An Invitation to Mathematics
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Unit - I
Polynomials
Lecture - 02
Lagrange Interpolation

Hello and welcome back, so last time we would talk about polynomials and we finished with this principle here, such that the polynomial of degree d is uniquely determined by it is values at d plus 1 points.

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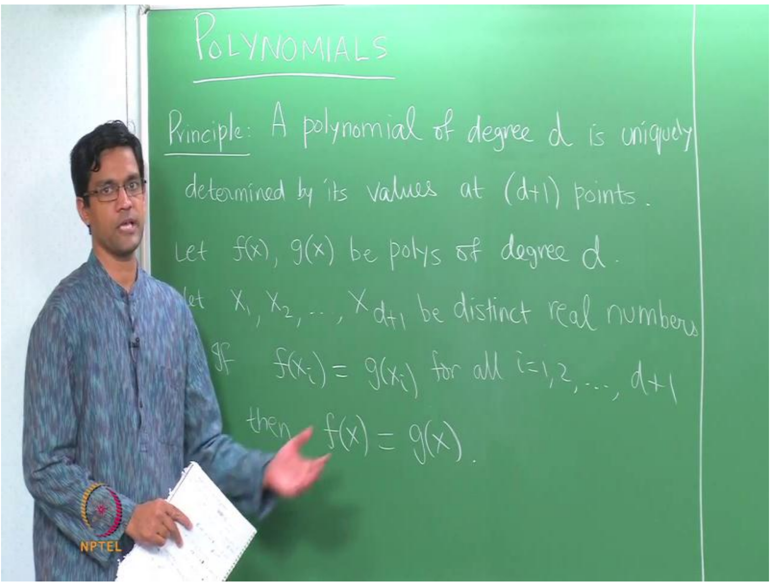

So, what exactly does this mean, so to make this little more precise, let f of x and g of x be polynomials of degree d. And suppose you have d plus 1 distinct points, thereby points of course, you mean no real numbers or complex numbers are whatever it is that you are plug in into the polynomial. So, let us say real numbers for our purposes, let me this distinct real numbers if...

So, if f of x i equals g of x i for these d plus 1 distinct points you know x 1 x 2 to x d plus 1, then the polynomials f and g must actually be the same, they must be equal to each other. So, that is what this principle means, when you sort of write it out in little more detail. So, let us see how do you actually prove a statement like this. So, what one does in many such things where you must prove that, two objects are equal is you sort of consider the difference.

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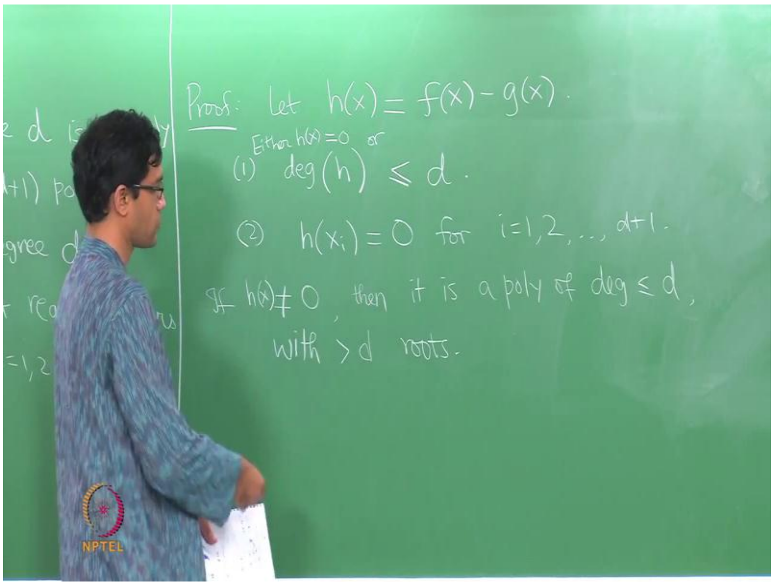

So, let us prove this statement, so you have a proof, let us define or let us consider a new polynomial called h of x which is just the difference between f and g. Now, what all we know about each, where here are the various things you know, fact 1 is the degree of the polynomial h. So, observe f and g are both degree d polynomials and when you subtract or add 2 degree d polynomials, what you might get is well possibly something of degree d, but could also be something of strictly smaller degree than d.

