Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-31
Proofs of Maximum Principles and Introduction to Schwarz Lemma

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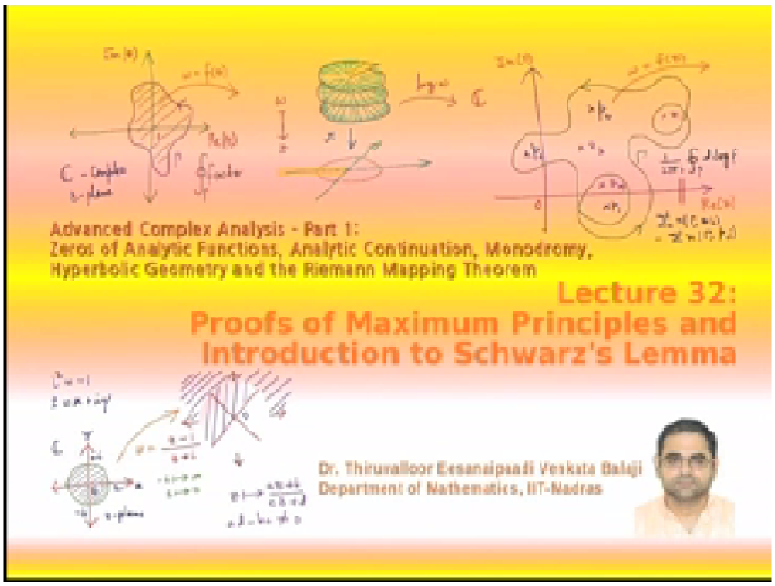

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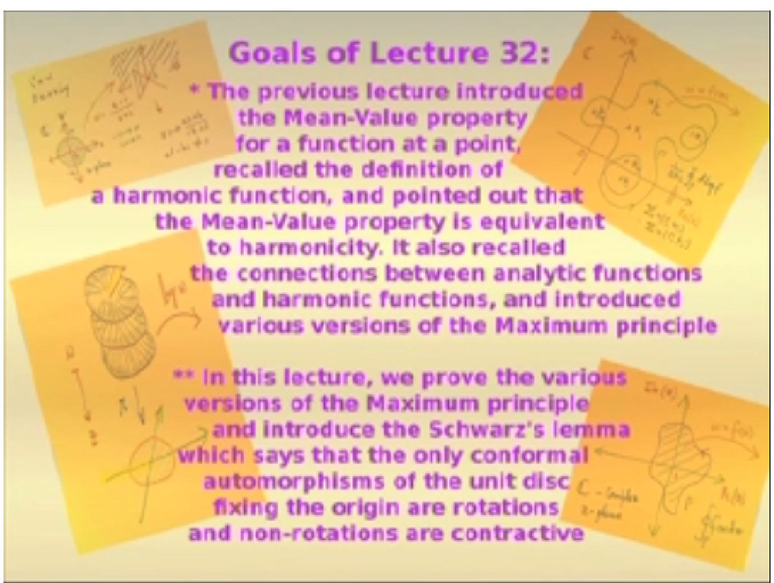

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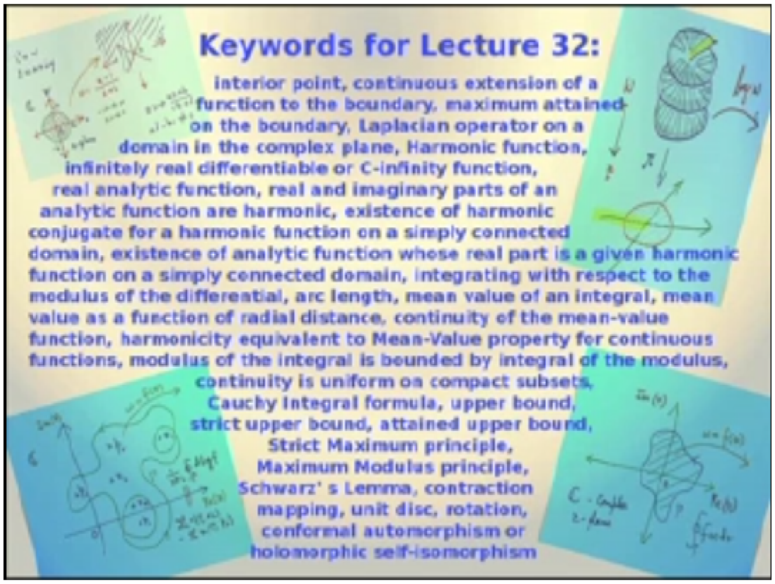

Okay, so let us continue with our discussion of how many functions and the maximum principle.

