Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-13
Doing Complex Analysis on a Real Surface_The Idea of a Riemann Surface

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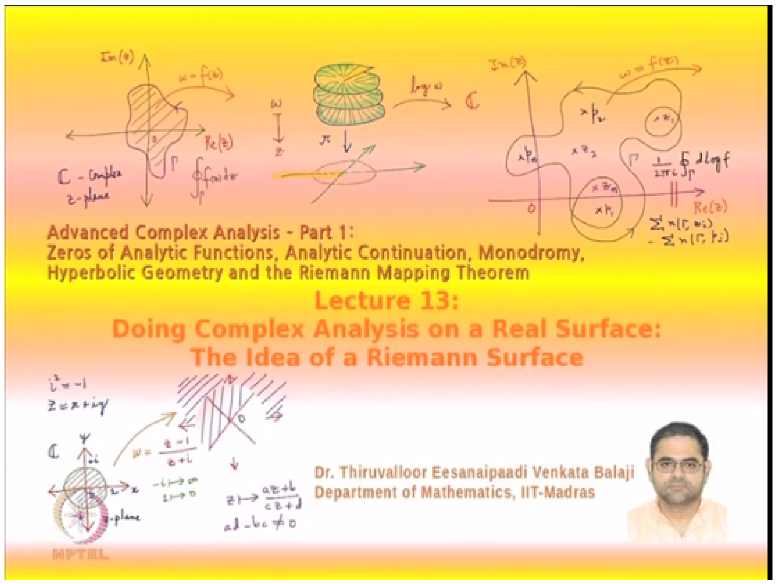

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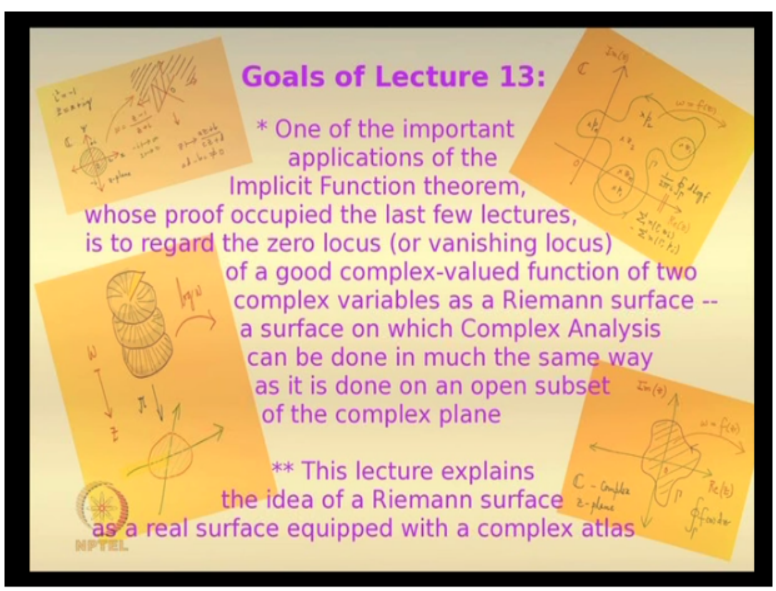

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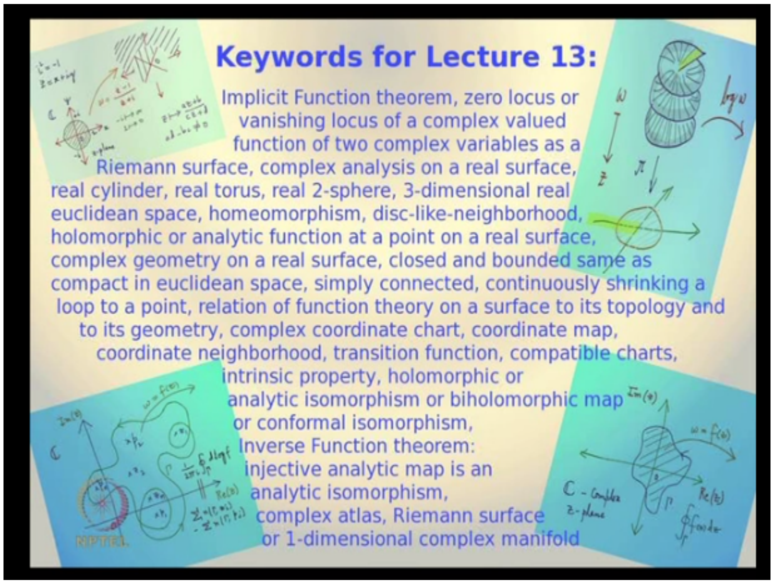

Okay so, what I am going to do now is to try to tell you about one of the important applications of the Implicit function theorem which is to look at this 0 locus of a function of two variables as Riemann surface okay. So, of course with suitable conditions on the function of two variables okay so, of course here I mean function of two complex variables okay.

