An Invitation to Mathematics
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Unit - I
Polynomials
Lecture - 06
Counting number of Monomials - Several Variables

Welcome back, so what we will talk about today is Polynomials in Several Variables. So, all along we have only talked about polynomials with a single variable X, but of course, that is you can generalize that.

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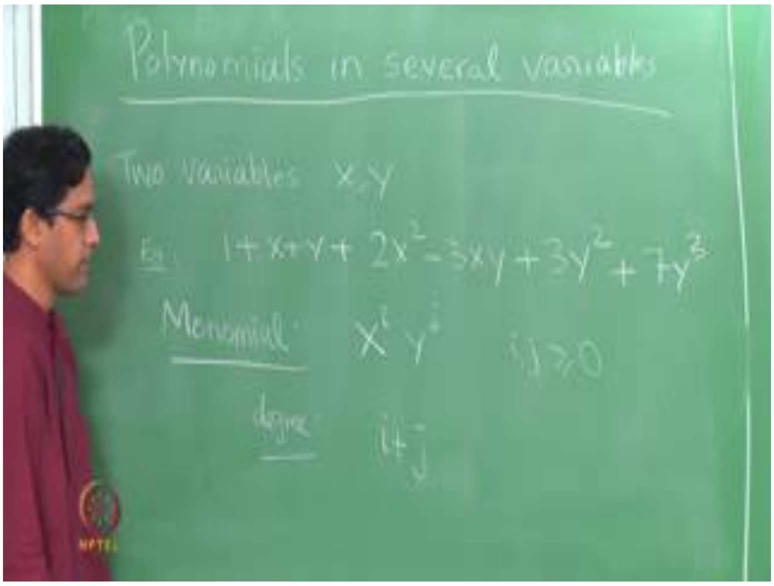

Let us do two variables, let us call the variables X and Y. So, here is an example of a polynomial in two variables, so let start with an example, so I wrote down a long example. So, 1 plus X plus Y plus 2 X square minus 3 X Y plus 3 Y square plus 7 Y cubed. So, well what is this? So, how do you define a polynomial in two variables for instance? Well, it is got terms of the form, some power of X multiplied by some power of Y, so those are terms which are referred to as monomials.

So, what is the monomial? Monomial is just the term of the form X power i, where i is some non negative integer times Y power j, where j is again non negative integer. So, i and j are 0, 1, 2, 3, 4 and so on, so that is of course, what is called a monomial and the polynomial of degree in two variables X and Y is nothing but, a linear combination of monomials, which means you take various monomials. You multiply them by some constant in front, you take finitely many monomials, multiply them by constants in front and add them up, that is what we have been calling linear combinations throughout.

So, an important notion here for a monomial is that of degree. So, just like for polynomials in one variable, the power of X, the largest power of X, which appears is called the degree of the full polynomial. If I take just the single monomial X squared, we said this monomial is of degree 2. Similarly, if we have two variables X and Y, you say the degrees are sum of the 2 power. So, the degree of this monomial X power i, Y power j is just i plus j.

So, if you sort of look at these things here the monomials which appear in this expression, I have 2 degree 1 monomials which appear. This is of degree 2, X Y is of degree 2, Y square is of degree 2, Y cubed is of degree 3. So, you have monomials of varying degrees.

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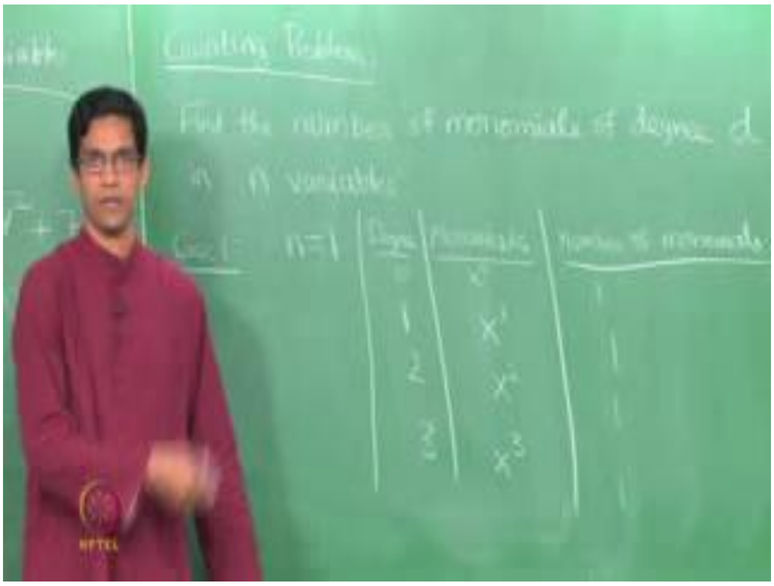

