An Invitation to Mathematics

Prof. Sankaran Vishwanath
Institute of Mathematical Sciences, Chennai

UNIT
Combinatorics
Lecture - 09
Permutations, Combinations and the Binomial Theorem

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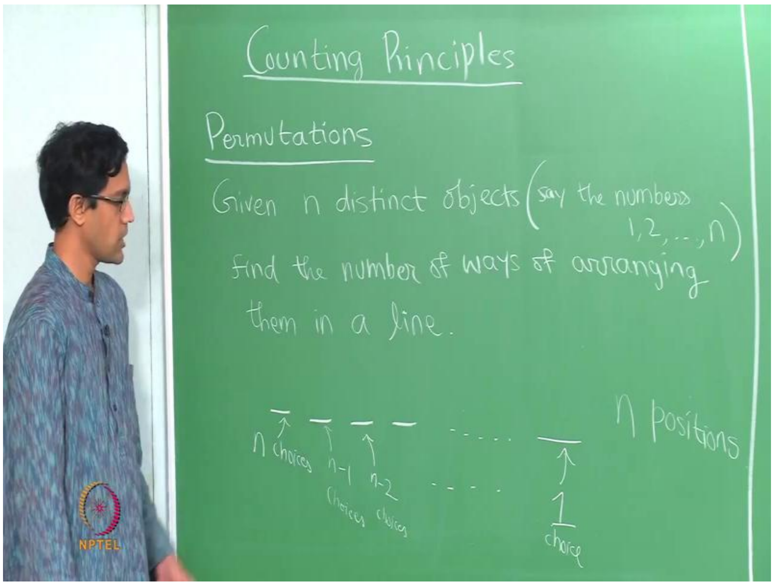

Welcome back, today we will talk about some basic counting principles. And so the most important two motions here are those of permutations and combinations, so let us start with permutations. So, what is a permutation? It is a way of arranging n given objects in all possible ways. So, let us write this out formally, here is a question given n distinct objects. So, where n is some number 1, 2, 3 and so on, so natural number given n distinct objects and often it is convenient to just think of those n objects as just being the numbers 1 to n say the numbers 1, 2, 3 till end.

So, we think of this as being the following problem you are given n positions. So, denoted this way along in a straight line and in each position or box what you want to do is to place one of these n objects. So, the number of ways of doing this is given by sort of the usual counting principles, which is sometimes called the product principle.

You first figure out how many ways there are of filling the first position, this first place here there are n you can put any of the n distinct objects. So, here there are n choices for filling the first position. Now, having filled the first position, if you sort of consider the second one there are n minus 1 remaining choices. So, now, once you fill the first one there are n minus 1 remaining choices with the first two filled, there are n minus 2 remaining choices for the third position and so on.

Still when you finally, get to the very last position you are more or less forced to pick the object that is left, so that is only one choice remaining. And the multiplicative principle or the product principles says the total number of choices is just the product of the number of choices at each one of these steps.

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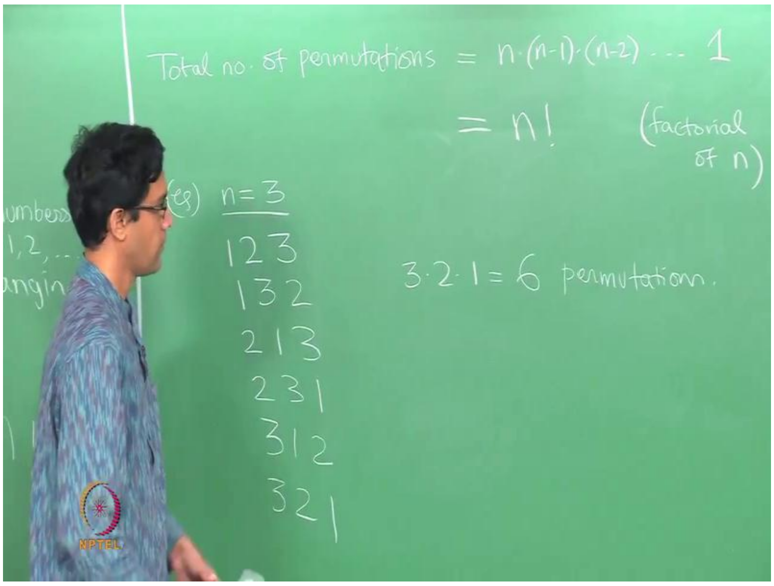