So, let us just be content with the following, the degree of h is at most d, it is less than or equal to d that is statement 1, statement 2 is that each vanishes at these d plus 1 points x 1, x 2 tell x d plus 1. Now, observe that the second statement means that these d plus 1 points are roots of this polynomial edge. So, this sort of already violates the thing we looked at last time, which is that a polynomial of degree d can have at most d distinct roots.

Because here is a polynomial whose degree is at most d, so it could not possibly have more than d roots. So, observe therefore so how do we now show, let just complete the proof formally, if h is not zero it is not the zero polynomial then... So, let me say, h of x is not 0 by which I mean it is not identically the 0 polynomial then... So, then it has degree, so I should just by little careful here. So, either since I kind of made distinction last time, so either h of x is a 0 or it is non zero and has degree at most d. So, if h of x is not zero, then it is polynomial of degree at most d with more than d roots.

In fact, with at least d roots, because here are d plus 1 roots, it could potentially have even more than that. So, it is a polynomial of the degree at most d with greater than d roots and that contradicts something that we already saw. So, that is really the proof that a polynomial of degree d is determined once you know it is values at d plus 1 points.

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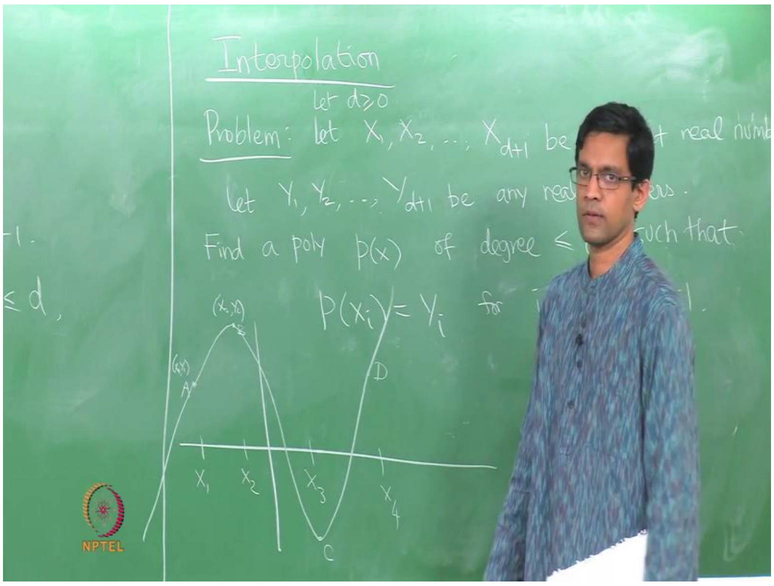

So, this sort of suggest the natural next question which is given prescribe values at d plus 1 points, can you find explicitly a polynomial which takes certain prescribe values at those points. So, what I mean is something like this, so here is the interpolation question, here is the problem, here is the problem statement, let x 1 through x d plus 1. So, what is d? Let d be a positive or negative integer, let x 1 through x d plus 1 be distinct real numbers.

So, those are the points at which I will prescribe some values, so what are those values like y 1, y 2, y d plus 1 be any real numbers, I do not need them to be distinct. So, problem is the following, find a polynomial let us call it p of x of degree at most d, which has the property that it takes the value y i at the point x i. So, in somewhat more graphical terms, what this has something like the following, you take some distinct real numbers.

So, here I have taken d equals 3, so I have four numbers x 1, x 2, x 3, x 4 and at each of these points, I prescribe some y values. So, I give you y 1, so may be that is y 1, x 2 I give you some value y value y 2, let say this is y 3 and x 4, I give some y value y 4. So, I have four value points, so this point is x 1 comma y 1, this is x 2 y 2 and similarly x 3 y 3, x 4 y 4. So, I have four points a, b, c, d and what this question really is asking here is to explicitly find the formula for a polynomial of degree at most three cubic polynomial at most a cubic.

So, I could potentially have a smaller degree polynomial, whose graph passes through these four points and looking for a polynomial whose graph will pass through these four points and further, I am looking for a polynomial of degree at most 3 that is the other important thing. So, that is the interpolation question, this is what is called a polynomial like this is set to interpolate between these four points.

