An Invitation to Mathematics
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Unit I
Polynomials
Lecture – 04
Taylor's Formula

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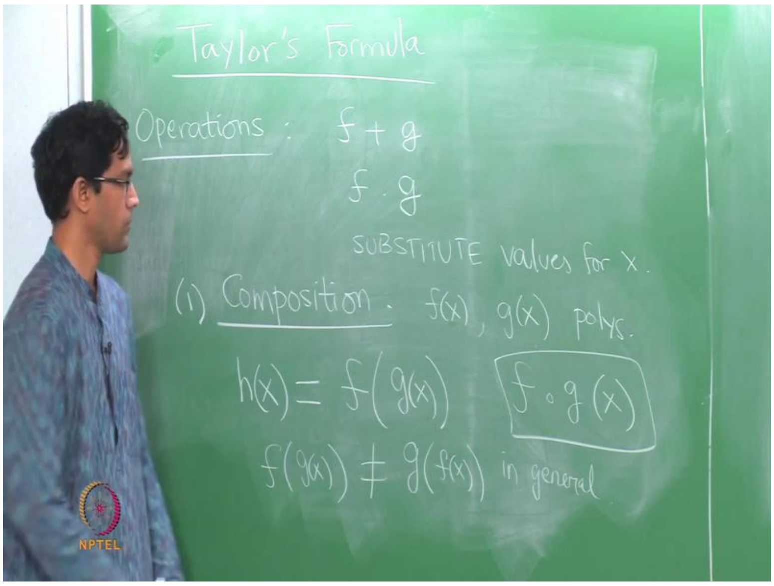

Welcome back, the next thing we will do with polynomials. So, I want to talk about something called Taylor's Formula. But, before I do this, let us just quickly recall, what operations we have defined on polynomials. So, given two polynomials, we had the following things, we could take, say if f and g are polynomials, we defined what the sum means, we recall what the product of those two polynomials means.

And well thus another thing we could do which is a substitution, substitute values for X. So, value X is the variable, so these are modulus of three things we have done so far. You can take sums, you can take products, you can substitute a value for the variable, but there are few more operations that one can perform on polynomials and let us start out by introducing two more operations on polynomials. So, one of them is called composition of course, again must be quite familiar.

So, if f and g, if f of X, g of X are polynomials, you can combine them to get a new polynomial, their composition called h of X is defined to be f of g of X. So, what is this mean? It just means in this polynomial f, wherever you have the variable X, you do not substitute a value for X, instead you just substitute the polynomial g of X in place of X. So, I hope this is familiar from I think seen before.

Now, this is called the composition of f and g, so sometimes this is written as f circle g of X, this is just notation for this. So, this is sometimes written like this, f circle g and key thing to note is that, composition very much depends on the order in which it is performed, so f of g of X may not be the same as g of f of X in general. So, f of g of X need not be the same as g of f of X in general, which means I mean it could be equivalent some special cases, but for arbitrary f and g, it will not be the case that there.

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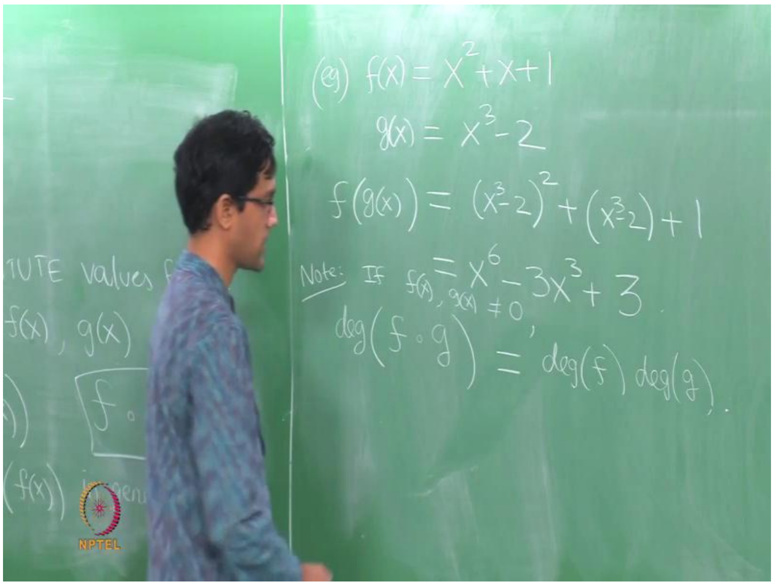

So, let just do an example of composition, just to get a feel for it is basic property. So, if I have f of X is X squared plus X plus 1. So, degree 2 polynomial g of X is X cube minus 2. So, this case a degree 3 polynomial, their composition what is f of g of X mean, it is just you take the polynomial f and plug in this polynomial in every slots. So, instead of X, I plug in X cube minus 2 for f of X. So, it is X cube minus 2 square plus X cube minus 2 plus 1, so that is f of g of X.