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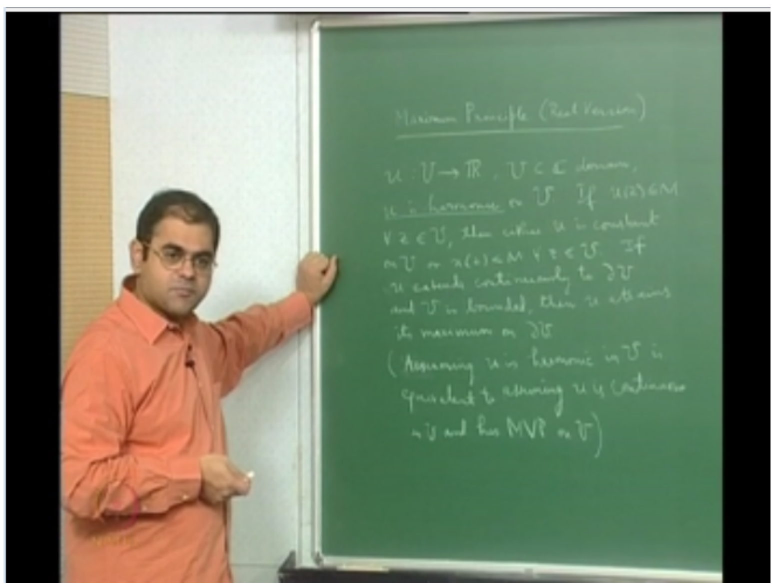

So, **so** if you recall we have this is maximum principle so this is the real version, so you have u defined on capital U with values in R U is a domain in the complex plane and use real valuehood function. And in fact you assume U is a function which is actually harmonic okay U is harmonic on capital U okay. And suppose u of z is less than or equal to M for all z in U.

Then either u is a constant on capital U or the bound is this in equality in strict or u of z strictly less than M for all z in okay. So, in other words a harmonic function and which is a non-constant from which is non-constant namely a non-constant harmonic function it can never attain a maximum in the in an open set okay an open connected set. So, and of course the point about an open set is z every point is a integer point right.

A set is open in topology if very if and only every point is an interior point so, what you are saying is that a harmonic function cannot attain a maximum at an interior point. And therefore you know if you can extend the harmonic function to the boundary to a continuous function. Then you should expect the maximum to be attained on the boundary okay. So, **so** let me state that also in other words.

So, if so, if u extends continuously to the boundary of u and capital U is bounded then unless then u attains it is maximum on the boundary okay. So, this is a real version of the maximum principle so, in fact the fact is that the condition that u is harmonic can be replaced by the condition that the by the equivalent condition that u is continuous and satisfies the mean value property has the mean value property at each point okay.

So, instead of assuming u is harmonic in u is equivalent of assuming to assuming u is continuous in capital U. And has the mean value property on it on capital U okay. So, this is something that I told you is a theorem that for a continuous function harmonicity on a domain is equivalent to it is having the mean value property at each point. And I think and we saw I gave you a proof in one direction namely if the function is harmonic.

Then it has a mean value property I use the quotient integral formula to prove that last time and the harder part is to show that the continuous function which has a mean value property is harmonic that is the statement that I need not proof and I am not going to give the proof of that okay. But I am going to use that fact that harmonicity is equivalent to continuity with the mean value property okay.

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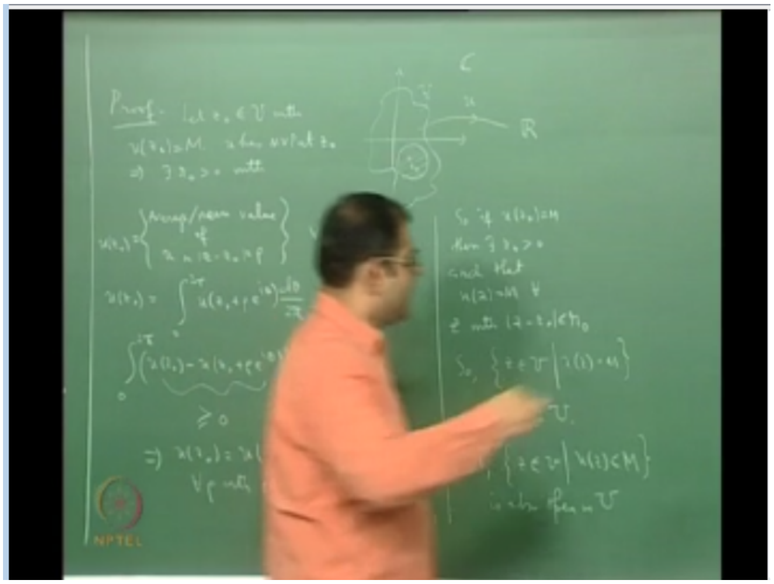