Because that is the kind of function that we dealt with when we did the Implicit function theorem. So, this recalls this requires you know uh trying to tells you what Riemann surface is so, that something that I will have to quickly recall for you okay. So, let me explain that.

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So, you see so I **so I** must say the following thing there is a in this same NPTEL programme and under which we are we are having these this course. There is another set of lectures for another course on Riemann surfaces which I have given and there you will find a detail exposition of about Riemann surfaces. So, one can I mean you can refer to that when you want more details.

But I will try to be as be possible and to just give you an idea of the kind of things that we need okay.

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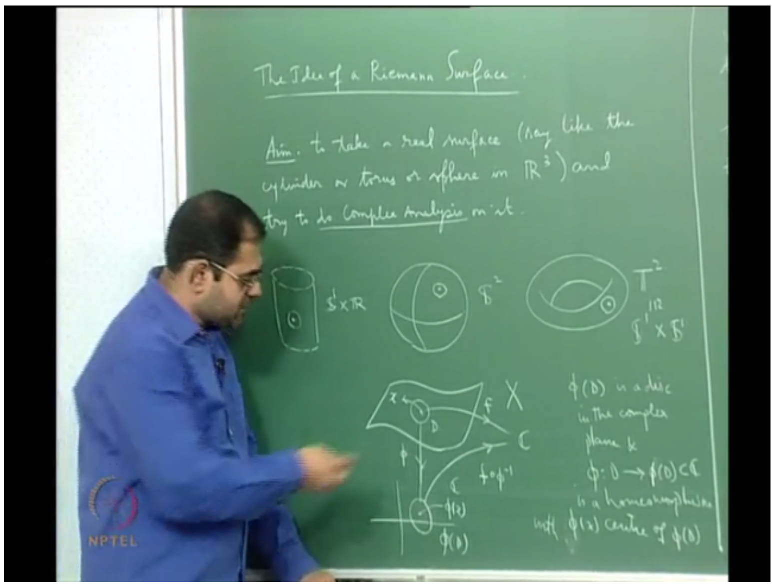

So, **so** what is the idea of a Riemann surface so, what is that so, the aim is to take a real surface say like the say cylinder or **or** torus or sphere in R3 and try to do complex analysis on it okay. So, **so** the idea is well you have for example cylinder host by in infinite cylinder which is which for example you can think of a s1 cross R okay s1 is the unit circle .

Unit circle cross a copy of the real 9 we will give you cylinder okay and then you can also think of the two sphere s2. So, which is I mean this is the two sphere in three space alright or you can think of also the torus which is t2 okay which is actually well this is same s1 cross s1 by that I mean it is homeomorphic to s1 cross s1 okay.

So, these are all you know surfaces that we can imagine sitting inside R3 three dimensional real space okay. And the point is it if you take any point on such a surface and take a small disc like neighbourhood. Then you can it is homemorphic to a disc in the plane okay it looks like a disc on the plane. So, you know if I take a point here and then I draw I cut out a disc like neighbourhood from it.

And palatinate I get disc on a plane I can do the same thing to a point here okay and I can do the same thing to a point here okay. So, of course the surface is curved but if you take as point on the surface then you take a small neighbourhood about that point. And you flatten it okay then it looks like disc or a neighbourhood of a point on the plane right.

And what is the aim, the aim is I one would like to do complex analysis all the surface okay what do a mean by doing complex analysis okay well got by doing complex analysis I mean define the notion of what a holomorphic function is define the notion of what a analytic function is and then study properties of analytic function try to prove lots of nice theorems about analytic functions.

And more importantly try to see how the properties of analytic or holomorphic function change as the objects changes. So, you expect holomorphic function on this to have something to do with the fact this is the cylinder you do not except the holomorphic functions on this it be the same as the holomorphic function on this at least the underline objects are difference.

So, you except some you except this difference to show up somehow when you study holomorphic functions. So, the aim is that you in trying to do complex analysis you try to define and study holomorphic functions on each of these surfaces. And then you try to see whether analysing such functions gives you some information about the geometry of these objects and thus for example distinguishing these objects okay.

So, the sphere has among the three these two are special when compared to this in the sense that they are compact okay because these two are both closed and bounded subsets of R3. So, they have they are compact alright there is this it this is a infinite cylinder okay. It is not bounded and you know subset of Euclidian spaces compact (()) (08:19) closed and bounded.