So, the total number of ways, total number of permutations of the numbers 1 through n is therefore, n times n minus 1. So, this is the factorial of n, so for example if you take n to be 3, so you look at the three numbers 1, 2 and 3 and so the number of ways arranging them in a line there are 6 of them. So, it is 1, 2, 3, so these are the three positions, so I put 1 in the first place, 2 in the second place, 3 in the third place, 1 in the first place, 3 in the second place, 2 in the third place and so on and that is the full list, there are 6 possibilities.

So, that is 3 into 2 into 1 that is 6 process 6 permutations. So, of course, that is the usual well known answer the n factorial. Now, one important think to keep in mind is that many of these sorts of problems regarding counting arrangements of various kinds often have equivalent problems which have a somewhat different description. So, sometimes it is important to have a way of realizing that two different problems are really the same problem. So, in the case of permutations here is another problem which is really the same thing in disguise.

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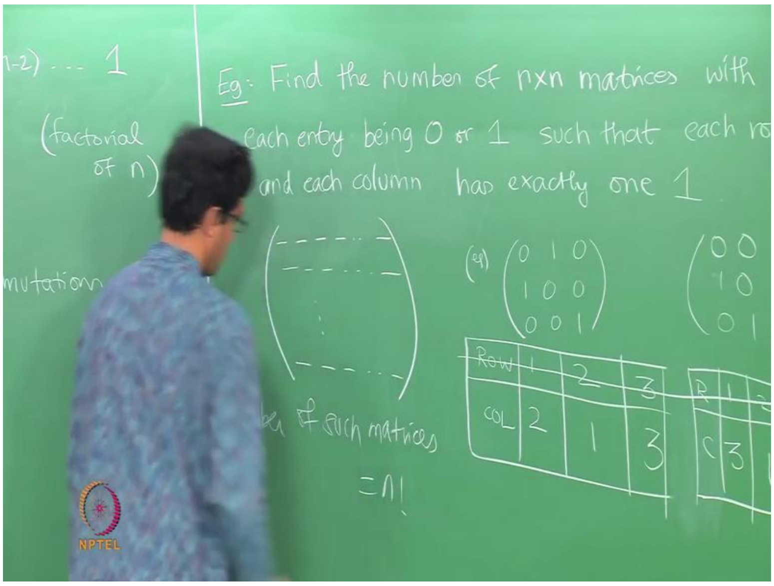

So, here is an example find the number of n cross n matrices logging n being the same number of n with each entry being 0 or 1. So, sometimes it is called 0 1 matrices, every entry is 0 or 1 with the following conditions such that each row and each column sums up to 1 or another way of saying it as exactly it contains a single one, every row has a single one and every column has a single one. And a matrix of course, is just an array of numbers or of entries, it has n rows and n columns that is what the n cross n means.

So, in a sense it is like you have n positions, but... So, in all there are n squared positions to fill, but what one has is restrictions you are only allow to fill 0s or 1s and the sums of each row and each column is a 1. In other words, in every row there is a single one occurring somewhere and every column with a single one occurring somewhere. So, what one does, so how does one solve the problem like this for instance?

So, this looks seemingly different from the problem of permutations. Now, to get a sense of this, so let us just see how would try and do this, so imagine you had such an arrangement. So, here is an example if you did have say for n equals 3, you could put a 1 in the second place, so that rows taken care of you can put a 1 here. So, here is an example of 0 1 matrix in which all rows and all columns have some 1.

So, how do you encode this information? Suppose, you are given a 0 1 matrix like this, what we could do is the following. We could note down in each row, the column number in which the 1 occurs. So, for example, in the first row the 1 occurs in the second column, the second row it occurs in the first column and the third row it occurs in the third column. So, this matrix here we would encode as follows, you would write each row number and each column number.

So, you note down the row and column numbers of the 1’s, so for instants here in row number 1, the one occurs in column number 2, in row number 2 it occurs in column number 1 and in row number 3, it occurs in column number 3. So, 0 1 matrix satisfying the required conditions is the same thing as this data here and observe that, this thing here always has the following form. The row is always going to be arranged as 1, 2, 3 till n if you are in the n cross n case and what can change, when the matrix changes what will change is not this set of entries.