And so observe, that we already know something from this principle right here, which is that if at all such a polynomial exists, we do not get no yet that it does. But, if it does exist, then it is suddenly unique that cannot be two different polynomials which satisfy this property and why is that because of what we just said. If I have, suppose I could find a polynomial p and maybe another polynomial q, both of which had degree 3 and at the same values at these four points, then according to that principle p equals q. So, note, so here is the first thing already that...

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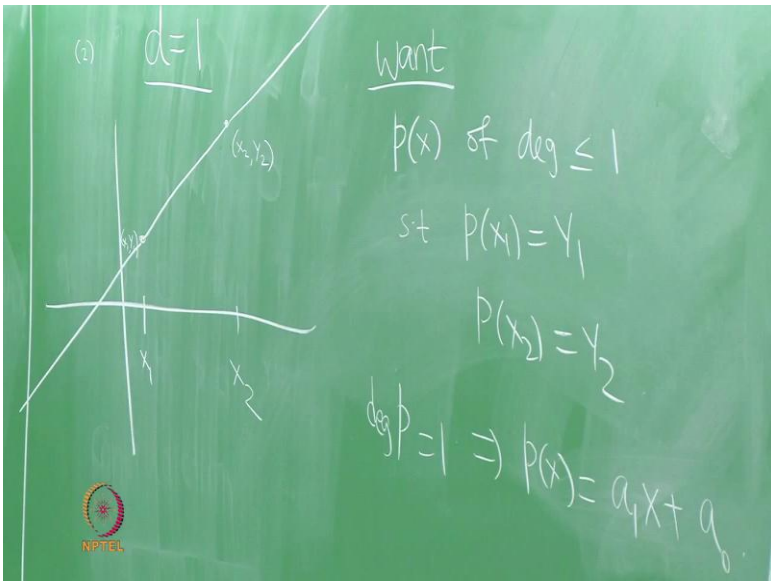

So, let us make some observations on this note, if such p exist it is unique, such p of x exists it must be unique, why by this principle that we just talk about, here that is 1. Now, let us do, to get a sense of this, we just do some special cases, so let us take let us do some special examples. So, let us take the case when d equals 1 in interpolation problem, what is that mean, it means that you need to find or what you have given firstly, you have given two points x 1, x 2 and you are given 2 y values, I am given y 1, y 2.

So, I have x 1 y 1 and I have x 2 y 2 and the questions says, can you find the polynomial whose of degree is at most 1, whose graph passes through these two points. So, what do I want, I want a polynomial p of x of degree at most 1 passing through these two points, such that p of x 1 equals y 1, p of x 2 equals y 2. Now, observe that a polynomial of degree at most 1, well let say a polynomial of degree 1 is just a straight line, the graph of such a polynomial is a straight line.

Because, what is the polynomial of degree 1 look like, so p has degree 1 means, p of x looks like, now let us say, so it is look like this, a 1 x plus a 0, thus a constant term and coefficient of x. So, the graph of such a polynomial is of course, just a straight line. So, this question really boils round to the following given two points find the equation of the straight line joining them. So, that is an interpolation question at d equal to 1 and of course, that is you know, that is a well known solution we know how to do this just using a usual notions from coordinate geometry. So, let says recall the solution to this problem.

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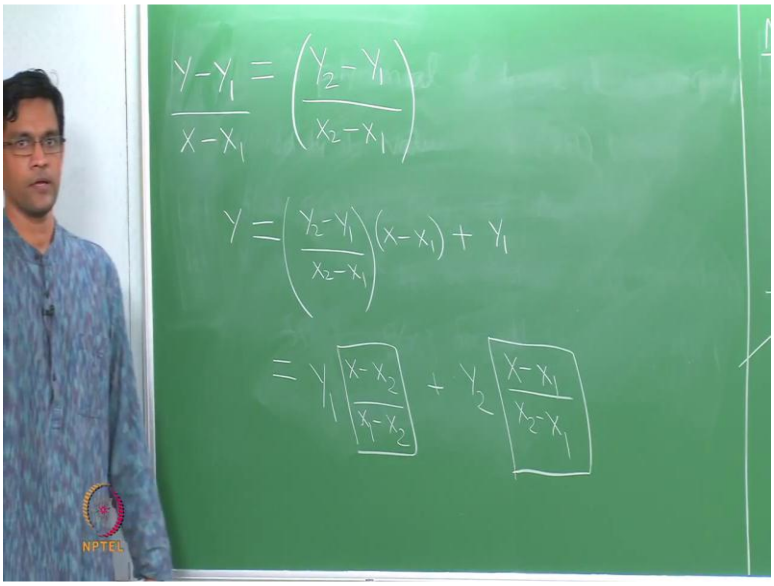