So, in other words I am just trying to say that maximum principle applies to any continuous function which has the mean value property because this is same as harmonic okay. So, what is the proof of this so, the proof of this is pretty it simple it of course uses the mean value property. So, you know so, the idea is that you know see so, you know you have so, if let me draw a diagram. So, you have so, here is a complex plane and you have some domain u well I am always drawing a bounded domain it need not be bounded at uh and then there is a point.

I take a point z0 in the domain alright and well the fact is that so, what is given to me is at u is real value small u is real value and it is bounded all the values of small u on this open set u which is the interior of this amoeba like region that I have drawn that is where the function u is defined small u is defined. So, there is a function like this, taking values in real line and all the values are small u are bounded above by M okay.

And what do I have to show I have to show that if u is not constant then u never attains the value M in the in that open set in that open connected set it is a domain. It is both open and connected as you see the connectedness is very important. It will be using the proof so, the other way of proving that statement is trying to show that if u assumes the value capital M at a point inside.

I will try to show that u is constant okay so, which so, the contrabass of that will be that if u is not constant then you cannot as assume the value M at any point inside. So, at every point inside the values of u will always be strictly less than M which is the assertion of the statement of the the assertion and the statement okay. So, suppose so, let me write this let us assume that z let z0 belong to u with u small u at z0 equal to capital M let us assume this okay.

And now you know the fact is that small u is a harmonic function but more importantly what I want is that it has the mean-value property okay. So, a harmonic function is a continuous function which has the mean-value property at every point. This is a equivalence okay by the way you know that harmonic means that the original straight definition of harmonic is that it should satisfy Laplace this equation.

It should have continuous derivatives of orders up to two and it should be continuous okay and it should satisfy Laplace this is the equation but let me again repeat it is an important fact that function which is harmonic has mean-value property and conversely a function which is continuous just continuous and has a mean-value property is harmonic.

And the beautiful thing about harmonic functions is how are they related to complex analysis they are related in sense that harmonic functions are always **on** on small disc like an neighbourhoods namely of on simply connected open sets. How many functions are always real value I mean real parts of analytic functions. And since analytic functions are infinitely differentiable it follows that the harmonic functions are infinitely differentiable okay.

And that is great because when you define harmonic function you only want the derivatives to occur up to R2 and the that the you should be continuous. But then it turn out to be infinitely differentiable okay that is because of the fact that harmonics functions are locally real parts of analytic functions, analytic functions are infinitely differentiable okay of course what is more uh amazing is the statement that.

You assume nothing about the derivative at all if you assume the mean-value property you are just assuming that the function is continuous and it has a mean-value property which is you take the mean of all the values on a circle surrounding a point sufficiently small circle surrounding a point. Then the mean value the average value you get is the value at the **at that** point at the centre of the circle.

And this wholes for all sufficiently small circles that is the mean-value property it is **it is** property which is defined by an integral. And it only and for the integral to be defined you only continuity so, you know if you have a function which is just continuous, and which has an mean-value property the upshot is that it is harmonic and as a result it is infinitely differentiable. So, it is really amazing okay.

So, anyway so, see at the point z0 you will have then small u have them mean-value property. So, it means that you know if I take a small disc surrounding z0 inside your domain so, you know so, I will have the following thing u you has mean-value property at z0 implies for all implies there exist an R0 greater than 0 with the average value or mean value of f on mod z-z0 equal to row okay.

The average value on the circle is actually equal to the not f it is u is equal to u at the centre of this circle. So, this is the main value property okay that and this is for all row greater than 0 row less than r0 so, for sufficiently for small circles centred at z0 the average value of the function on the circle is equal to the value at the centre that is the mean-value property.

But an how is this define this is defined as 1/ so, it is defined by integral you integrate from 0 to 2pie the **the** function values u of z0+row e to the I theta that is how a point on this circle looks like as theta varies from 0 to 2pie you get the whole circle centred at z0 radius row. And then you integrate with respect to d theta by 2pie, this is an mean value and this is equal to u z0 okay but of course you know I assume that u zo is M okay.