Now, here is an interesting counting problem. So, I want to do the following, I want a count, the number of monomials of degree d. Find the number of monomials of degree d, of course, when I say this, I should also tell you, how many variables I have. So, observe this depends also on the number of variables. So, let just take examples, so here I only talked about polynomials of degree 2, but of course, it is easy to see, what you would do in general, if you add X, Y, Z. You would just say a monomial is something of the form X power i.

So, here is what a monomial would look like, if we had three variables. So, let us do this for one variable, suppose I only have, so when I say n variables n is 1. So, I only have a single variable X, I am going to do the following, I am going to write down the monomials of each degree. So, if I have degree 1, there is only X power 1, if I have degree 2, I can only do X power 2, if I have degree 3, I can only do X power 3 and so on. Similarly, if I have degree 0, I can only do X power 0. So, here it is very simple, there is only one monomial of each degree and so on. So, for every degree, there is exactly 1 monomial.

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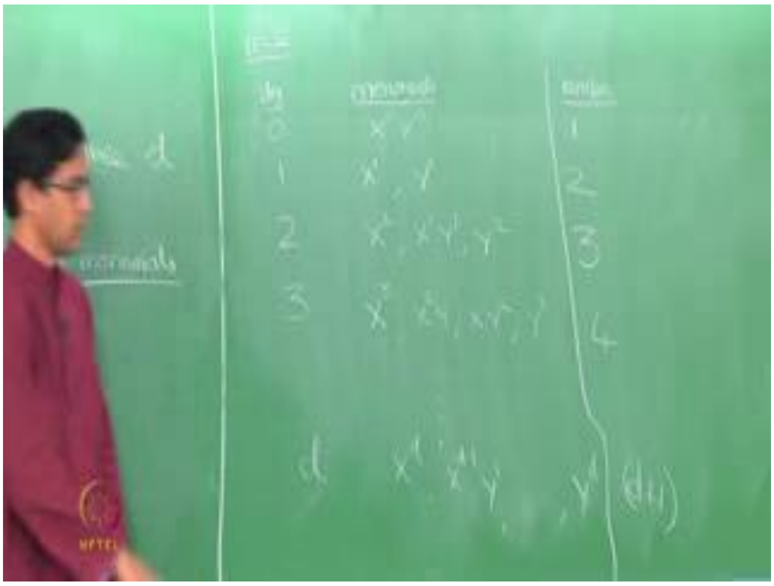

Now, let us do the same thing, if n equals 2, so I have two variables X and Y, suppose I have n equals 2, I have two variables X and Y. Let us do the same thing, let us write down for each degree, let us write down all the monomials and let us count the number of monomials. So, again degree 0 is easy, I can only have a monomial X power 0, Y power 0, that is the only way in which the sum of these numbers can be a 0.

So, as only 1 monomial of degree 0, degree 1 I can either have X power 1 or I can have Y power 1, so there are two possibilities. If I want degree 2, so I can either take X squared, which means the total degree is coming only from the X or I can take X power 1, Y power 1 or I can take Y square. So, a 3 possible monomials, each of which would have degree exactly 2. So, this is now 3, do a same thing degree 3 is X cubed, X squared Y, X Y squared, Y cube. So, there are 4, 4 monomials of degree 4 and so on of degree 3.

So, again here it is pretty clear, how this is going to go. If I have degree d, then here of the monomials X power d, I subtract 1 power of X, increase 1 power of Y. So, 1 keep doing this still I finally land of with Y power d and there exactly d plus 1 such monomials. So, we have manage to figure the answer out in both these cases, if you only have one variable, thus exactly one monomial of each degree, if I have two variables, there are d plus 1 monomials of degree d.