So, what the equation of the that line, well here one way of writing the equation of the line, you say y equals, so I am going to use what is called the slope form of the equation of the line. So, I know the slope is y 2 minus y 1 by x 2 minus x 1 that is a slope and so let us write it in slope form. It says y minus y 1 by x minus x 1 is the same as y 2 minus y 1 divided by x 2 minus x 1. So, this is the equation of the straight line which passes through these two points.

So, let us I just simplify this a little bit, so I am going to write it as follows, I am going to write it as y equals something. So, I am going to simplify this, so I am going to get y 2 minus y 1 times x minus x 1 plus what you get plus the y 1. So, I am going to rewrite this once more in the following form, I will write it as something times y 1 plus something times y 2. So, I want do the following, I want to write it as y 1 times some expression which does not have y 1 or y 2 plus y 2 times some expression.

So, I am going to try and rewrite it in this form, so let us do that, this looks like y 2 times x minus x 1 by x 2 minus x 1, x minus x 1 by x 2 minus x 1 and I have a y 1 which occurs here as well as y 1 there. So, if you sort of see what the y 1 coefficient looks like, so here is another way of rewriting the same expressions. So, I just done a little bit of manipulation, we should check that this is correct.

So, it is y 1 times x minus x 2 divided by x 1 minus x 2, so that is equation of the line. So, here is the polynomial p of x that we want, so let us y is just another name for the polynomial p of x. So, here is the solution to the interpolation problem at for the case d equals 1, so we just going to do something. So, we will soon see that in fact this has a very natural generalization, so I am going to call these two polynomials as something. So, let us call this polynomial in the first box in which multiplies y 1 as p 1 of x and the thing which multiplies y 2 as p 2 of x. So, observe that p 1 and p 2 have the following property…

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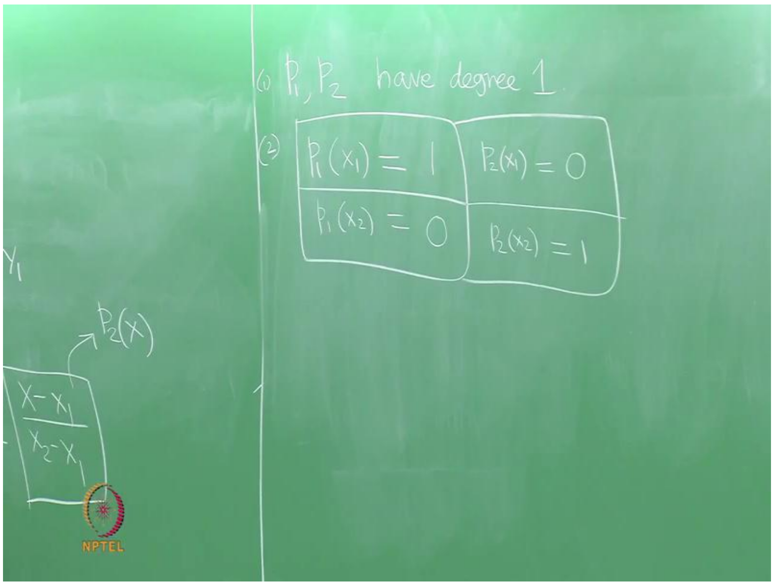

That is polynomial p 1, p 2 have well, let us write down their properties, they are both of degree exactly 1, have degree 1 and they have the following interesting property, if you take p 1 and evaluated it at x 1 and x 2. So, this is p 1, if you put x 1 for x, then what you get here is just 1, whereas if you put x 2 you get a 0. So, this is what p 1 is and if you try the same thing with p 2, it is the opposite it is 1 on x 2 and 0 on x 1.