So, this is certainly closed but it is not bounded okay so, this is not compact and therefore you know so, these are topological properties. When you have many other properties for example some other topological properties for example among the three this is the surface that is simply connected the notion of simply connects surface is the property that you take a loop on that surface.

Then you can continuously swing that loop to a point without going away from the surface okay. And you can see that I can do this on the sphere I cannot do this always on a torus. Because you know if take a loop that goes around the tube like thing okay that makes of the torus. When I can never swing it to a point okay similarly if I take a loop like this I can never swing it.

Because of the hole in between okay so, this is no simply connected where this is simply connected. And similarly this is not simply connected because if I take a loop that goes around because of the hole in between I can never swing it to a point on the surface without leaving the surface. So, this as these two are compact this is not compact, this is simply connected, this is not simply connected, this is not simply connected okay.

So, there are so many topological properties and then there are I mean you can go on to do some kind of differential geometry okay. Where you try to study the curvature of your object then you that see if I draw this sphere has a constant curvature alright. And then if you look at the torus it depends on what kind of curves I take on the torus alright.

So, there are lot of things it geometrically you can on these things on these objects. And you the question is always to analyse to see how when you analyse function on these objects. How properties of functions reflect these geometric properties see that is the whole quest of doing analysis with geometry in mind okay. So, we always try to do analysis with geometry in mind namely you try to do analysis on object which means study functions on object.

And then try to see how properties of these functions how bring out the geometric properties of the object. By geometric properties one means of course topological and much more complicated structures on the object okay fine. So, the aim is to do complex analysis on these objects or more generally you know I could just think of some surface in three dimensional space okay.

You can think of a surface like this of as a wave if you want okay is move to wave. And then well if I take a point on the surface and then you will take a small disc like neighbourhood about that point. Then the question is you can flatten this out and again it will look like a point on the plane. And then you can ask how to do complex analysis on this.

So, the aim is I will have to define what is meant by an analytic function at a point. So, to begin with I have to define the notion of when a function is analytic at a given point. And then if I then I gives that definition to say when it is analytic an open subset okay. And then I can also use that to say when it is analytic on the whole surface. In this way I will get a analytic functions on the whole surface.

And I study all these analytic functions I study their properties and the in the final hope is that when I study the properties of these functions the they that should bring a about that should out some geometry hidden in these surfaces okay. So, some of the geometry should be captured by that so, the aim is therefore if you give me a surface like this. How do I define a analytic function at a point of the surface okay.

So, let us look at the natural way to do this okay the natural way to do with this is well . So, here is my surface X alright X you could think of X either cylinder or the sphere or the torus okay. And there is a point p there is this point p on this on the surface alright may be I will use x because the surface is denoted by capital X.

And then I have this I call this as D because this is a disc like neighbourhood and what do I mean by saying that it is a disc like neighbourhood. It is actually homeomorphism to a disc on the complex plane okay namely if I translate the statement then it is a disc like neighbourhood into formal mathematics. I am just saying that I have chosen a homeomorphism may be a topological isomorphism of this neighbourhood okay with say the unit disc in the complex plane.

But if you want the point x going to this the origin okay or more generally I could of course chosen it to be a homomorphic to any disc finite radius and with x going to the centre of the disc okay. a homeomorphism so, let me call it as ffe okay which is defined on this D okay into the complex plane okay. So, fie fie of D is a disc in the complex plane.

And fie from D to fie of D inside c is a homeomorphism so, what I have done is i just translated the statement that D is a disc like a neighbourhood of the point small x on the surface okay which means that I have chosen a homeomorphism of D with a disc in the complex plane. And of course I am not writing it fie of x goes so, x under fie goes to fie of x and in fie want you can think of fie of x to be centre of the disc okay.

So, you know with fie of x centre of the disc fie of D okay and so, this is what I mean by disc like neighbourhood of a point. And now suppose on this disc like neighbourhood suppose I have a function F define on this with values in complex numbers alright. So, I have complex value hood function defined on this disc alright and what is my aim.

My aim is to tell when that complex value hood function is analytic at the points small x okay remember our aim is to do complex analysis on the surface which means I have to define and study analytic functions on the surface okay. But to define an analytic function on the whole surface the first step is to define an analytic function at a point okay. Once you define analytic function at a point.

Then you define an analytic function on a set an open set to be function that is analytic at each point. So, the problem is first firstly to define an analytic function at a point. And so, what does it mean it means suppose you are given a function in the neighbourhood of a point okay. When will say it is a analytic at that point that is a question so, here is my points small x on the surface capital X.

I have this function F define on this neighbourhood on this disc like neighbourhood o fthi spoint small x which takes complex values okay. And my aim is I want to define when F is analytic at small x okay and the see the it is very easy to see how so, what I am go to do some I am going to just say for example draw diagram here. So, here is my complex number complex plane and this is my fie D.