So, I want to go of course, expand this out and full, this is X power 6 minus 4 X cube as X cube from there, so it is a minus 3 X cube plus 4 minus 2 plus 1. So, if you expand this out and full, this is what it turns out to be and of course, one can similarly compute in this case g of f of X to see that in fact, it is not the same. Now, what are the key properties of this kind, observe that the degree of f circle g, meaning composition of f and g.

In this case one of them is degree 2, the other was degree 3 and their composition had degree 6 and this is 2 in general. The degree of a composition if f and g, so note if f and g are non-zero polynomials, if f of X, g of X naught zero polynomials, then the degree of their composition is in fact the product of their degrees. So, that is going to be true in general, it is a very easy to see, I just leave this an exercise for anyone, who wants to check this.

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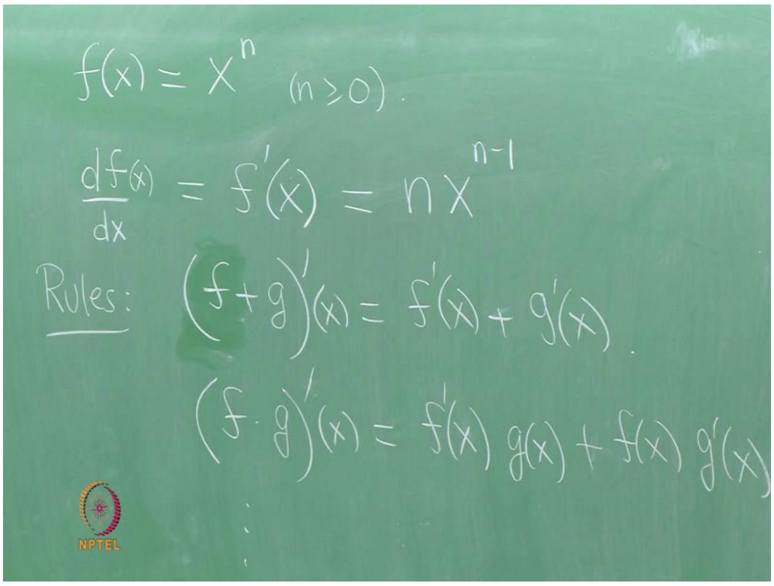

Now, so I will talk about composition, it is one operation on polynomials, here is the second operation that of derivative. So, again it is the same notion of derivative that probably you seen in calculus, except on polynomials you can sort of be define much more simply. It does not require developing a big theory of what limits are and so on and so forth, but it can be just given by simple rules in general. So, derivatives are much simpler, when you only worried about polynomials.

So, what is the derivative of X power n, for example? So, if I take a polynomial f of X, which is some power of X. So, n here is for a since it is a polynomial, I am talking about n greater than equal to 0. The derivative of this polynomial, so denoted value can either call it d by d X of f of X or as f prime of X. So, these are both notations is just n times X power n minus 1.

So, the derivative of a polynomial X power n is just n X power n minus 1, it is a usual rule for the derivatives and what I want to do is sort of see, what happens when I keep doing this repeatedly, so this is what you would call higher order derivatives. So, observe that all the usual rules for the derivatives are of course valid, it all for me derivative still means whatever it means in calculus, so all the usual rules or derivatives are valid.

So, in particular, if I take the derivative of the sum of two polynomials, the answer will be the sum of the derivatives. So, observe that the rules for the derivatives are still the same. So, if I take d or let us call it, if I take f plus g, sum of two polynomials and try to find it is derivative, then it is just the sum of the derivatives or if I take the product of two polynomials and try to take the derivative.

So, observe taking derivative of the product is given by what is called the product rule, which is you take the derivative the first term keep the second term as it is, this is the product rule and so on. So, any other rules that you remember for derivatives in calculus of course, all the same apply here, you can pretty much feel free to use all the rules.