So, you know I can actually write this as integral 0 to 2pie u z0 –u of z0+row e to the i theta the whole into d theta by 2pie equal to 0. I can write it like this that is because you know u zo is a constant if the first integral will just give me u z0 because if I integrate d theta over 2pie from 0 to 2pie. I will simply get 1 okay so, now what you must understand is that this is the integrand this integrand is you know.

This always greater than or equal to 0 okay that is because u z0 is M your z0 is M and all other u always are less than or equal to M that is already given to me okay. So, the difference is always non-negative so, I have a non-negative so, you know I have this fact I have this situation where I have a of a of real valuehood function on a closed integral okay which is non-negative.

And the integral is 0 okay but then that means the integrand has to be identically 0 okay this is something that you know. So, **so** this implies that u of z0 has to be equal to u of z0+row e to the I theta for all row with 0 less than row less than r0 okay. And of course so, **so** of what have I proved I prove the following thing have proved the following thing. I have proved that if u attains the value M this maximum value M at an interior point z0.

Then there is a whole disc surrounding z0 where u is constantly equal to that value M okay. So, on this whole disc okay as I increase as I allow and of course you know it is very important that theta of course is varying from 0 to 2pie okay. So, if I fix a row and let theta vary from 0 to 2pie I will get the circle of radius row. And then if I make rows small then I will get the whole disc.

I will get all points in the disc centred at z0 and radius r0 okay so, what all these tells you is that so, if u of z0 is M. Then there exist r0 greater than 0 such that u of z is M for all z with mod z-z0 with ro this is what it this is what we have proved okay. If u attains the value M at the point z0 then there is a whole disc centred at z0 contain in your domain with radius ro okay where u attain u where u is equal to where u attains the same maximum M okay.

So, what this tells you is that so, this so, I have basically I have use the mean-value property at z0 I am applying the mean-value property at the point where u has attains the maximum okay. And I am using the fact that the integral of an if the integral of an non-negative function is 0 okay. Then that non-negative function has to be 0 itself okay so, the moral of the story is what as the tell you this tells you.

This tells you the following thing the if you take the set of all z in u capital U such that u of z is equal to M. It tells you that this is open in you is an open set because what I have proved whenever z0 is in this set. I proved that there is a whole disc centred at z0 which is in this. So, every point of this z is interior point of this set and that means is this set is open okay on the other hand you also have the set of all points of u where uz is strictly less than M. This is also open in u okay this is also open why because you see why is that true.

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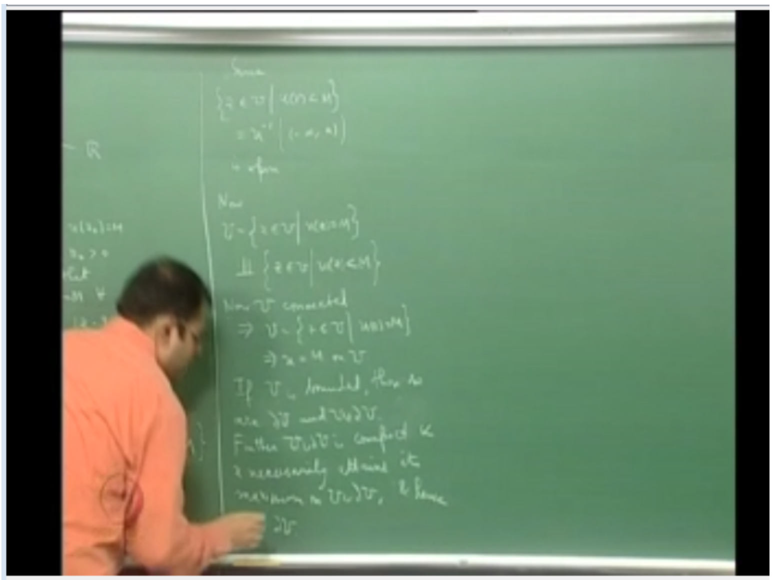

Because u of z this set of all points in u where u of z is strictly less than M is just u inverse of –infinity, M okay. It is a inverse image of the this interval –infinity M on the real line I mean this is precisely all those points where u takes a value strictly less than M. And you see u is continuous function and this is a open set and you know one of the characterization of sub continuity is that the inverse image of an open set is open under a continuous map.

So, therefore so, **so** this set is open but then what is the union of these two open sets the union of these two open sets is all of you okay. So, you get u is equal to the set of all points where small u attains the value capital M and disjoined union with set of the other points where small will u is strictly less than M because anyway it is given that all the values of u are less than or equal to M.