This is my disc the image of the disc the image of this disc like neighbourhood D here is a disc. And it goes to the point which is well this point is fie of x that is the centre of this disc that is where that is that fie maps small x2 okay. So, of course this fie actually okay of course please do not confuse it with an null set a single for the null set okay. So, you know now what I am I going to do see the complex analysis is that.

We have studied only allows as we have done complex analysis on the plane right if you have function defined in neighbourhood of appoint on this plane. Then I know what it means to say that the function is analytic okay. So, I can use that now to tell you when F is analytic at x because you see pie take fie inverse mind you fie is a homeomorphism it is a topological isomorphism which means fie is you know by adjective it is continuous.

And the inverse of fie is also continuous that is what homeomorphism so, this so, there is this map in this direction also which is fie inverse that is the continuous map that is also topological isomorphism. The inverse of a homeomorphism is also homeomorphism so, you have a topological isomorphism from here to here. And then I can followed by this function F so, what I get is I will get a composition like this.

And this composition is fie inverse followed by F okay so, I go by fie inverse and then I apply F. And why is that helpful because that is the function from an open disc in the complex plane to the complex plane. And for such a function I know how to define when it is analytic so, it is very simple so, what I will do is.

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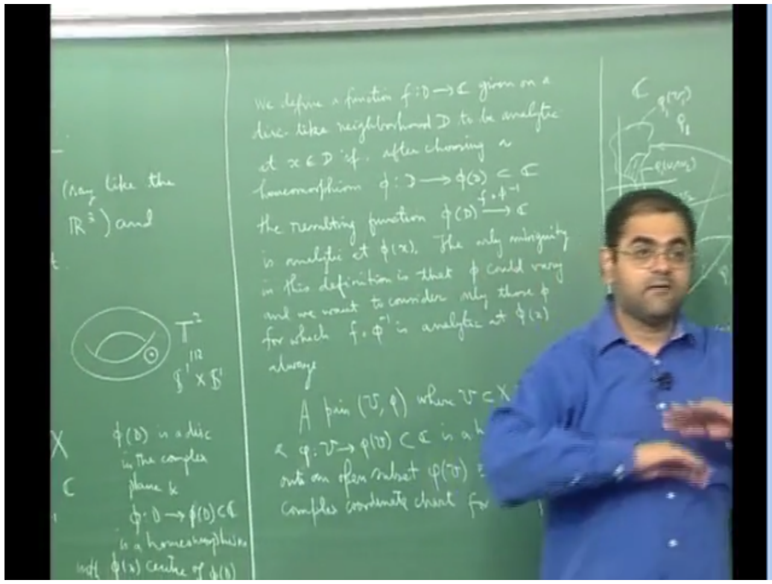

We define a function F from D to c define given on a disc like neighbourhood D to be analytic at x belonging to D okay. If after choosing a homeomorphism fie from D to fie of D inside c the resulting function fie of D to c given by first go by fie inverse. Then go then apply F is analytic at fie x so, you see it is a very simple definition.

I want to say when F is analytic at x okay but this point x and this point D I mean this point x in this neighbourhood this disc D disc like neighbourhood D or identified with the point fie x which is the centre of the disc fie D okay. So, instead of saying F is analytic at x I will say that this composition function is analytic at this point fie x which is corresponds to x. It is a very natural definition okay.

So, this is a nice way of deciding that when a function is analytic at a point okay. But there is a small issue the issue is that you know that the only ambiguity is that I use this homeomorphism fie of this disc, disc like neighbourhood with disc in the uh complex plane okay. But there could be many homeomorphism of this disc like neighbourhood with other disc like neighbourhoods, or other neighbourhoods in the complex plane okay.

And then the question is that if I change fie okay my function F circle fie inverse will change fie of D itself will change okay. And fie of and the point fie of x will change therefore. And my function F circle fie inverse will change okay and then if I my definition is sensible then I should either always get that this is analytic at fie x for every fie that I choose or it is not analytic at fie x for every fie that I choose of course.

But I of course I want only to choose fie such that it is always analytic so, there is a condition for that the condition is that you know instead of if you change fie okay. Then the change in fie should induce a holomorphic isomorphism between subsets of the open subsets of the complex plane that is the condition okay. We technically we use the word that the coordinate charts.

Before differ by transition functions which are holomorphic so, **so** let me explain that see if you take a open subset of the surface and you give me a homeomorphism of that open subset with an open subset of the plane okay that is called a coordinate chart okay. It is called a complex coordinate chart that is because you are because by giving this isomorphism of this subset homeomorphism of this open subset is the open subset of c what you are doing is that every point here.