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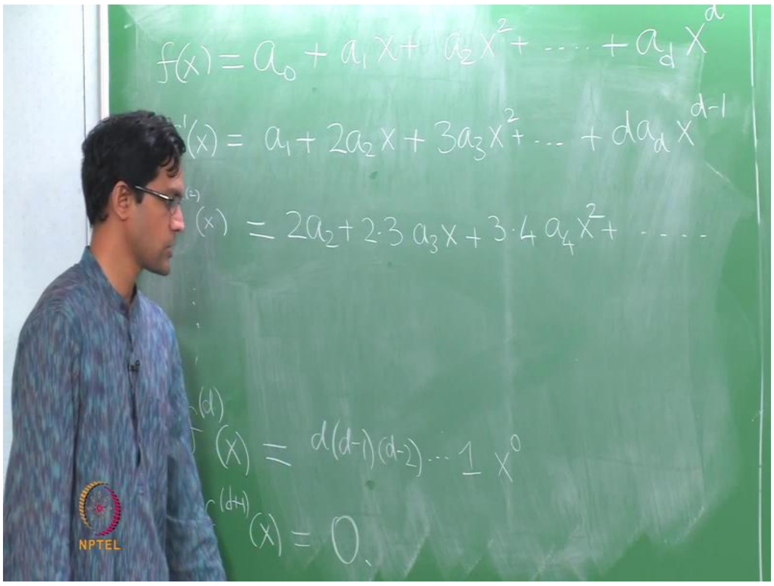

So, now observe the following nice thing which happens, when you keep taking derivatives of polynomials. So, suppose I have a general polynomials, so I just write an arbitrary polynomial has this expression, so degree d polynomial and if you sort of take one derivative, what we call as f prime of X, then a derivative of constant is 0. So, it is a polynomial to start with while the constant term now becomes a 1, so this is really a 1 plus the derivative of this is x square is 2 X and so on, d a d X to the d minus 1.

Now, of course, you can keep doing this, you take the derivative of the derivative, so this is often what is called the second derivative f double prime or sometimes just written as f 2 of X, where the two on tops is in brackets. So, again when you take a derivative again, the a 1 is a constant, so it vanishes. So, it is start with the 2 a 2 plus x square will now again give you 2 X as derivative. So, this is 2 into 3 a 3 X plus the next thing will become let say 3 into 4.

Finally, keep doing this d times, you take a derivative at every step you keep doing this d times, what you finally, end up with is, now only this last term survives, when you do d derivatives. And the derivative of this taken d times is just a d multiplied by d into d minus 1 in to d minus 2 and so on, d minus 1 at every step the power of X will sort of claim down and times X power 0, which is just a 1.

And if you do it once more, since the preceding answer is the constant this just going to be a 0, so this is not the full hierarchy, if you take a polynomial and you keep doing successive derivatives, repeated derivatives to it. What you will get is, it will be a degree d polynomial to start with the degree d minus 1 polynomial d minus 2 and so on. Till it finally, becomes a constant, a degree zero polynomial and the answer final answer is just 0. So, now, here comes the, here is the problem that we want to solve which leads to what is called Taylor's formula.

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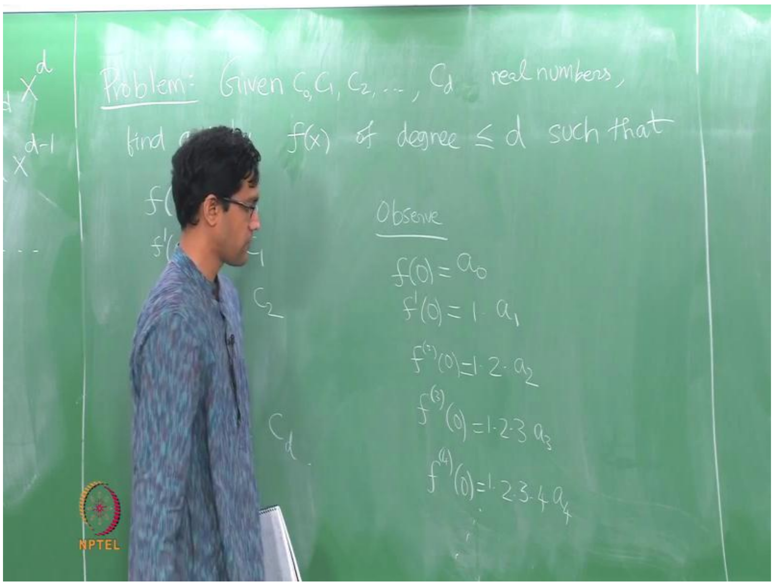

Again, it shares some aspects with the interpolation polynomial. Given real numbers, so let me call them C 1, C 2, so d plus 1 real numbers not distinct or anything arbitrary real numbers, find a polynomial, may be just call it as f of X, so here it sort of nicer to re index this. So, instead of calling them C 1, C 2 till C d plus 1, let me call them C 0, C 1 till C d and you see in a minute while, let just call these d plus 1 real number as C 0, C 1, C 2 or not.