So, those points where you takes the value M form 1 piece and those who take the values strictly less than M form another piece. And the union of these two 0 but what have we just we have seen we have seen this also open this also open is a two open sets okay. And u is union of two open sets but u is a domain capital U is a domain it is connected and a connected cannot be written as a disjoint union of two open sets two proper open sets.

Because any set that is written as a disjoint union of two non-empty open sets is already disconnected writing a set as union of two pieces which are disjoint from each other and which are open is already disconnecting the set okay. So, **so** the connectedness of u will tell you that one of these has to be empty okay. So, but you know z0 lies here so, this set is non-empty therefore this has to be empty.

So, it will mean u is equal to this which means u is constant okay and that will be the end of the proof so, now you connected because it is a domain is an open connected set implies u is exactly the set of all points where u small u is equal to M this set is another set is empty okay. This implies that u is equal to M on U so, that finishes the proof that if u attains the value M at an interior point.

Then it has to be constant okay alright now there is one more statement the other statement is if u extends continuously to the boundary and u is bounded okay. Then u attains it is maximum on the boundary okay and what is the proof for that the proof is very simple you know if you take **if** if capital U is bounded okay then it is boundary is also bounded. So, if you take u union the boundary it is a bounded and closed set.

It is closed because you have the boundary to it so, it is compact any subset of Euclidian space which is both closed and bounded is compact. So, that is what I am using the boundedness of u okay the boundedness of u is also going to give you the boundedness of dou u okay. So, it gives you the boundedness of u union dou u which is the closed set. So, which therefore it becomes compact.

And you know continuous function on a compact set is uniformly continuous and attains it bounds okay. Therefore u has to attain it is maximum on u union dou u but we have already proved that it will not attains it is maximum in the interior unless it is constant. Therefore it has to attain it is maximum only on the boundary and that is the proof for the second part of the statement okay.

So, the second part of the statement just follows from the first part of the statement okay. So, **so** I will just write that down if u is bounded then so is then so are dou u and u union dou u which is and so yeah let me write that few is bounded then so are dou u and union dou u further u u union dou u is compact and u necessarily attains it is maximum on u union dou u.

And hence on dou u because it see it cannot attain a maximum in the interior the only case when it attains a maximum in the interior it is **ca** is when it is constant. But if it is constant then it is extension to the boundary will also be constant by continuity. So, it will attain that maximum that constant value the constant value will be the also a maximum value it will be the maximum value.

And that will also be on the boundary okay so, on the other hand if it is non-constant then certainly it cannot attain the maximum value in the interior. So, it will attain the value on the boundary alright fine so, that finishes the real version of the so, I mean the point I wanted remember is that a continuous function which has the mean-value property on a domain has the maximum principle okay.

A continuous function which has a mean-value property on a domain satisfies the maximum principle that is what you must understand it cannot attains it is maximum in the domain it has to attain a maximum only on the boundary provide you can extend it to the boundary okay. And of course boundary should be bounded right so, alright so, this is the real version.

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Now let me go to the complex version so, **so** here so, the statement is a same except that instead of taking a real valuehood function you take a complex valuehood function okay. So, **so** f from u to c u inside c in to domain and you assume f is harmonic on capital U and of course saying that f is harmonic on u is equivalent to saying both the real and imaginary parts of f are harmonic on capital U.

So, a complex valuehood harmonic function is by definition something for which both the real and imaginary parts which are real valuehood functions they are harmonic okay. So, I have the same statement only thing is that since is complex valuehood I have to write I cannot write u is I cannot write f is less than or equal to M. Because it is complex valuehood I have to use the modulus of f.

So, let me write that down if mod of z is lesser than or equal to M on U then and either f is constant on U or mod fz is strictly less than M for all z in U that is the maximum the modulus of f will not attain it is maximum in the interior okay. And basically that means that if you find a if you take any point with a certain and take the modulus of the function and that point.

You can always find another point where the modulus of function is bigger when you say function does not attains it is maximum it is mean that given it is value at any point. I can find another point where it is value is bigger that is when that is what it means to say a function does not attains it is maximum okay. So, it exceeds it is function values at every point by function values at some other point suitable point oaky.

So, **so** if so let me write the other part as in that case if U is bounded and f extends to continuously the boundary then f attains it is maximum can mod of a attains it is maximum on the boundary okay. So, this is the complex version so, what is the proof the proof is that, we just use the apply the real version so, the proof is well so, again proof is a same thing.