So, you have these two polynomials which have, you know these two nice properties that their 1 and 0 on x 1 and x 2. So, that sort of suggests a general way of talking this problem, so let us just do interpolation in the general setup, so let just do it for any d.

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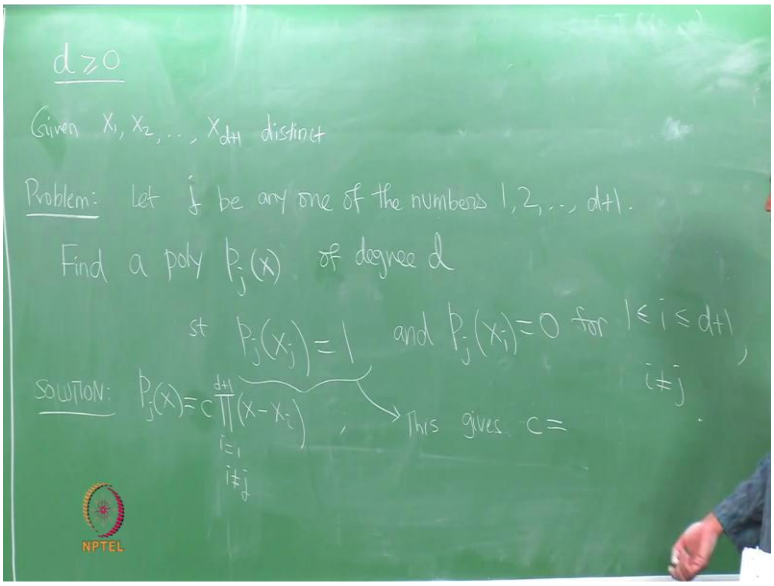

So, let just try the general interpolation problem, let us take any d positive and so what are we given, we are given those d distinct points. What we will try and do is a following, let us try and find the polynomial which has the following property, it is 1 on exactly one of these points and 0 on all the other points. So, here is a problem, you take any one of these as, let j be any one of and it will be any one of these number, fix any one of them. And let us find polynomial called p j, such that it has the following property of degree d, such that p j is 1 on x j and 0 and all the other x i for all i between 1 and d plus 1 except for j.

So, we will see in the moment why this is useful, but it is a sort of a generalization of what we have done write that, we found the polynomial which should be 1 and exactly one of these points and 0 on all the other points. So, first let us try and solve this problem, how do we find the polynomial which vanishes on all the exercise, other than on the single point x j. So, here is the solution, we remember we have already done this last time, if you know a root of a polynomial, you know that exercise root of this polynomial, then we know that this polynomial can be written as x minus x i times something.

So, observe that p of x or we calling p j of x, since it vanishes and all the exercise, it must each of them must be a factor. So, it is should look like, you know for example j is 3 then p j of x should looks like x minus x 1 should be a factor, x minus x 2 should be a factor and so on, in general all the x minus xi's. So, it is write it as product, so p j of x must certainly have all of these as factors, product of all the x minus x i's except for i equal to j.

And notice that this is already of degree d, so this is already a degree d polynomial, because it has exactly d factors and what I want is that, it is a polynomial exactly of degree d. So, I cannot really have any more factors here, what I could have some constant in front. So, let say p j of x might look like some constant times the product of x minus x i, so already we are able to get a quit a lot of information just by knowing that here are d roots of this polynomial.

And now to find this constant we use this lost piece of information, it says that if you plug in x equals x j, then you should get a 1 on the left hand side. So, now, observe putting this conditioning this gives us the value of c. What is the value? If you put x equals x j, the left hand side is a 1, the right hand side is the product of x j minus x i. So, the constant c must in fact d 1 over the product x j minus x i.

So, this is the product, overall i from 1 to d plus 1, but i is of course not equal to j, that is the choice of the constant which will make this a 1. So, well what we have done, we have at least manage to solve this problem, if you want to find the polynomial of degree d which is 1 on a single point and 0 elsewhere, well here is the answer. It is just product x minus x i, i goes to want to d plus 1 divide it by product x j minus x i, product i goes to d plus 1 and observe it is a lot like what already work here the... So, in a sense this is the instants of that principle for d equals 1. So, now, why is this useful?