So, I want the following conditions to be satisfied, I want my polynomial to have degree at most d and it should be f evaluated at 0. If you plug in X equal to 0, I want it to give me the number C 0, if I take the derivative of x and evaluated at 0, I want to get the numbers C 1. The second derivative evaluated at 0, should give me the numbers in two and so on till the d th derivative evaluated at 0 should give me C d.

So, in the interpolation problem what we were prescribing was some X values and some Y values and one thing the polynomial which will take those prescribed values which will take the prescribe Y values and those X values. Here, it sort of different there is a really only one X value which is in plane, which is the value X equal to 0, at every point what you really doing is substituting X equal to 0.

But, what you are now given is various things you given the value of the Y value at that point, you are given the value the derivative at the point, you can give the value the second derivative at that point and so on and so forth, given values of all derivatives, but at a single point. In the interpolation problem, you were actually given only the values, there are no derivatives involved, you are only given the values, but at d or d plus 1 different values.

So, but in many ways it is similar, both of them really ask for something like this, you give d plus 1 pieces of data and ask for a polynomial, which has those you know satisfies the data especially. And in fact, it turns out this problem much simpler to solve than the Lagrange interpolation. So, observe that, it is a more or less, all the things we need are already there on the board. So, if you take f of 0, then observe that plugging in X equal to 0 will makes all subsequent term 0.

So, what you get here when you plug in X equal to 0 is only the very first term, you will only get a 0. Similarly, if you take the second line, you plug in X equal to 0, all subsequent terms are 0, leaving you only with the constant a 1 and second derivative will at X equal to 0 will only give you the constant 2 a 2 and X can will give you constant 2 into 3 a 3 and so on.

So, observe that, so this is quite straight forward from what we doing that f of 0 is just a 0, f prime the derivative at 0 is just what, you say 1 times a 1, the second derivative at 0 is just 2 times a 2, third derivative at 0 is 2 into 3 a 3, it just write one more fourth derivative at 0 would just be 2 into 3 into 4 a 4 and so on. So in order to make this little more symmetric, let just throw in a 1 everywhere, 1 into 2, 1 into 2 into 3, 1 into 2 into 3 into 4 and so on.

So, what we gather from here is that in fact these constant a 0, the co efficiency a 0, a 1, a 2, a 3 and so on are just directly obtain in terms of the values of the higher derivatives at 0.

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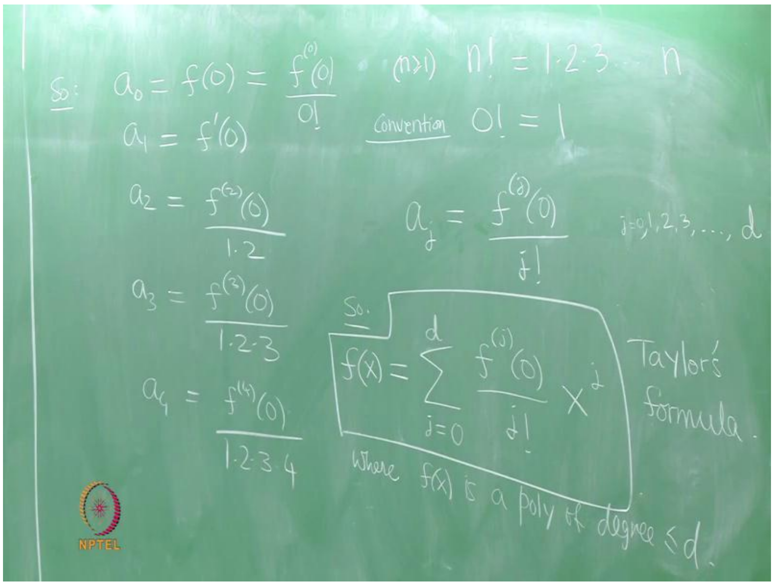

So, what we conclude, so in fact, we conclude that a 0 is just, a 1 is just the first derivative, a 2 is second derivative divided by 1 times 2 by 1 into 2 into 3, 1, 2, 3, 4 and so on and these denominator are precisely, what we are called the factorials. So, observe that in general, if n is a natural number, n factorial just means the product of all the numbers from 1 to n and so this is just another way of writing this is.