You are giving a complex coordinate namely the coordinate on the target so, it is called a coordinate chart okay. So, specifying this fie specifying this pair fie and D is what is called a coordinate chart okay more generally you can specify a pair consisting of a homeomorphism defined on in open subset. This D I have chosen his disc like neighbourhood but I could have chosen an open subset alright.

And such a pair is called coordinate chart and the whole problem is that my definition of analyticity of a function at a point should not change. If I change the coordinate chart if I choose different coordinate charts and the fact that I mean the condition that will ensure that this does not that this happens is the condition that the two coordinates charts they differ by a holomorphic isomorphism.

So, let me explain that more generally so, **so** let me say this the so, the only ambiguity in this definition is that fie could vary okay. Fie could vary and we want to consider only those fie for which f circle inverse is **is** analytic or holomorphic at fie x always okay. So, let me writes little more a pair U, fie where U inside x is open and fie from U to see to fie of U inside c is a homeomorphism onto an open subset fie of U okay.

Fie of U (()) (28:46) just of c okay is called a complex coordinate chart for each point of U okay. So, they example is if you take this disc like neighbourhood surrounding the point small x. And you take this homeomorphism of this disc like neighbourhood with a disc in the complex plane then this pair D,fie that is the complex coordinate chart.

It is a complex coordinate chart for every point of D because for every point of D it gives you point in the complex plane which has coordinates okay. And it does it in a manner which is topological isomorphism. Because fie is a topological isomorphism it is a homeomorphism okay so, instead of choosing disc an open disc like neighbourhood the point and this homeomorphism.

And more generally choosing any open set okay and calling such a thing as a chart alright and then the question is so you know the more general picture will look like this.

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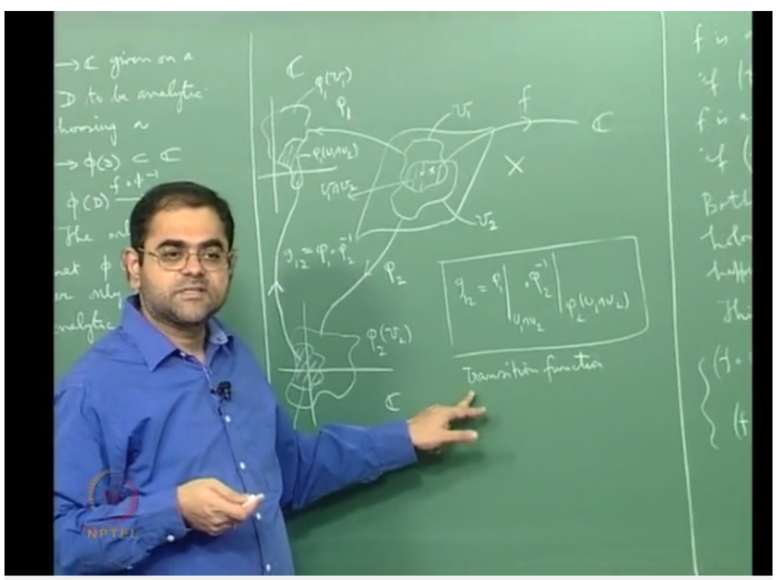

The more general picture is that I will have this surface x see I will have this point small x okay. And well I will have some open set U alright of X and I will have a coordinate chart which consist of U and a homeomorphism fie into the complex plane. So, this will go into the complex plane and it will map this open set into some open set of the complex plane.

So, this will be fie of U okay so, this is an isomorphism topological isomorphism of open subset containing this point x with an open subset of the complex plane right. This special case that I wrote down there was when this look like a disc and this was actually a disc okay. But it can we instead of disc just I am even on open set okay and the reason for that is well analytic functions are and not necessarily defined only on discs there.

The most general set on which you can define an analytic function is an open set okay. And of course if is an open connected set I mean if it is an open set it is a union of discs anyway okay never the less never the point is you know suppose I have a now suppose I have a another chart okay. So, I take another charts so, I have something like this so, here is a so, you know let me number it or okay.

Let me call this is U1, let me call this is fie1 this is fie1 of fie1 okay. Now suppose I takes another open subset of x say U2. Suppose I take another open subset of x and suppose I give you another chart namely I give you another homeomorphism fie2 from U2 into s of z of the complex plane. This is fie fie2 of U2 this is also an open set of the complex plane.

And well this fie 2 is also homeomorphism so, this is a another indent so, U2 is another neighbourhood of the point the same points small x. And fie2 is identification of that neighbourhood open neighbourhood U2 with an open subset fie2 of U2 of the complex plane. So, you know what is happening is that the neighbourhood of that point common to these two sets they have pair of coordinates.