You assume that it there is a interior point where mod f attains the value M and sure that f is constant. So, suppose z0 is a point of U with mod f z0 equal to M okay where I have to show that f is constant right so, **so** this means f z0 is e to the i alpha M okay so, see mod f z0 is equal to M and you assume that of course I am assuming that m is not 0 okay yeah so, I mean even if M is 0.

I could have taken alpha to b e0 so, f of z0 is a complex number with modulus M alright then I can write f of z0 as e to the i alpha times M okay for suitable alpha alright of course if M is not 0 then I can divide by M and I will get f of z0 by M has modulus 1. So, it is uni modular complex number so, it is of the form e to the ialpha. So, I will get f z by M f z0 by M is equal to the ialpha.

And well if f z0 is 0 then **I** I just take alpha equal to 0 okay so, in any case this **tis** equation holds for suitable alpha right. Now of course if f if mod f z0 is 0 it means f is z0 itself is 0 okay fine so, I can write this now what I am going to do is you look at e to the –ialpha f of z look at this function okay. Now this function is just f of z multiplied by a constant e to the –ialpha is some constant.

It is a unimodular complex number it is a complex number of modulus of 1 so, you have multiplied a function by a constant and harmonic function multiplied by a constant it is again harmonic function. Because after all function is a harmonic if it is for example if it satisfies Laplace equation and multiplying a function that is harmonic by a constant is going to keep it if you must harmonic it is going to give rise to again a harmonic function.

Because the differentiable operator the constantly come out the differentiable operator okay so, this is also this is also harmonic alright. So, look at this which is also harmonic okay and if you look at so, in particular real part of that is also harmonic. Because you know our definition of a complex function being harmonic is that both the real and imaginary parts should be harmonic. So, the real part of this also harmonic okay

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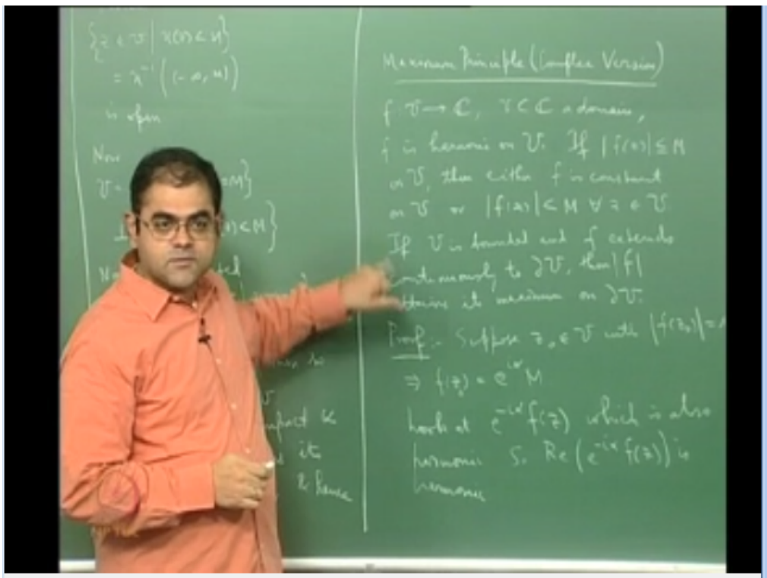

Now you see if you look at the real part of e to the –ialpha fz this is less than or equal to modulus of the real art of it is the –ialpha fz okay. Because any real number is less than or equal to modulus okay and this is certainly less than or equal to modulus of e to the –ialpha fz because you know for any complex number the modulus of it is real part I sless than or equal to it is modulus okay mod real part of w is less than or equal to mod w okay for any complex number w.

But this is the same as mod fz because modulus of e to the –ialpha is 1 alright and mod fz is always less than or equal to M okay. So, what I have got is that I have got this is a real value harmonic function which is bounded by M. And what is it is value at z0 at z0 it is value is exactly M okay real part of e to the –ialpha fz0 is actually real part of M which is equal to M okay.

So, I have harmonic function which has values less than or equal to M and it has attained a value M at an interior point. Therefore by the previous case the real version of the maximum principle what I can conclude is that this real part of e to the –ialpha f has to be constant okay. So, by the real version of the maximum principle real part of e to the –ialpha fz is equal to M. It has to be a constant okay.

So, the real version of the maximum principle says that whenever you have e real valuehood harmonic function on a domain if it is if it contains a maximum value in the interior it has to be constant. So, this function the real part of e to the –ialpha fz is an harmonic function of the domain capital U it is bounded by M all it is values are bounded by M. And it is attains the value M at a point z0 inside the domain.