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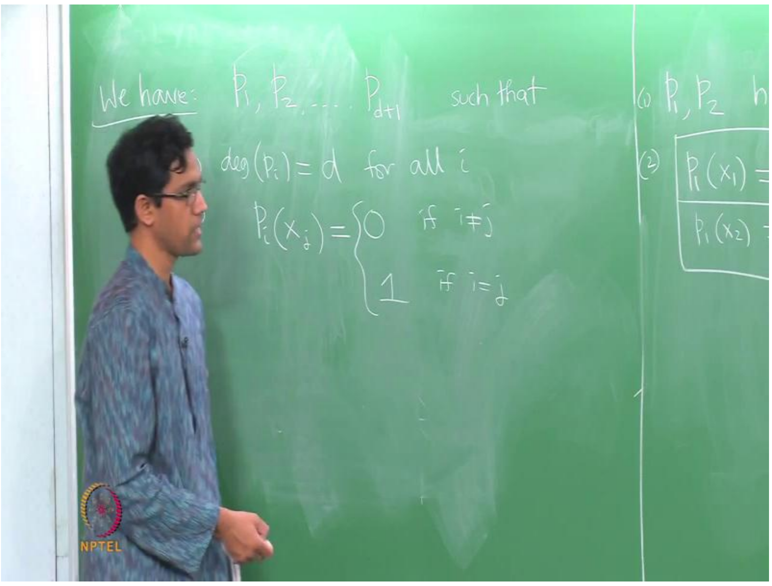

So, what is what do we have, so we now have mini polynomials, we have polynomials p 1, p 2, p d plus 1. We have a formula there which works for every value of j, which have the following property such that what do we know about them and they all of degree d, degree of p i equals d for all i. Well by all i, I mean all i from 1 to d plus 1 and property 2, which is sort of the key property is that p i vanishes on x j. So, p i evaluated on x j will give you 0, if i is not equal to j and if give you 1, if i equals j.

So, it has it is really a nice property about it is and use of this polynomials p i you can solve a general problem. So, recall what is the general interpolation problem which has, find the polynomial which takes the value y i at x i.

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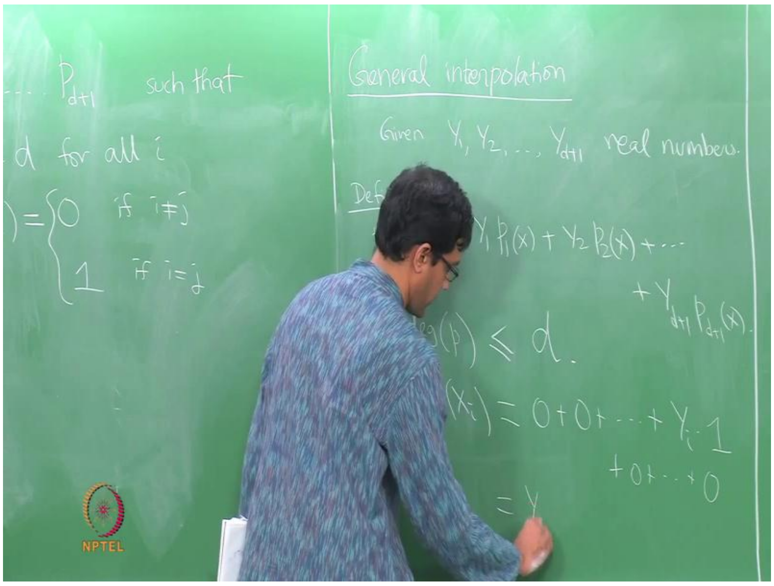

So, here is the solution to the general interpolation problem. So, general interpolation what I given your also given these real numbers y i. So, we are also given the values not necessary distinct real numbers, so what we do is define a polynomial p of x. So, let us define one of the p of x to be well we know all p 1 through p d plus 1 we just say it is y 1 times p 1 of x plus y 2 times p 2 of x tell y d plus 1 times p d plus 1 of x.