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So, of course, the next thing to do is look at three variables. Suppose, I have X Y and Z, let us do the same thing. Let me write out the monomials and then, we will count the number of monomials. So, again degree 0 is easy, thus only 1 at 0, Y power 0. So, here we have the following monomials, there is 1 of degree 0, there are three fellows of degree 1, there are six of them of degree 2.

Now, here we get to the interesting part, how does this go on? So, of course, it is going to be very difficult to figure out the general degree d case, unless we have some more systematic way of approaching this problem. So, let us do the following. So, suppose I want to write out all monomials of degree 3, involving the variables X, Y and Z. Here, is what we will do, let us only write down the monomials which involve X and Y, those which do not have Z in them.

So, that problem we have already solved, we know which of the monomials of degree 3 in only X and Y. So, those are these four fellows X cube, X squared Y, X Y squared and y cubed. So, let us write those down. So, it is X cube, X squared Y, X Y squared and y cubed, those do not involve the Z. Now, let us do the following, suppose we also want things which have let us see, so now, we also want to know which of the ones which have a Z in them. So, these are the ones which do not have a Z. So, we already have a count of 4 there plus the ones which do involve of Z.

Now, let us look at the ones which involve the Z, what are they going to look like, well there all going to have at least a Z power 1. So, they are going to each of the polynomials here which are, so I am going to write them down here, they are all going to have at least Z. So, let me pull out that power of Z and then, figure out what is left. So, if I write out the monomial of degree 3, which has Z power 1 unit, what is left must again b, a polynomial in X Y inside, monomial in X, Y, Z, but of one lower degree.

Because, Z power 1 already accounts for degree 1, what I am looking at is just the remaining degree. So, what I do is, I take each of these preceding polynomials and I multiply them by Z; that is all I need to do. So, I take each of them. So, I have X squared time Z. So, I written in the Z X square, Z Y square, Z multiplying Z square Z multiplying X Y, Z multiplying Y Z and Z multiplying X Z.

So, each of them has some Z in it, so that Z, you pull out and what is left is just a degree 2 monomial. So, something of one preceding degree and that number we know there were exactly six of those. So, we know the answer now, it must be 4 plus 6. Now, this thing that we just did here, this basic idea, the procedure is very important.

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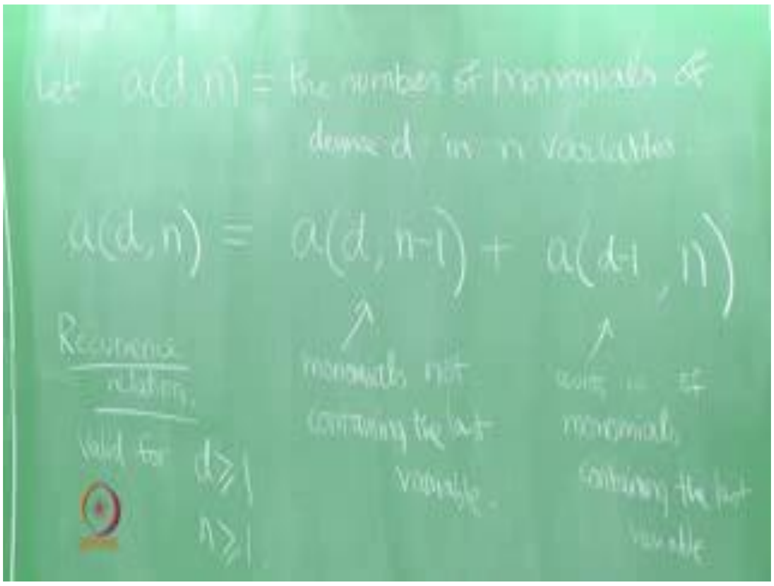

So, let us do the following, let us give this number and name, let us call the number of monomials of degree d in n variables, let us call it a of d comma n. So, that is the number. So, what we have just proved here, if you sort of look at the argument here, it says the following, if I want to say how many monomials there are of degree d using all the n variables. Here is what I can do, I pick one of those variables, I take the last variable and I look it the monomials, which do not involve the last variable not containing the last variable. So, these only involve the first n minus 1 variables.