Because you see if I take this intersection namely U1 intersection U2 okay if I take intersection then for every point in the intersection including points small x. I get a if I take it is image under fie1 I will get a point in the complex plane so, that will give me a coordinate alright and if I take it is image under this chart okay. Then I will get a another point so, I am actually giving you two coordinates okay.

And well this shaded region the intersection okay this is U1 intersection U2 this of course you know the intersection of two open set is a again an open set okay. And under homeomorphism an open subset is again identified it this mapped homeomorphically on to an open subset of the target. When you have a homeomorphism of one space into another it is a topological isomorphism of one space into another.

Then an open subset of the source space is carried by this homeomorphism onto an open subset of the target space and if you restrict the homeomorphism to that open subset. Then it is again a homeomorphism from that open subset to it is image. So, if you take this U1 intersection U2 what will happen is that is an open subset of U1 and of U1. And under this map fie1 which is a homeomorphism it will go to well it will go to some open subset of fie1 of fie 1.

So, it will go to some shaded region here okay and this shaded region will just fie 1 of u1 intersection U2. It will just it is just like fie1 gives you an homeomorphism namely a topological isomorphism of U1 with fie1 of few and it will also give you when restricted to U1 intersection U2 homeomorphism of U1 intersection U2. And it is image which is fie1 of u1 intersection U2 in the same way.

If I take the image of under fie2 of U1 intersection U2 I will get another open set here I will get an open subset here. And what is this, this is just fie2 of few1 intersection U2 it is just the image under fie2 of few1 intersection u2 which will carry u1 intersection U2 isomorphically topologically isomorphically that is homomorphically onto this subset okay.

So, what will happen is you see therefore I have a map like this so, I have a map from so, let so I can define a map like this okay. And this map is you first apply so, it is this map is from fie 1 of few1 intersection u2 to fie2 of U1 intersection U2 okay. And this map is you first apply few1 inverse okay and then apply few2 right. So, just for notational reasons let me look at the direction.

So, let me call this is g12, g12 is well yeah let me call this is fie 1 circle fie inverse okay. and therefore it has to go I would apply fie2 so, it is going to be it is like this not like this but it is like this. So, I apply fie 2 inverse and then I apply fie1 okay mind you when I write fie1 circle fie2 inverse it is not defined everywhere okay fie2 inverse this I am considering it only on this shaded region which is the image under fie2 of U1 intersection U2 okay.

So, what I mean by g12 is g12 is actually fie2 inverse restricted to fie2 of U1 intersection of U2 followed by fie1 restricted to U1 intersection U2. This is what it is but if I write it like this it will looks terrible okay so, I am applying I am taking this shaded set which is fie2 of U1 intersection U2. I am applying fie2 inverse to that mind you I can apply fie2 inverse to that because fie2 is a it is a topological isomorphism.

So, it is inverses also are topological isomorphism so, I apply fie2 inverse so, I am restricting fi e2 inverse to this set that is what I would written here okay. So, this vertical line denotes restriction of a map to a subset right and then what I am doing is that so, if I take fie2 invers4 and restricted to this subset. It will carry me carry it will carry this into this shaded region which is U1 intersection U2.

And then there I apply fie1 so, which means I am applying the map fie1 restricted to U1 intersection U2 that is what I would written next. This is the map fie1 to U1 intersection U2 and this is called a transition function it is called a transition function. And you know what is the importance of this transition function it is a following see suppose I have a function defined in an neighbourhood of the point okay.

Suppose I have function defined in a neighbourhood of the point you know to say that the function is a analytic at that point. I have to use a chart that is what my definition says alright so, you know if I have a function defined on U1 intersection U2 with values in the complex plane okay. Then to say that the function is a analytic at the point x okay I can either choose this chart or that chart alright.

Because to say that a functions analytic at a point I have to use a chart that is how I do it alright. And the point is a I want to do it in such a way that it is consistent okay namely I do not want it to be analytic with respect to one cart and not analytic with respect to another chart that means there is some restriction on the charts. When the overlap and that restriction is that this transition function should be holomorphic isomorphism okay. So, see so, let me write that down and explain why.

