An Invitation to Mathematics
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UNIT - I
Polynomials
Lecture – 03
The 0 – 1 Idea in other Contexts - Dot and Cross Product

Welcome back, last time we looked at the business of Lagrange Interpolation. Now what you want to do this time is sort of given other example of the basic idea that went into the solution of the interpolation problem.

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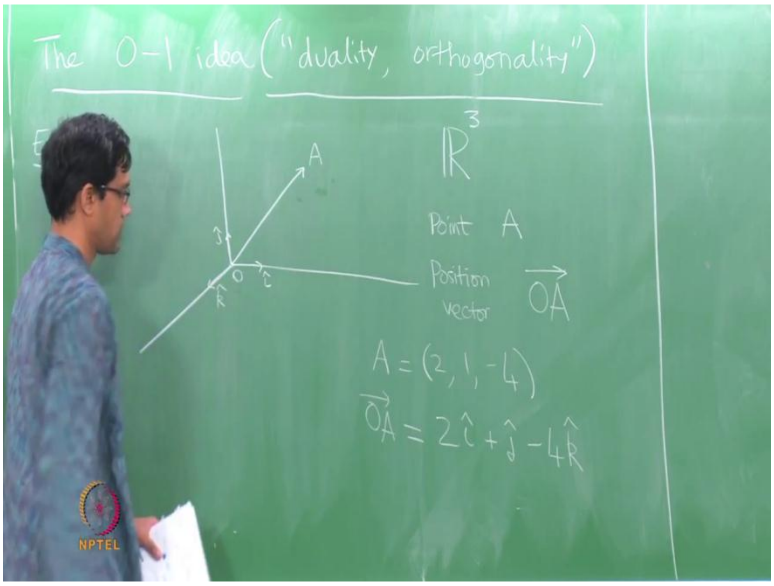

So, let me just call this the 0-1 idea, so in various context in which it appears it goes under, you see terms like duality or orthogonality and so on mention. So, these are all words which will sort of appear whenever this idea is used in various places in Mathematics. Now, let us I just call the 0-1 idea for now and we look at an example of this.

So, what is the 0-1 idea, what I mean by that? In order to solve the interpolation problem, you are given d plus 1 points and we solved it by first doing the following. We found polynomials which were 0 everywhere else except at a single point where it was a 1. And the final answer was more or less obtained by combining these various polynomials. Now, let us two similar thing in a different contest, now for this example it will require a little bit of recall of what vectors meant and you know what vectors are in three dimensions.

So, I want to really going to allow, you know the entire theory of vectors just recall the basic things that I need. So, here is the typical picture I have, so this is what you call R 3, R stands for the real numbers, R 3 is just three dimensional space. So, I have three dimensions and what is a vector, so what is a point in R 3, a point typically is given by three coordinates x, y and z. So, I have a point and the position vector of that point, so by the position vector I just mean, well I join the origin to that point.

So, let us call the origin is O and the position vector of the point A is just by besides is moralized this ray, let us point from the origin to the point A. So, the position vector, so A is the point and it is position vector which is just, it is usually denoted OA, it is the ray from the origin to that point. So, now, here is what I want to do, so mainly we just give an example of this. So, if the point is let say A is, let say the point 2, 1, minus 4, then the position vector of A is usually written as follows in vector notation.

We say it is 2 times i cap plus j cap minus 4 k cap is the conventional ways of writing this were. What are i, j and k? They just refer to the unit vectors along the x, y and the z directions. So, this is the usual notations, so I have i which is along the x j which is the unit vector along the y and k which is a unit vector along the z directions. Now, there are natural operations that one can perform on vectors.