So, you combine the solutions here by multiplying them with the y i's is what is happen called the linear combination of the p i's and now observe that this polynomial has all the properties that we want. So, what are the properties of p, first we observe the degree of the polynomial p. So, each of the p i which is of degree d and what you doing is your short of tacking a linear combination, your multiplying them by the various constant and adding the answer that could very well end up being the polynomial of degree smaller than d as well, it is less than are equal to d for sure.

So, that is property 1 and property 2 which is the key thing that we want the value at x i. So, we have observe what is a value when you put x equals x i, well all these terms other then the ith term will actually be a 0 when you put x equals x i that is the key property here, only p i of x i will contribute a one all the other terms will give you a 0. So, in fact only the ith term comes up, so this is the basically a lot's of 0s the ith term contribute gives you y i times the value of the polynomial is 1 plus a bunch of 0s again. So, that is the exactly the y i that we wanted to be, so this is precisely the solution to the interpolation problem.

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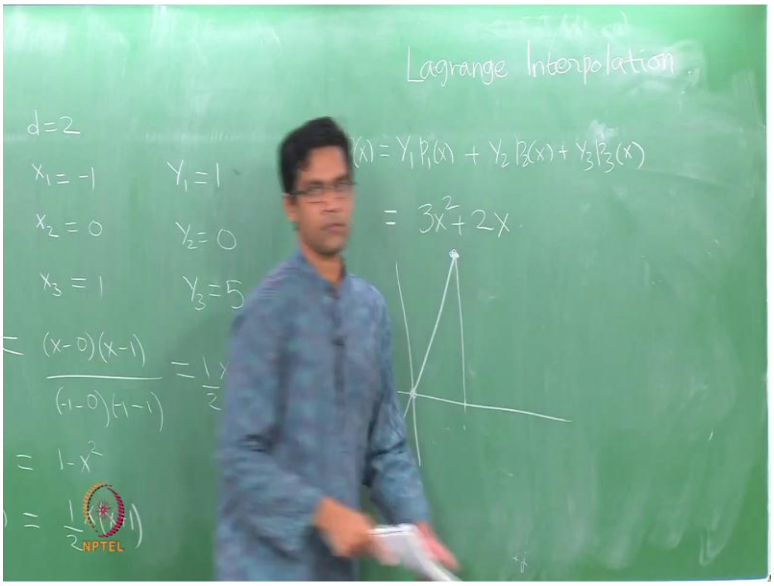

So, just given example here suppose I want d to be 2 and I give you three point x 1, x 2, x 3 to be minus 1, 0 and 1 and we take some values of y, this some arbitrary values 1 0 and 5 like a say we need not we distinct here just turn out to be. So, here let just in all do this whole thing, so observe that the y is do not really enter the picture until the very end. So, I do not need to very about them right now, if I know the x is I can compute my three polynomials p 1, p 2 and p 3.

So, let us calculate those three polynomials, so p 1 of x according to this prescription is it is x minus 0 into x minus 1 divided by this minus 1 minus 0 into minus 1 minus 1. So, that is my formula for p 1, so this is nothing but the short of work is out it is half x into x minus 1 similarly p 2 of x. So, you short of do the same thing again, so let me just write down the answer, it is a 1 minus x square p 3 of x similarly terms out to be half x into x plus 1 not of do this all I have to done is just plug in into my formula write there.

And the final step the interpolating polynomial which is the linear combination of these three would just we obtain by multiplying the first answer by a one, the second answer by a 0 and the third answer by a 5. So, the interpolating polynomial p of x remember is nothing but y 1 p 1 plus y 2 p 2, y 3 p 3 of x. So, in this case if you short of just work this out terms out to be 3 x square plus 2 x.

So, I am just leaving the details something you should early work out in yourself, observe that this in fact does do the job, if you plug in x equal to 0 it gives you a 0, if you plug in x equals minus 1 it is 3 minus 2 which is a 1 and if you plug in x equals 1 it is 3 plus 2 which is a form. So, this guy in fact does do the and if you short of look at the graph of this follow it is. So, in fact the some short of parabola like this the value at... So, at minus 1 the value is something is 1, 0 the value is 0 and at 1 the values.

So, it is a parabola which short of pass of through these three points, this point here, the origin and this point there. But, here is the general principle and so this procedure here that we talked about those by name called Lagrange interpolation something that we will work in complete generality, no matter what your value of these.