Therefore it has to be constant so, it has to be and that constant value has to be the same as the constant value at z0 which is M. So, this function is exactly M okay now look at e to the –ialpha fz this is actually real part of e to the –ialpha fz+itimes imaginary part of e to the –ialpha fz okay there is just expressing a complex number as the real part+itimes it is imaginary part and then but this is equal to M+itimes imaginary part of e to the –ialpha fz okay.

So, I **I** have this but on the other hand what is the modulus of this the modulus of e to the –ialpha fz cannot exceed if the modulus is same as mod fz. Because mod of e to the i –ialpha is actually 1 and that is less than or equal to M okay what do these two equations tell you see I have a complex number whose real part is M and it is modulus cannot exceed M. The only way is the imaginary part has to be 0.

So, this will tell you that the imaginary part e to the –ialpha fz is 0 okay. Because the modulus is square root of M square+this square of this imaginary part and if and that can never be less than or equal to M unless it can only be equal to M and that case the imaginary part should vanish. Therefore you get this and the movement the imaginary part is 0 the function is equal to it is real part.

And that is equal to M so, you will get e to the –ialpha fz equal to M and this will tell you that fz is a constant it is just M e to the –ialpha e to the ialpha. So, that completes the proof okay and again as for the later statement is concerned the proof again uses the fact that if capital U is bounded then the boundary of U dell U that is also bounded and U union dell U will become both bounded and closed to be compact.

And mod of which will be a continuous function on this compact set will attains it is bounds and therefore you know that if f is not constant then mod f is always strictly does not attains it is bound in the interior. So, it has to attain it is bound on the boundary okay. So, there finishes second part the second part just follows as if the real case okay. So, in particular what you must understand is that if you take an analytic function okay.

This is what you would have seen in a first quotient complex analysis you take an analytic function analytic function also satisfies the maximum principle okay namely if you take an analytic function which is all mind you analytic function is also harmonic function. Because the real and imaginary parts of an analytic function are harmonic okay but the only extra condition is that the imaginary part is that harmonic conjugate of the real part.

That is what makes it analytic if I simply take two real harmonic functions and write them as U+iv that will not give me an analytic function unless the v is harmonic conjugate of U okay. So, even for analytic functions the maximum principle applies and that is what you would have seen in an earlier course in complex analysis. But what I want to you to appreciate is the important fact that at the base of all.

This is the mean-value property I mean it is just if a function has the mean-value property okay then if it continues in has the mean-value property. Then it automatically has the maximum it satisfies the maximum principle the maximum can only be attain on the boundary not in the interior okay. So, I need to come to so called Schwarz lemma which I need to use let drawn.

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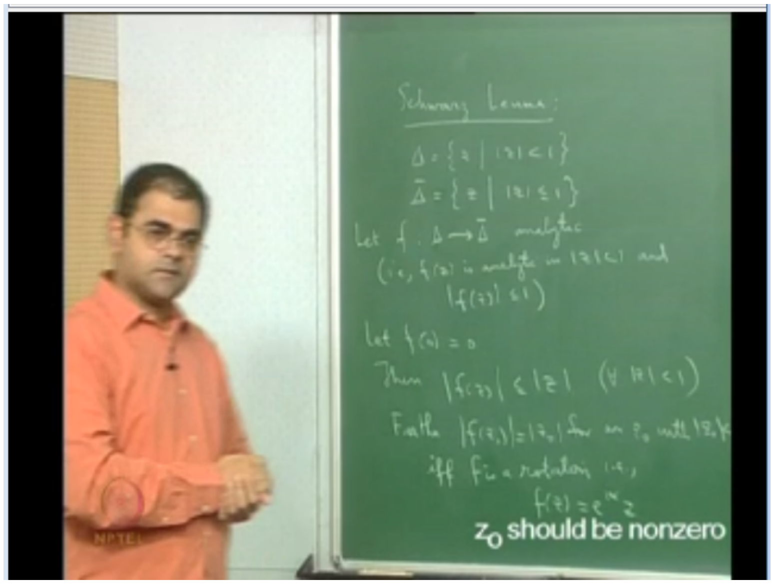

So, **so** here is the Schwarz lemma so, what this Schwarz lemma so, you know Schwarz lemma is well so, delta is the unit disc okay. And you have and of course delta closure is it is closure the close unit disc and you are looking at analytic functions define don delta and taking values in delta closure okay. So, f from delta to delta closure analytic okay so, that means you have what you, you are saying is f is analytic in mod z less than 1 which means f is defined on delta.

