after all it is Moebius transformations. So, it has map a circle to the line or a circle but here what is doing this calculation actually shows that it is mapping the unit circle to the unit circle okay.

And what about the point 0, 0 goes to when I put z= if I put let me not put 0 let me put z0 where the z0 go to, z0 will go to 0 if I substitute z0 I will get 0. So, but where is z0, z0 is inside the unit circle okay, the z0 is inside the source unit circle and 0 is inside the target unit circle. Therefore this will tell you that the interior of the unit circle will also go to the interior of the unit circle, see what is happening is that you have on the z plane you have the unit circle.

This is z plane you have the unit circle and on the omega plane also you have the unit circle and you know unit circle on the z plane goes to the unit circle on the omega plane because of this calculation under the map z going to w or omega given by z-z0 by 1-z0 bar z. Once a Moebius transformation maps a circle onto a circle then there are only 2 choices the interior of the circle can either go to the interior of this circle or the interior of the circle will go to the exterior of that circle that is only possibility.

But what I am saying is there is if you take the point z0 here which is inside the circle that is going to the origin. So, the point in the interior of this circle is going to a point in the interior of this circle therefore what this will tell you by property of Moebius transformations is at this whole **cir** unit disc is going to be mapped in the unit disc. So, this will tell you is that z going to w=z-z0 bar by 1-z0 bar z maps the unit disc onto the unit disc okay.

So, it is a holomorphic automorphism of the unit disc and then you know if I take if I multiplied by e power i alpha, e power i alpha is only a rotation okay and a rotation will just map the unit disc back onto itself alright. Therefore if I take this map of this form it is certainly a holomorphic automorphism of the unit disc okay implies z going to e power i alpha z-z0 bar by z-z0 sorry.

This to be z-z0 by 1-z0 bar z is an element of the holomorphic autmorphism of the unit disc. So, what I have proved is that this is contained in this, I have proved that any element like this is certainly a holomorphic automorphism of the unit disc conversely I have to show any holomorphic automorphism of the unit disc is like that and for that I will use this corollary okay, I will just use this corollary to show that, I will continue later because I just have to compose by a suitable transformation of this type and apply Schwarz’s lemma okay.