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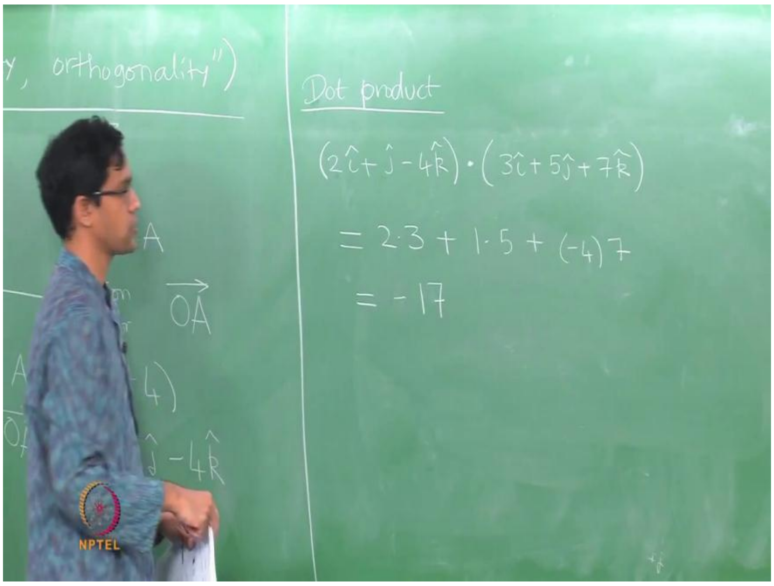

So, there are two are them called dot and cross product, so for now just recall what that dot product of vectors meant of two vectors. Well, it is given by the following thing, let say again just do it by the example, just recall what dot product means. So, suppose I have a vector say 2 i is even take the one that we wrote down plus j minus 4 k that is the vector. So, dot products sometimes also called the scalar product, dot with let say 3 i plus 5 j plus 7 k, here is another vector.

The dot product of these two vectors is defined to just be, you sort of do the following you multiplying the x components together was 2 times 3 plus the y components or 1 and 5, 1 times 5 plus the z components are minus 4 times and so the whatever that is. So, that 6 plus 5 11 minus 28, so that is going to be minus, so that is the definition of the dot product and recall again, that the dot product has contain some geometrical information regarding the angles and lengths of vectors.

So, here are two vectors, the dot product sort of encodes some information about the angle between them and what there lengths are and so on. Now, what I want to poles is a problem of the following kinds, so I want to take three points in R 3. So, I have on here I have point A, I want two more points B and C. So, what you given are three points in R 3.

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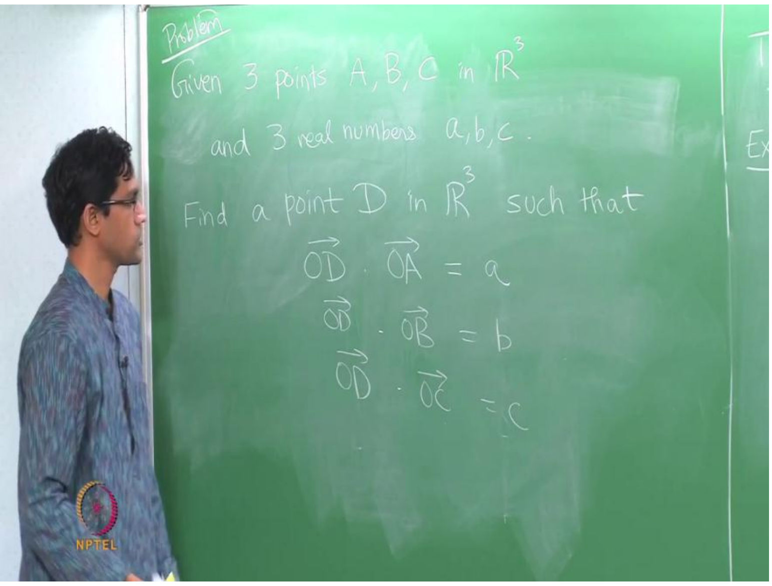

So, given three points, so in three and what are elsewhere, so I might want some restrictions, I want them to be sufficiently independent in some sense, but let us not worry too much about these things right now. So, given three points in our R 3 and three real numbers, so what shall we call them a, b, c just arbitrary real numbers.

So, here is the problem given three points in R 3 and given three real numbers a, b and c find a point d in R 3, such that it satisfying the following conditions, the position vector of D has prescribe dot products with these three other position vectors. So, I have three position vectors OA, OB and OC, I want OD to have dot product a with OA, dot product b with OB and dot product c with OC.