You are always going to list them like that, what will change will be the column numbers, like the 1’s may appear in different columns. So, for instance instead of this matrix if you had say 0 0 1, 0 let say 1 0 0, if you had a matrix like this, then instead of this table you will construct the table which is row column, row number 1, row number 2, row number 3 and in the first row, the 1 is in the third column, the second row it is in the first column and the third row it is in the second column.

So, if you just see that the first thing is the same, so let us for now delete this piece of information it is always going to be the same. So, it is always going to be 1, 2, 3. This matrix is really encoded by this sequence here 2 1 3 and similarly, this matrix here is encoded by the sequence 3 1 2 and so on. So, given a permutation of 1 2 and 3 think of putting those numbers in this, the column numbers think of putting them as a column numbers and then, the row numbers are 1 2 and 3 and then, you can reconstruct the matrix.

So, I hope it is sort of clear at least from this example that, the set of permutations and the set of 0 1 matrices are really, they have the same number of elements. In fact, there is a nice one to one correspondent between them.

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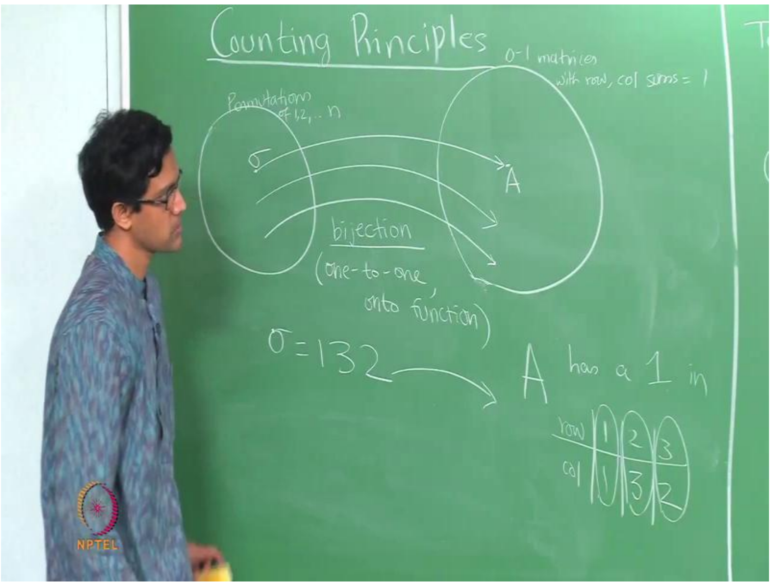

So, on the one hand we have solve it just draw it ((Refer Time: 10:14)) this, you have the set of all permutations of 1 through n, permutations of the numbers 1, 2, 3 till n. And on the other hand you have the set here consisting of 0 1 matrices with row and column sums 1 that is the other set. Now, what we have done according this prescription here is really the following, given an element of this set in another words given a permutation.

So, let us call the permutation something, so I call the permutation as sigma by which I mean sigma could be a sequence of 1s, 2s and 3s. For instance, sigma could be the permutation 1 3 2 if you are taking n equals 3 and corresponding to the sigma what you doing is associating to it a matrix here, let us call this matrix as A. So, what is the association given 1 3 2, you construct the matrix A with the following property. A has a 1 in row number well in row what, in the first row you will have 1 in the first column, so in row, column.

So, in row 1 you see what sigma is that is a 1, in row 2 sigma is a 3 and in row 3 sigma is a 2. So, these are the positions of the 1's, it is in the position 1 1, 2 3 and 3 2, so that is the association. Now, similarly given a matrix satisfying those conditions you can reconstruct the sigma in pretty much the same way, you note down where the 1's occur and then, forget about the row numbers and just read think of the column numbers as giving you the permutation of 1, 2 and 3.

So, things like this, so what we have done really is establish of one to one correspondents between this set here and the set of 0 1 matrices. So, everything here you are associating something there and there is also an inverse map. So, this is what you call a bijection. So, what one has really done here is established a bijection between these sets. So, recall the term bijection means that it is a map between two sets which is one to one and onto, so this is a one to one and onto function.