Now, the monomials of degree d which only involve n minus one variables is of course, the number is just whatever we are calling a of d comma n minus 1. So, that is those which do not contain the last variable plus the ones which do contain the last variable. Well for that, the idea was the following you know that the last variable will surely appear to at least power 1. You pull out that last variable and you see, what is left.

So, just what we did here, you pull out the Z and what is left is again something in n variables, which has one smaller degree. So, plus number of monomials of degree in n variables, but of one smaller degree, take every one of them and hit it by Z, where Z is the name of the last variable. So, this fellow here actually counts, this counts the number of monomials, which involve the last variables contains the last variables. So, by taking care of these two disjoint cases, what you are obtained is the total count.

This is all possible monomials of degree d is a once will do not contain the last variable and that counts the number which contains the last variable. So, this is now exactly, what we now we kept calling it a recurrence relation, when we talked about the Chebyshev and the Legendre polynomials. This is now a recurrence relation again, but it is a recurrence with two variables, so it is somewhat more complicated.

So, observe, this is again a recurrence relation, meaning it determents the value of a d n in terms of somewhat lower values, either n is smaller or d is smaller. Now, firstly, we should say this recurrence is valid for, what values of d and n, d should be at least 1 and n should be at least 1. So, on the right hand side I have d minus 1, which means this is at least 0 and also have n minus 1 occurring, so is at least 0. So, now we have a recurrence relation which is valid for a both d and n at least 1.

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And whenever you have a recurrence relation, you need some initial conditions as well. Meaning, you need have what happens when d is 0 or unique know what happens n is 0. Further, we know the following things, let me look what is a is 0 of n and let me also ask, what is a d of 0, these would be are our initial conditions. So, a is 0 n would be let us read this, it is a number of monomials of degree is 0 in n variables.

So, we have already done this, there is only one monomial of degree is 0, which is X power 0, Y power 0, Z power 0, every variable power 0. So, there is only one fellow for all n is 0, so that is easy.

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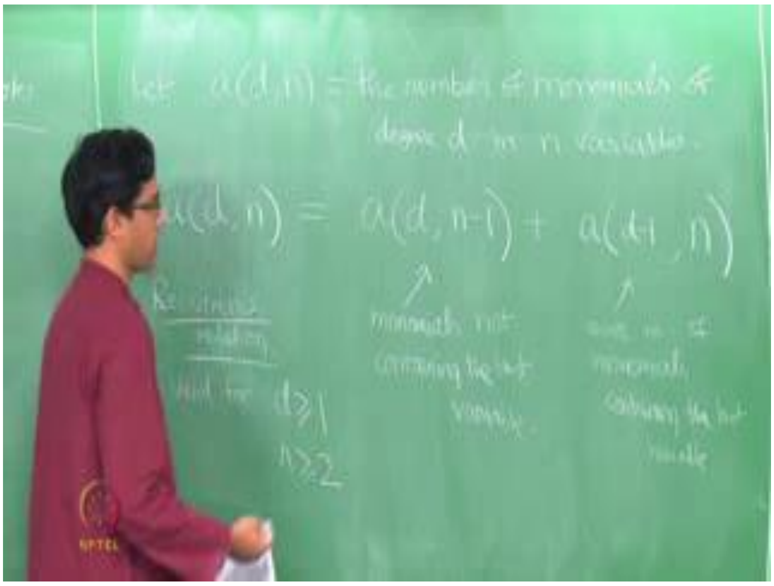

Now, whatever the next thing a of d comma 0, so d comma 0 is, that is probably not something that you want to do, because we would have to define this. So, let me just modify this. So, let me say, I want at least one variable, I do not quite know what it means to have 0 variables. So, then this equation is only 2, if n is at least 2, because on the right hand side I have a n minus 1 occurring. So, which mean this should be at least 1.

So, let me do this, what is a d 1, this would be the number of monomials of degree d in a single variable and that again we have done. Thus, if you only have one variable X, then for each degree d there is a exactly one monomial which is X power d. So, this also 1 for all, so, I should said for all n at least 1 here and this for all, so these are the initial conditions.