So, again it is sort of like the polynomial condition I have in a three given vectors and I want the dot product to be certain values or those three products. So, let see what would the 0-1 idea refer to in this context, it is the following, let us before we solve the general problem in the earlier case, we sort of found those special polynomials. So, similarly here let us solve a special version of this problem, so let us solve the case when a is 1 and b and c are both 0.

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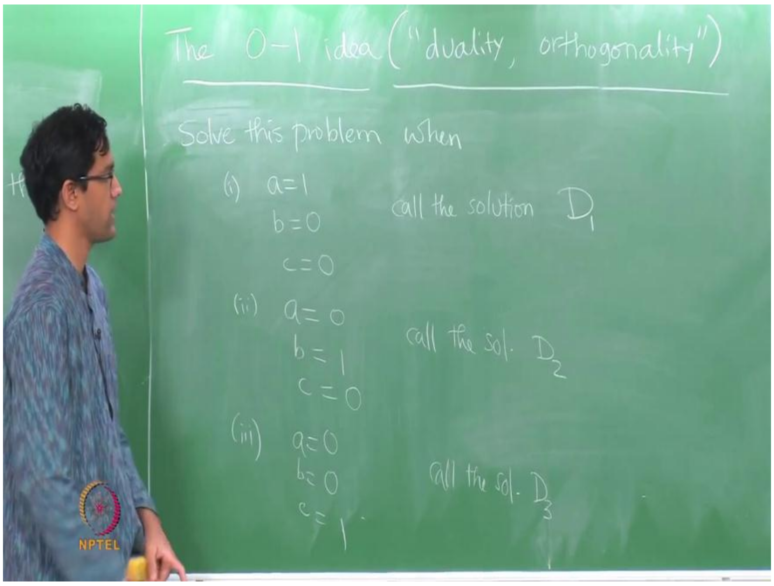

So, let us do the following, let us solve this problem, when then a equals 1, b equals 0, c equals 0, in other words I want to find a point D such that the vector, the position vector of D has inner product, has dot product 1 with OA and has dot product 0 with the other two founds, so that is the first thing. So, let us call the solution as the point D that you get let us call the solution instead of D is called D 1. So, similarly I will let solve another problem, let us find a point b whose dot product is 1 with OB and 0 with the other two vectors.

So, suppose we can solve this problem, call the solution as D 2 and we solve the third problem which is dot product 0 with the first two vectors and 1 with the third vector and call that solution as D 3. So, instead of solving the general problem we solve three simpler problems, we find D 1, D 2 and D 3 whose dot products has 0s and 1s that is sort of what this in the 0-1 ideas. Now, suppose we could do this, we still do not know how to do this, but assuming we could do this.

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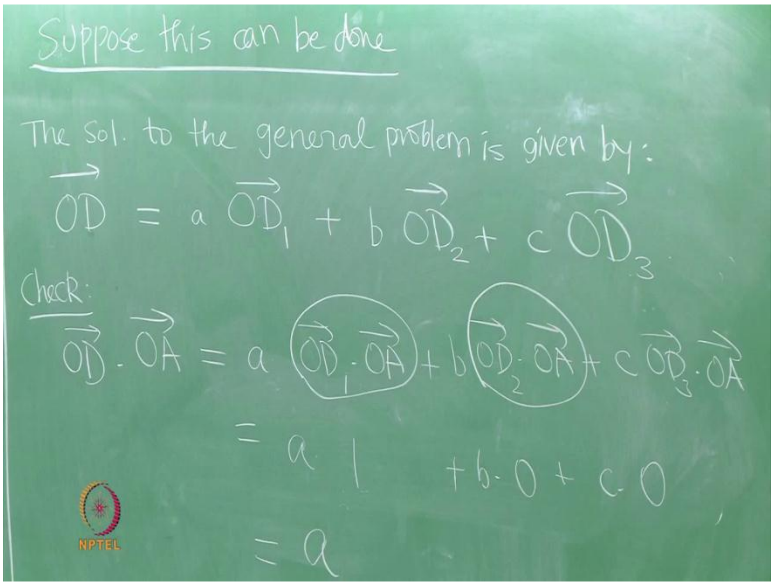

Then here is how we solve the general problem, the solution to the general problem by which I mean the original problem with the three given real numbers a, b and c how do you solve the original problem if you know these three solutions is the following, we just take is the following solution to the general problem is given by... So, you take, you define the position vector OD as you have three position vectors OD 1, OD 2 and OD 3 we just take a liner combination besides a times this plus b times that.