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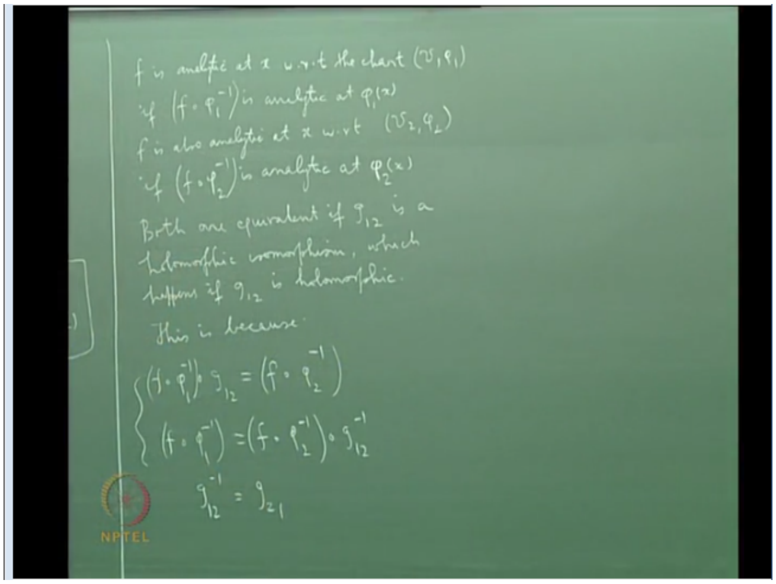

F is analytic at the point x with respect to the chart U1,fie1 well if what is the condition I so, you know this point x to a point here okay which is fie2 fie1 on the x. And what is the definition of F being analytic at x I compose F with fie1 inverse and by say fie1 inverse followed by F is analytic at fie1x. Now that is my definition here so, the definition is F circle fie inverse it is analytic at fie1x okay.

But then F is also analytic at x with respect to the chart U2,fie2 if F circle fie2 inverse is analytic at fie2x. What does this mean this means I am trying to define F to be analytic at the point x using this chart which means what I do is that I take this point x I take it is image here. I will get a point fie2x and now I am saying is now you look at this composite function which is first apply fie2 inverse.

Then apply F now that composite function is define form an open subset of the complex plane and it is a complex valuehood functions. So, it make sense to say it is analytic alright so, you see what is happening is now I had to two charts. I had two charts I am getting two definitions of the same function being analytic at the same point okay. So, what you can I mean you can see from this that as many charts as I can find at that point okay.

I will get as many definitions of the function at being analytic at that point but you see the property of function being analytic at a point is intrinsic. It should not depend on anything okay all nice property is like continuity differentiability okay analyticity. There all intrinsic to the function you should not these are all properties that should not change if you change coordinates. If you make a nice change of coordinates these properties will not change.

So, the property of function being analytic should not change so, that means all these definition should be consistent with one another. It should not happen that you know my fie1 and fie2 are charts I mean these two charts are such that you know with respect to this chart. The function is analytic at x but with respect to this chart the function is order analytic at x such things should not happen okay that tells me that the kind of charts.

I can consider they should be restricted in some way and the restriction is a following it is a following both are equivalent. If g12 is a holomorphic isomorphism which happens if g12 is holomorphic okay so, you see so, let me make a couple of statements you see first of all I am saying that look at the nice condition that this transition function is holomorphic okay mind you this function is from this shaded region to that shaded region.

So, it is a function from an open subset o fthe complex plane to another open subset of the complex plane okay. And it is actually a homeomorphism because it is a composition of homeomorphisms this way from this shaded region to that intersection is a homeomorphism. Because it is given by fie2 inverse okay and from that shaded region to this shade this shaded region is again a homeomorphism.

Because it is given by fie1 so, this is a composition of homeomorphisms so, it is a homeomorphism okay. But in particular it means that it is one to one mind you okay so, if you put the condition that g12 to is holomorphic. Then you have a one holomorphic function okay but you are seen in the previous lectures that if you have one to one holomorphic function it is a holomorphic isomorphism just putting the condition that holomorphic function is one to one will ensure that the inverse function is also holomorphic okay that.

This is what we saw when we studied the inverse function theorem okay so, the condition that g12 is holomorphic is sufficient to guarantee that g12 is actually holomorphically isomorphic namely that the inverse of g12 g12 inverse that is holomorphic okay. And why does that hell to say that these two definitions are equal that is because of the simple calculation. You see if I take g12 and then apply F circle fie1 inverse okay.

I will get F circle fie2 inverse see because you see g12 is fie1 fie2 inverse okay forget the restrictions do not worry about the restrictions. Because I mean writing them also will make the notation complicated g12 is fie1 composition fie2 inverse okay. If I plug it in here I will get F composition fie1 inverse composition, fie1 composition fie 2 inverse and that will and because fie1 inverse composition fie1 is identity.

I get F composition fie2 inverse so, what this tells you is that this function F circle fie1 inverse it differs from the other function F circle fie 2 inverse by a function which is the holomorphic isomorphism. So, you know if this is holomorphic and then because this is holomorphic the composition will become holomorphic. So, this will become holomorphic conversely if this is holomorphic I can move g12 to the other side by operating by g12 inverse.