But, more simply stated all it saying is that you have an correspondents, it mean elements here and elements there, every element here corresponds to something there and you know, similarly every element there corresponds to each elements. So, when two sets are in bijection they of course, have the same number of elements, so the number of matrices which has the properties exactly also n factorial.

So, the number of such matrices also n factorial, so this motion of bijection is rather important we will come up again and again when one is trying to work with counting problems. It allows you to sort of recognize a somewhat more complicated problem as being the same as potentially a simpler problem.

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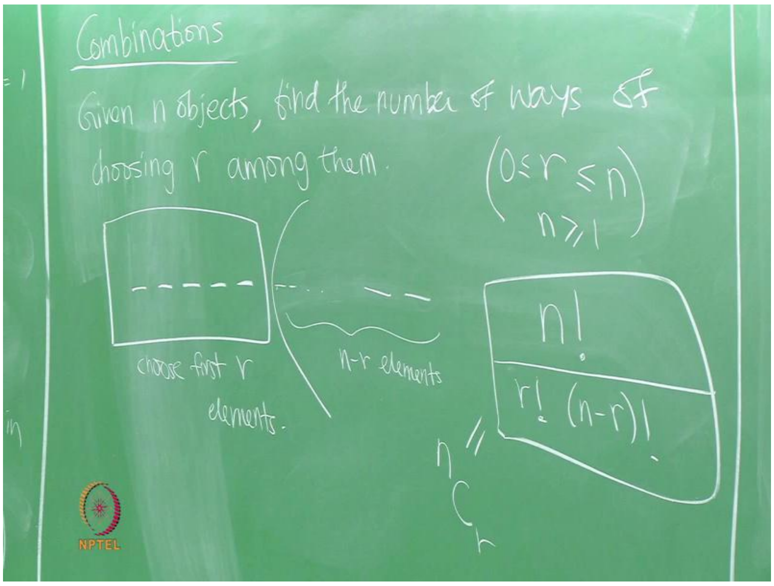

So, I just talked about permutations the other important notion is those of combinations. So, combinations again the typical problem here, so here r is some number between 0 and n, n is a number that is equals to 1. So, given a set of objects and the combination is not ways of ranging, but rather just ways of choosing are out of these are objects and again this is I presume something that you must have same before.

So, the way to solve this problem is sort of by going through permutations. So, what this really amongst two is to say well, let us take these let us arranging these n objects. So, here is the n positions, you arrange these n objects in all possible ways, so you look at all possible permutations and once you have a permutation, you just choose the first r elements of that permutation. So, choose first r elements and sort of put them in your basket that is the choice you need.

So, the question is of course, many different permutations will leave to the same choice of these are elements. So, how many different permutations will leave to the same choice of r elements, well the answer is if you take a permutation you are given some permutation and now, you construct another permutation in which you permute these first r elements. You permute them any which way you want, then in that new permutation that you generate the first r elements are still the same, the just occurring in a different order.

Similarly, if you take a given permutation and then you construct a new permutation by permuting the remaining r n minus r elements. So, here are the n minus r elements that are not chosen, now you permute those n minus r elements in any order you want the final choice of r elements that you are going to pick will remain the same, because you only going to worry about the first arguments. So, what this arguments says is that well you have n factorial permutations in all.

But, out of those r factorial times n minus r factorial, r factorial is the number of permutations of these r objects, n minus r factorial is the number of ways of permuting these n minus all objects. Now, that is the amount of repetition that you have when you count these. So, the final number of distinct choices of r elements that you can generate from the set of all permutations is just n factorial divided by r factorial over n minus r factorial. So, this is now finally, the number of choices, so again like a said I hope this thing is somewhat familiar from before. So, this number is often called n C r. So, number of combinations of r objects out of n objects.