So, I have these three points D 1, D 2, D 3 and here is the liner combination a, b and c where those are the given three numbers, this liner combination we claim is going to be a solution through the general problem. So, let us check this, let us check that this is in fact rule. So, what do we need this point D to satisfied, it needs to have describe that products with OA, OB and OC. So, let us compute what is OD dot product with OA, let see whether this answer is exactly a.

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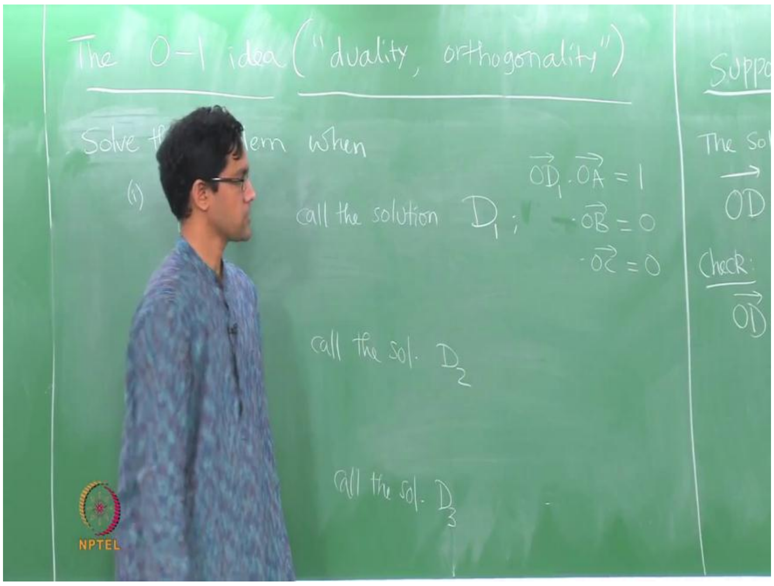

So, observe what is the property of D 1, so D 1 I said it is a solution to the problem when a equals 1, b equal 0, c equals 0. So, what is this mean by saying the solution is D 1 it means the D 1 has the following property, the position vector OD 1 has dot product own with OA and 0 with the other two fellows that is what D 1 does. So, let us compute this, so OD dot OA would be OD 1 dot. ((Refer Time: 11:29)) So, it is a times OD 1 dot OA, b times OD 2 dot OA and c times OD 3 dot OA and observe that you know sort of again doing by the same thing for D 2 and D 3 that, this would be a 1 this dot product is a 1, while the other two dot products are in fact, 0s. So, it is plus b times 0 plus c times 0, so the answer is exactly a.

So, similarly if you worked out what the dot product of OD with OB was, then only the second term would give you a 1, whereas the first and third terms would give you 0 and similarly for the third time, you only get 1 in the third term and 0s in first two terms. So, essentially if you could solve three simpler problems, you can solve the original problem by just taking a liner comprehensive and observe this exactly what we did for interpolation as well.

Because, there we solve, you know we found this polynomials p i's and then we just have to take a liner combination with them and that is all the more general original problem. So, the last step remained is really the following can we solve this problem, is this any easier to solve then the original. So, it often happens that these simpler problems are in fact, simpler. Meaning, they can actually we solved by some easier method like in the case of the Lagrange interpolation, the finding those p i is actually very simple, because it is 0 at d points and non zero at only one point.

So, because it was 0 at those three points, we were able to use the fact that it must have each of them as a factor, like each of those x minus x i's would be a factor of the polynomial p and so on. So, that greatly simplify the calculation we were able to quickly figure out what the p i, the form the p i must have. Now, here is another instance where similar thing happens, it is actually easier to solve these problems. So, let just do it by example, so some sort of really one trying to demonstrate what the broad idea is in this case.