Because mind you g12 inverse is also a holomorphic isomorphism and that will tell you taht this will be holomorphic. So, this is same as saying and this is equivalent to you know F circle fie 1 inverse is F circle fie2 inverse composition g12 inverse. And g12 inverse is if you look at it carefully it is g21 the way we have define it and g12 is holomorphic if and only g21 is holomorphic that is because an injective holomorphic function as a holomorphic isomorphism okay.

So, the beautiful condition is that if you are only restricting to deciding the analyticity of a function by using charts which whenever they overlap they satisfy this condition that the transition functions are holomorphic. Then for all these collection of charts the holomorphic the analyticity of a function the holomorphicity of a function will not change okay. So, the moral of the story is that to do analysis a surface you just give me a collection of charts oaky which covers every point.

And in order that the notion of a holomorphic function at a point does not depend on your chart make sure that whenever two charts intersect they are compatible in this sense namely that the corresponding transition functions of holomorphic okay. So, if you therefore if you start with a surface and cover it by such a collection of charts which are compatible to one another. Then you will be able to concretely tell when a function on that surface is holomorphic at a point.

And therefore when it is holomorphic on an open set and so on so, you can study you can define and study holomorphic functions. So, what do have what this discussion tells you is that if you have a if you want to do complex analysis on a surface you have to endo that surface with a collection of charts which covers all all of the surface which covers every point of the surface.

And with their condition that whenever two charts intersect the corresponding transition functions are holomorphic such a collection of compatibility charts is called an Atlas okay. And a surface given an atlas is called Riemann surface. And is called Riemann surface because it has first studying where U1 and basically that is the structure that you mean to define and study holomorphic functions on the surface okay. So, let me write that down.

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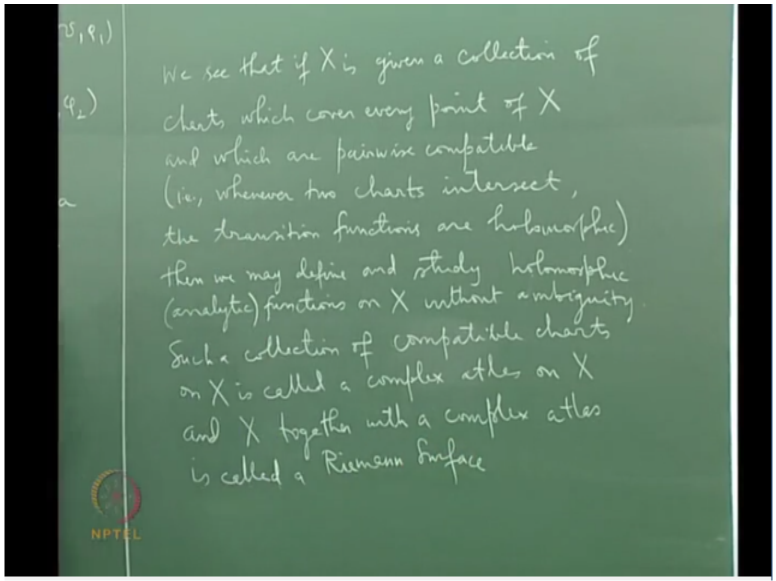

You say you see that if x is given a collection of charts which cover every point of x and which are compatible which are pair wise compatible that is whenever two charts intersects the transition function are holomorphic okay. Then we may define and study holomorphic functions of course sometimes I use the word analytic sometimes I am use the word holomorphic.

But they are one and the same okay so, we make define and study holomorphic at this analytic functions on x without ambiguity such a collection of charts on of compatible charts on x is called a complex atlas on x. And x together with a complex atlas is called a Riemann surface okay. And the Riemann surface is the correct structure that allows you to do complex analysis this is answer first okay of course.

The important point is that you could have different sets of you could have different atlases giving different Riemann surface structures on the same set x okay. And then it becomes a question of geometry as to how, how many such structures are there in whether they are isomorphic to one another or not and how these structures reflect the geometry of and the topology of x and so on and so for things.

So, very interesting area of research and study okay so, I stop this lecture by just saying that the reason why I did all this is it tell you that if you take a function of two complex variables and if you put some nice smoothness conditions on the on that function namely you put the condition that at least one of the partial derivatives does not vanish at each point.

Then the locus of zeroes of that function becomes a Riemann surface and why it becomes the Riemann surface is because of the Implicit function theorem. So, the Implicit function theorem allows you to look at the 0 locus of a complex function of two variables oaky as a Riemann surface. And therefore on the set on that 0 locus you can do complex analysis okay that is an important application of the Implicit function theorem. I will explain that in detail in the next lecture.