Now, this clamatorial problem here is best expression in terms of a table. So, what we should really do is write a table with various values of d and n. So, let me just do this d is on top n is on the bottom, the degree can go from 0. So, the degree goes from 0, 1, 2, 3 and so on, the number of variables goes from is at least 1 here. So, degree goes from 0, 1, 2, 3; the number of variables ranges from 1, 2, 3, 4.

Now, what we are said here is a following, a is 0 n means, if I put d is 0 and n to be anything the value is 1. So, what I am tabulating here are the values of a d n. So, this is the table of values of a d n. So, I am tabulating them, what is change the first column contains all 1's. So, what this says is that, the first column, so if you plug in d equal to 0 and take any value of n, it is all 1's and this condition here says that, if you plug in n equals 1 and then, take any value of d, the values are again 1.

So, certainly the first row and the first column have all 1's in them and now, what the recurrence relations says is, if n is at least 2, which means, you are at least from the second row downwards and if d is at least 1. So, which means you are at least from this column here. So, a d n for all these empty boxes is what the recurrence relations tells you, it says that, if you want to compute the value of a d n in any box, if I want to know the value here for instance.

What I need to do is to look at the one lower value of d, so the value here is the some of the value in the box above it and in the box to the left of it. Solve it saying is that, it is just the some of the values in those two boxes. So, by is not a doing is iteratively you can find the values in all the boxes for instance. Now, I know if I want find this box here, it is the sum of 1 and 1. So, that is a 2, this box here would be sum of 1 and 2.

So, what I have done now to fill the table is more or less use the property that the value in any box is the some of the value in the box above it and in the box the left of it. So, here I have you know this row is 1, 2, 3, 4 and so on the next rows 1, 3, 6, 10 and so on. So, these are the things at we already had computed and now, observe if you want return of for four variables, the number of monomials of any given degree, in four variables that is given by this formula here by this table 1, 4, 10, 20, 35, 56, in five variables which given by this and so on.

So, at the moment this might look like some strange numbers that we are generating, but hopefully it is not all that unfamiliar.

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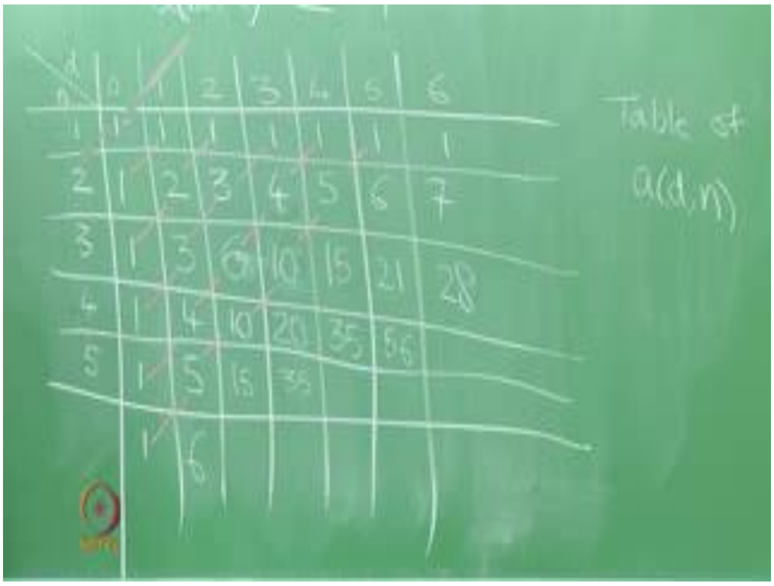

So, in order to sort of get a better perspective on this, once you sort of read this in the following way, so imagine is a 1, 1 1 by read it diagonally and so on just a 1. So, if you sort of read this table in this diagonal fashion, then I hope these numbers are somewhat more familiar in from other contacts. So, will again return to this, when we talk a little bit more about counting principles and we will see, how the general principles will allow us to solve this counting monomials problem, without using this recurrence just by a more general procedure. And but in the mean time, you can sort of try and think about what these coefficients really are.