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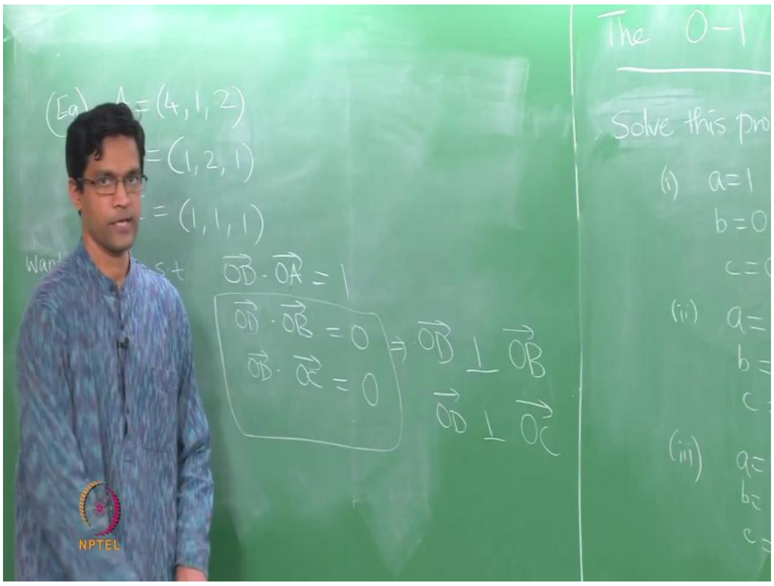

So, here is my example I will take my point to be I need three points, so I will take 4, 1, 2 I will take A, point B which is 1, 2, 1 point C which is 1, 1, 1. So, let say those are the three things that I have given and let me see how could I solve, I will just demonstrate how to solve the first problem, how do we find D 1. So, suppose I want to find the point D 1 such that let me just call it D, such that the dot product of OD with OA, OB, OC is 1, 0 and 0. I am trying to find the point D which is well has inner product has dot product 1 with OA and 0 with the other two vectors.

Now, just like in the polynomial case the fellows with whom it has value 0 sort of give you a lot starting, you know they form a very nice starting point, from that you can get quite a lot of information about the polynomial. So, observe here that the fact that is dot product 0 is, well again this equation little bit of recalling what a dot product means being dot product 0 means that this actually perpendicular to this vector.

So, dot products really mean the following that OD is therefore, orthogonal to or perpendicular to the vector OB and it is perpendicular to the vector OC, so this is the property of the dot product that it is really if it 0, it means the angular is 90 degrees between those two vectors. And so this vector OD that we are looking for is, it is a vector which is perpendicular to both OB and OC.

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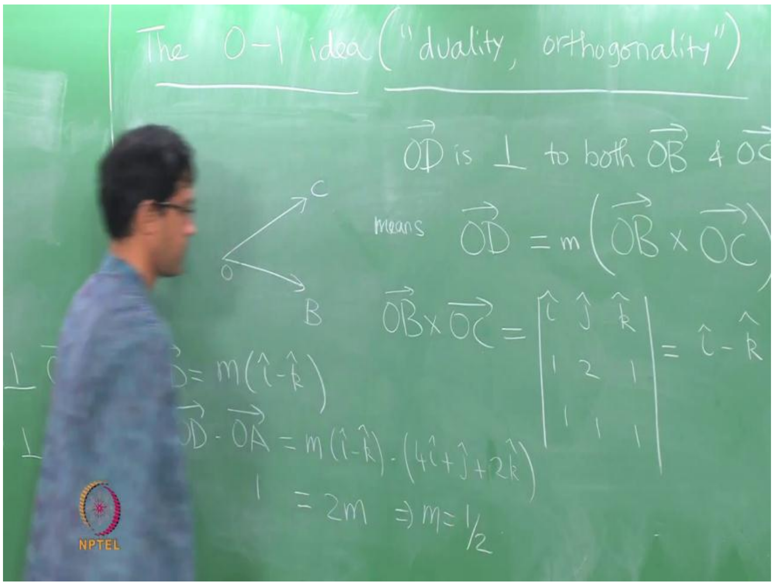

So, if I have two vectors let say OB, OC and I am looking for a vector which is perpendicular to both of them, then this vector is given by, well the easiest way to get it is by what is called the cross product of these vectors. So, observe OD is perpendicular to both OB and OC means, OD in fact, lies along the cross product of these vectors, so I will just briefly recall what cross product, the rule at least what is called the right hand tool for the cross product.