And it takes value in delta closure so, mod fz is less than or equal to 1 I mean if you want to state this in words you will say let f be an analytic function on the unit disc with modulus less than or equal to 1 okay. You want to state it geometrically you are looking at an analytic function which is defined on a unit disc and taking a values inside the closed unit disc okay. And let us also assume that 0 goes to 0 f of 0 equal to 0 okay.

So, assume this right then this Schwarz lemma is lemma that come that compares the modulus of fz and the modulus of z okay. So, **so** this lemma is mod fz is always less than or equal to mod z okay for all mod z less than 1 okay. So, in some sense what it means is that in terms of lens it is contraction it is a contraction map in the sense that if I start with a complex number z in the unit disc okay.

It is length is mod z okay but if I takes it is image fz it is length will become smaller so, which means that the mapping is kind of contractive okay. But it is not entirely that only in terms of length is contracting but there is also twist okay. So, the question is the other part of the Schwarz lemma is a case1 when equality occurs so, the next of Schwarz lemma tills that if you get equality even for a single point in the unit disc.

Then f is a rotation okay further mod f z0 is equal to mod z0 for an z0 with mod z0 less than 1 if and only f is a rotation that is f of z is equal to e power ialpha okay. So, this is Schwarz in other words any mapping from the unit disc taking values in a closed unit disc if it is not a rotation I mean you take an analytic map you take an analytic function defined on the unit disc and taking values in the closed unit disc.

And assumes 0 goes to 0 map that preserves 0 preserve the origin okay if it is not a rotation then mod fz will always be strictly less than mod z it will be strictly contractive okay. The only case when it is not contractive when it is a rotation and in and if it is a rotation then you know mod fz will be equal to z for all z. Because mod of e to the ialpha will be 1 so, if equality occurs for a single z0 it will occur for all z with mod z less than 1.

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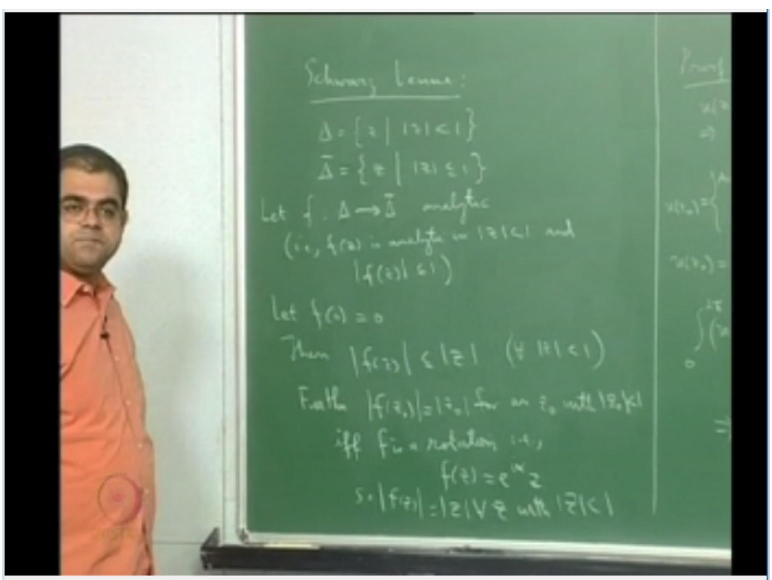

So, let me write that okay and it is a powerful lemma which is used it is the proof is rather simple because it uses the maximum principle but it is very powerful lemma and finds the lot of using conformal mapping and many other situations. So, what you must understand is in the case when f of z is a rotation it is a rotation about the origin then f becomes a one to one conformal map namely it becomes the holomorphic isomorphism of the you know unit disc on itself okay.

And in fact you know and in fact this is the these are the only holomorphic isomorphism is a unit disc on to itself. The only holomorphic automorphism of the unit disc which preserve the origin or the rotations that is the like that is seems like a converse but actually it is also a corollary of the Schwarz lemma so, **so** let me write that also.

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Let me write this corollary unique automorphism any holomorphic automorphism of the unit disc that fixes the origin so, automorphism is a self isomorphism okay automorphism means self isomorphism it is a holomorphic isomorphism from the unit disc back it back to itself. And it should fix the origin okay is a rotation so, this is the corollary of a this is the corollary of Schwarz lemma okay. The only conformal automorphism of the unit discs or rotations those that fix the origin or rotations okay so, we look at a proof of this in a next lecture.