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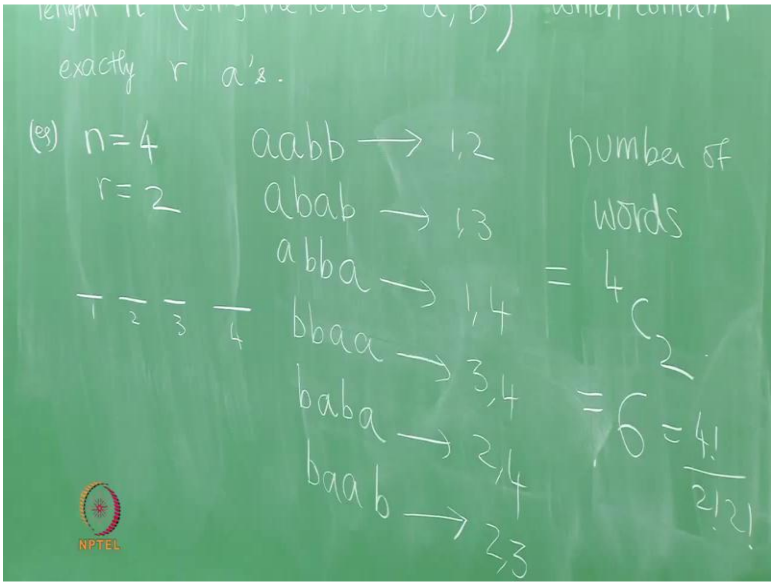

And so again just like I mention in the case of permutations often there are other problems which are really problems of this nature into guys. So, here is another problem you have say two alphabets a and b. So, let us imagine I have two letters a and b and what I want to do is to construct words of length n. So, find the number of words of length n which only use the letters a and b and you given some further conditions such that which contain exactly r a's, r occurrences of a, so to get a sense of what this mean.

So, let us taken example suppose I take n equals 5, r equals 3 what this once or what this asking for. So, restrictions like this simple I say 4 and 2 I want words of length 4 which contain two a's. So, for instance a word which has two a's, so word is just 4 letters from together that is, what the word of length 4 means and I want two a's among these. So, which means the remaining two are b's, so here is one choice, here is another word, here is another word.

Now, similarly have b a b a and let say what do I have now, I have b a a b, so what we have done is return down list of words which have the length 4 and which contain exactly two words and there are six of them, six such words and hopefully I got everything right there in any more words that one can construct. Now, as return this does in quite seen to be related to combinations, but one can actually thing of this pretty much as the following problem, imagine you have four positions which are be filled with a's and b's.

Now, what you want here is to figure out how many ways there are of filling it in with exactly two a's for instants. So, out of these four positions I need to fill two positions with these and so if you just number the position as 1, 2, 3 and 4 to obtain a word like this all we need to do is figure out which two positions will be fill with a's. So, each word of this kind gives me the following piece of information it tells me two positions. So, this is the same information as the two positions which have a's in them.

So, for instants the word a a b b as a's in positions 1 and 2, so you should think of a a b b as corresponding to... So, let us just write this out, so here are the six words we wrote out and a a b b has a's in positions 1 and 2. So, I think of it as 1 2 this is a b a b has a's in position 1 and 3, this has a's in position 1 and 4, this has a's in positions 3 and 4, this is 2 and 4 and this has a's and position 2 and 3. So, word like this is really the same thing as a choice of two positions.

So, what is that, that is equivalent to the following thing your given four numbers 1, 2, 3 and 4 which are the position numbers amongst them you have to pick two numbers which are the two positions where ba's occur. So, the number of choice is therefore, again number of words is the same as the number of choices of two positions which is 4 C 2 which we note to this 6. So, one can just compute it is 4 factorial divided by 2 factorial ((Refer Time: 22:19)). So, now, this business of word is often an important thing to keep in mind it occurs in many different places.

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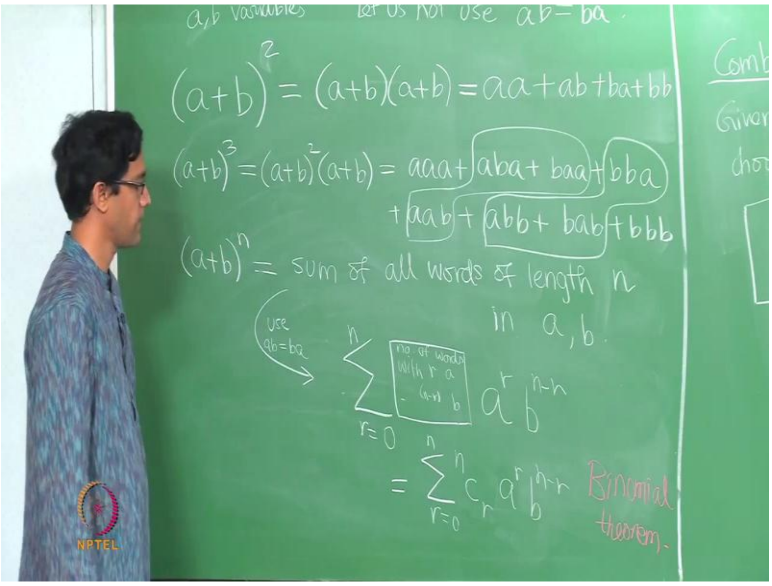