You sort of look at OB and OC and where does it point along, you sort of go from OB to OC with once right hand and the thumb points in the direction of the cross products. So, that just gives an idea what the direction of the cross product is, so it means that OD is in fact parallel to the cross product. So, in other words it must be, you take OB cross with OC, it is a cross product of vectors and you could possibly, you know it could be some multiple of this.

Let us call it some multiple m, some real number m times the cross product. So, it points out or could point in, if we have multiple m is negative, but at least what this is done is the quickly given as was starting points for the form of OD. So, all we need to do is figure out what the cross product of these vector is. So, let us do that, so there is again the standard formula for a cross product. So, in terms of, but determinant, so I am not going too much into this right now.

So, hopefully this is familiar if not it something that you know, just as small ingredient which you should able to refresh quickly, you sort of write a determinant i, j, k and let us say what are the vectors you write 1, 2, 1 write 1, 1, 1. Then, sort of this is the formal way of remembering what the cross product looks like, you sort of says i times 2 minus 1 and so on.

So, I am not getting into the rule itself, this is not too important for what I am going to do, let me just write down the answer itself, it is i minus k. So, compute the cross product of OB and OC using whatever rule you are familiar with, the answer turns out to be i cap minus k cap. So, I know therefore, that OD is some multiple of that vector, now which multiple is given by the other condition that I know an OD I know that OD is suppose to have dot product 1 with OA. So, that determines that multiple for me.

So, observe OD is m times i minus k, so I will compute it is dot product with OA and will what is the answer let us do this OA is 4 1 2. So, I have m times i minus k dot product with 4 1 2, so this gives me... So, the dot product rule that I just wrote out it is 4 minus 2 such 2. So, I get 2 m and this is supposed to be a 1, so this tells me that m is... So, in other words this d that you looking for is just half of i minus k, so that tells you what the solution is.

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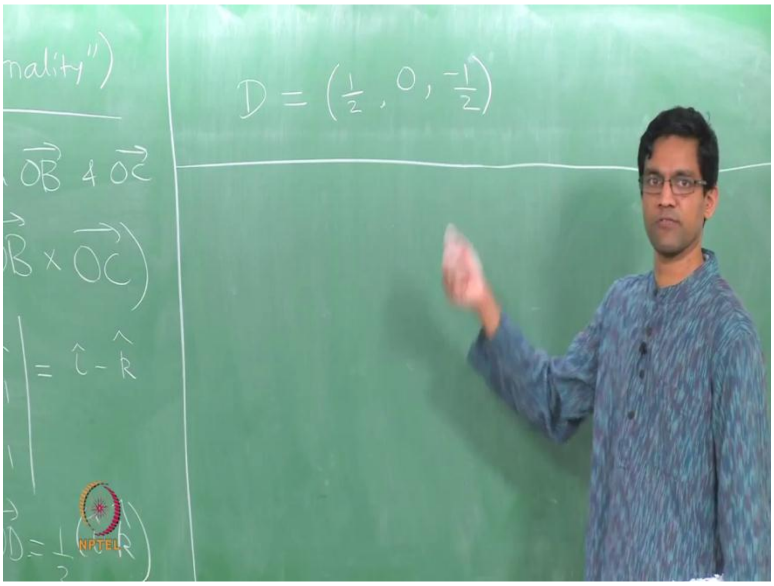

So, the point D is that is the solution finally, the position vector is half of the i minus k means that the point is just half 0 minus half. So, here is a way of solving the 0 1 problem, you just sort of take the cross product of remaining two vectors and then to figure out which multiple of the cross product it must be you sort of see the dot product with the remaining vector and in, that will tell you what the scaling inference seems to be. So, this is just an instance of this 0-1 idea of trying to find something which is 0 on everybody except on one follow on which it is a one.