So, just say a j is nothing but the jth derivative of f at 0 divided by j factorial. So, this holds for j equals to 1, 2, 3 till d, this is actually j is 1, 2… So, observe from here first derivative divided by one factorial is just a 1, second derivative divided by two factorial, third derivative divided by three factorial and so on. But, in fact, one can sort up just to have more uniform expression, you can also defined zero factorial as 1.

So, this definition really is only n is at least 1, but the usual convention is to define zero factorial also as 1, this is your convention in which case a 0 also has is given by the same expressions. So, observe this is nothing but you take f you can give that the function itself as being the zeroth derivative of itself divided by zero factorial which is a 1. Zeroth derivative just means, you do not do any derivatives, you just leave it as a function itself.

And so a 0 is, if you wish you can think of it as the zeroth derivative evaluated at 0, divided by zero factorial.

So, in fact, with this convention, this is also true for j equal to 0. So, if you wish, you can now change this to say, it is true for j going from 0 to d. So, what is that mean, the jth derivative evaluated at 0 is in fact given that exactly what we require to be C j. So, conclusion is that the polynomial f of X that you want is nothing but so it is the sum. So, it is use the summation notation if j going from 0 to d of, so f of X is just given by the sum of it is higher derivatives at 0, divided by j factorial times X power j.

So, this formula here is what you would call Taylor's formula. So, this is for where f is a polynomial of degree at most d and this is what is called the Taylor's formula. So, this something that holds in general for any given any polynomial of degree at most d, what we have said is, it is co efficient can all be obtain in terms of derivatives of the function. So, the function itself can be written back in terms of the co efficient like this.

So, just to complete the solution to the problem that we mention, if you instead of r given describe what the derivatives are then you just use you just plug these values in there and you figure out what the functions is.

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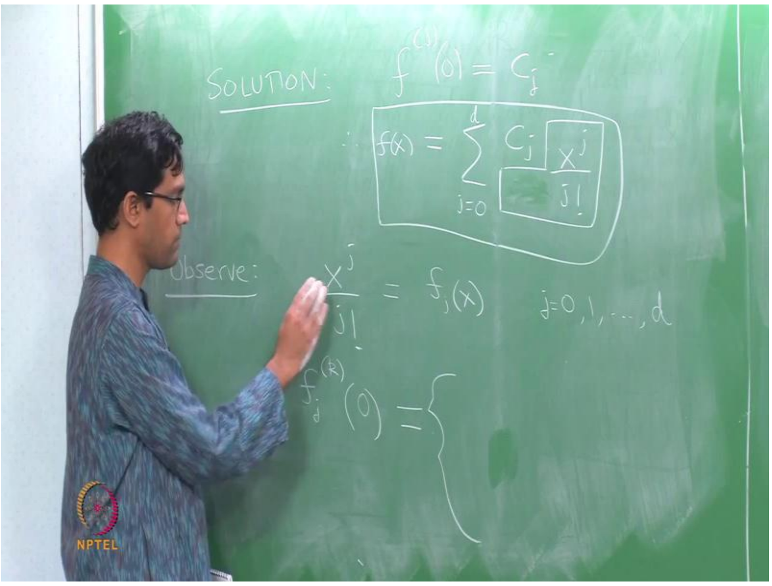

So, what is the solution to this problem here, so the solution to the problem as suppose there is the following that I know what the jth derivative at zeroth, it is given to be g j. So, therefore, my function f is nothing but summation C j by j factorial X power j this is the function that you want, the polynomial of degree at most d with prescribed higher order derivatives at 0.

So, all it is solve this problem now just, so one remark here, observe this may not have the degree exactly d, because you know the dth derivative could be a 0. Suppose, my C d is a 0, it means that X power d occurs with co efficient 0, so this could very well have degree smaller than d. So, all we can says at most d is the best we can say, because we do not quite know what this last real number is.

Now, the other thing here is to observe that, so observe just one more things, since we start of talk about the 0 1 idea at some point, if you look at, so let us just things of this as follows, think of it as c j multiplied by X power j divided by j factorial. So, let me just thing of it like this, so you look at this type X power j by j factorial, let us call it a polynomial something, let us call it f j of X.