So, well including in this usual identity that one sees is in algebra. So, if you want to compute for a example a plus b the whole squared or a plus b cubed and so on. So, at something that one half and just, so just think of a and b for now as variables. So, we are often interested in trying to find how this expanse out a plus b whole power n, now we will just do this by example we will we look at a plus b whole square. So, there will just give you the well known identity I have a plus b terms a plus b.

But, here is I am just going to do it with the slight change, I will just you know I will use a distributed law write it as a times a. So, I am not going to write it as square for the moment plus a times b plus b times a and plus b times b and of course, recognizance this a square plus b square plus 2 a b. The only difference now is I have written it out like this without using the fact that a b is the same as b a I want to for now keep the distinction. So, let us not use, so let us not use the commutativity property, in other words let us not use this property a b equal b.

So, again if you want to compute a plus b cube it is a plus b square for the multiply by in a plus b. So, I know what a plus b square looks like to that I need to short of attach a on the right and the b on the right and add them up. So, it is a a a, a b a, b a a, b b a these are the four terms which I get if I attach and a to the right of these four terms and then if I attach b I get plus a a b plus a b b, b a b plus b b b. So, that is a plus b hole cube and again notice it just the familiar identity if you use a b equals b a, so the thing of a and b as computing variables this a cube plus b cube plus let see a square b, a square b these three terms I just really a square b similarly these three terms are a b square. So, that is again the familiar identity.

So, observe this is short of. suggesting the following thing. So, observe I just wrote out all the words there with two a's and two b's. So, this is not quite that, but it short of clear from looking at what we are doing that here what I obtain this all possible word’s of length three in a's and b's. I have return every possible word of length 3 in a's and b's. So, the next step is of course, length 4, but it is short of clear by doing this repeatedly that a plus b whole power n is going to give you the following this is just the sum of all words of length n in a and b.

So, observe a plus b squared is just there are four terms which are basically all possible word’s of length 2, here there are 8 terms which are all possible word’s of length 3 and so on in general you have 2 power n terms consisting of all possible word’s of length n in a and b. And now here is the thing if you now use the commutativity property, so let see how this will let us now allow ourselves to use the fact that you can move the a and the b pass each other, if you do this then what happens is the following.

So, we allow the ourselves to use a b equals b a, then what happens is the following then many of these words will now start collapsing into 1. For example, here words which have two a's and one b will all become a square b. So, in general it is going to be an expression, so the question is suppose I want to know I have say a power r b power n minus r. So, I want to know what is the how many times just this terms occur well every possible word of length n which contains r a's and n minus r b's every one then will collapsed give you the same term.

So, the coefficient of this thing here is just the number of words, the total number of words with r a's, r occurrences of a's or n minus r occurrences of b. But, that exactly the thing that we have computed, so this is the sum, now overall possible choices of r that is what this is going to become. But, this is exactly the thing that we computed we said well in general there I did four choose two, but it is clear in general what it is, is exactly n choose r.

Because, a word of length n in which you have r a's is just determine by the positions where the, a is occur. So, this is just n choose r a r n C r a power r and that is exactly what we call the binomial theorem. So, that is indeed the binomial which tells you how to expand a plus b power n and this short of gives you much clear a ideas to how these numbers which really appear when you try and do a counting of certain types of configurations.

Why do these the numbers of combinations appear, when you try and do this algebraic identity, when you try and expand this identity, reason it appear is, because it really counts all words of a certain kind and that is exactly given by the number of combinations. So, next time we will short of look at some further properties of combinations.