So, for each value of j between 0 and d, I have a polynomial called f j of X; it is a very simple polynomial in this case, it just X power j divided by j factorial. Now, what property does this have, so if I take this polynomial f j and I subjected to say an arbitrary number of derivatives. So, I take f j and I say, I take k derivatives of f j and I evaluated at 0. So, my question is, what is this answer, what to I get, when I take this polynomial, subjected to k derivative, so I applied derivatives k times and then, evaluate the answer at 0.

So, observe that here what happens, if k is smaller than j, if k is simply smaller than j, then taking k derivatives will give you some constant in front, but there will be some power of X left over, which when you evaluated X equal to 0 will give you a 0. So, this will give you a 0, if k is strictly smaller than j, if k is strictly bigger than j, then here is a polynomial of degree j. But, you are taking derivatives for too many times, you are taking derivatives more than j times.

So, of course, the whole derivative itself 0 in this case, you do not even need to plug in X equal to 0. So, if case less then j or greater than j it is in fact 0, but if you subject to exactly to j derivatives, then this is going to give you j into j minus 1 into j minus 2 and so on, each time you take a derivative. And when you plug in X equal to 0, you will get exactly 1, let just rewrite this f j k of 0 is 1, if k equals j and 0, if k is not equal to j.

So, this is again starting to look a lot like, what we did in the case of interpolation in the example of vectors and so on, it sort of like this 0 1 idea. So, these polynomials f j's are like with respect to the operation of taking derivatives, you are not really plunging in or there is no dot bracket. But, now here the operation is with respect to taking derivatives, f j has the property that it is most well behaved when you take exactly j derivatives and plug in X equal to 0.

When, you do anything else, when you subjected to any to k derivatives where k is not equal to j, the answer is always j, so this is again the 0 1 idea in action, if you wish. But, here it is much simpler I mean it would not be particularly eliminating to go this ways straight up. So, there is one slight variant of this that I want to mention, here is the variation of this problem.

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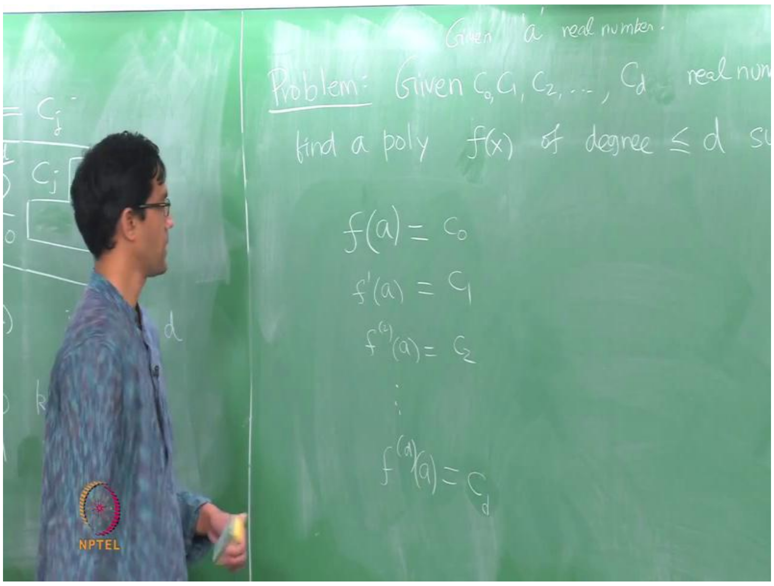

Suppose I given d real numbers, you want to find the polynomial f of X of degree at most d, which has the following property that If you take j derivatives are all it is right way in the same way. If I evaluate f at there is also real number a which is given, so the evaluation of f at a gives you C naught, first derivative at a gives you C 1, second derivative at a gives you C 2 and so on, till the dth derivative of a gives you the number C d, where a is also some real number.

So, what is a, a is also some real number, given a which is a real number and so this is basically instead of plugging in X equal to 0, what you are doing is really plugging in X equal to a everywhere. And so given all higher order derivatives at the point a, how do you reconstruct the function f itself. So, the problem now is fine f and that is what written in this.

So, find the polynomial and I just leave this as an exercise, which you can moralize do by pretty much following the same set of steps. So, just try copying the same argument that we did with X equal to 0, terms out thus also simpler thing that one can do, but anyway that something that we look at the sometime.