An Invitation to Mathematics
Prof. Shankaran Viswanath
Institute of Mathematical Sciences, Chennai

Unit
Matrices
Lecture - 30
Linear Equations, Lagrange Interpolation Revisited

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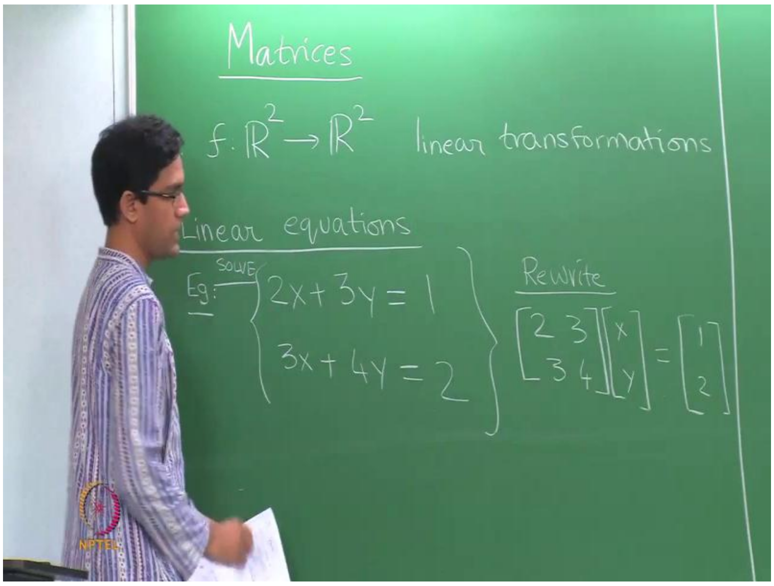

Today we will start talking about Matrices. So, recall we have already encountered matrices in our study of linear transformations when we are looking at maps from R 2 to R 2 which are, what we called linear transformations and in this context, matrices appeared as encodings of linear transformations. So, linear transformation was essentially determined by four numbers a, b, c and d and arranging them in a 2 cross 2 matrix form somehow was the correct way of encoding this linear transformation and we saw that the operation of composition of linear transformations was precisely captured by the operation of multiplication of matrices.

So, that is already a context in which we have encountered matrices, now here is another context in which you probably have already seen matrices. So, this is likely to be a somewhat more familiar context, which is in trying to solve systems of linear equations. So, here is an example let us consider the system of two equations 2 x plus 3 y equals 1, 3 x plus 4 y equals 2 and there are two unknowns x and y. So, the question is to solve, so solve for x and y.

And so of course, there is always the way of trying to eliminate one variable in favor of the other and so on, but here is the sort of the easy way of doing it, somewhat clean approach which is to rewrite this equation in matrix form. So, we rewrite the system of equations in the following way, we consider the four coefficients 2 3, 3 and 4 form the matrix and this equations can just be written as this matrix multiplied by the column vector or the 2 cross 1 matrix x comma y.

So, the product of these two gives the coefficients 1 and 2, so this system is exactly equivalent to this equation written in matrix form and as it is easy to check the definition of matrix multiplication says you need to do 2 times x plus 3 times y that is the first coordinate that you get when you multiply them out and the second one is 3 times x plus 4 times y. So, and when you equate these two, all you get is the equations 2 x plus 3 y is 1, 3 x plus 4 y is 2.

So, these are two equivalent ways of writing the same thing, but this suggests a way of solving it. So, what is the usual way of solving such systems of linear equations, we multiply both sides of this equation by the inverse of this matrix. So, what is the inverse mean?

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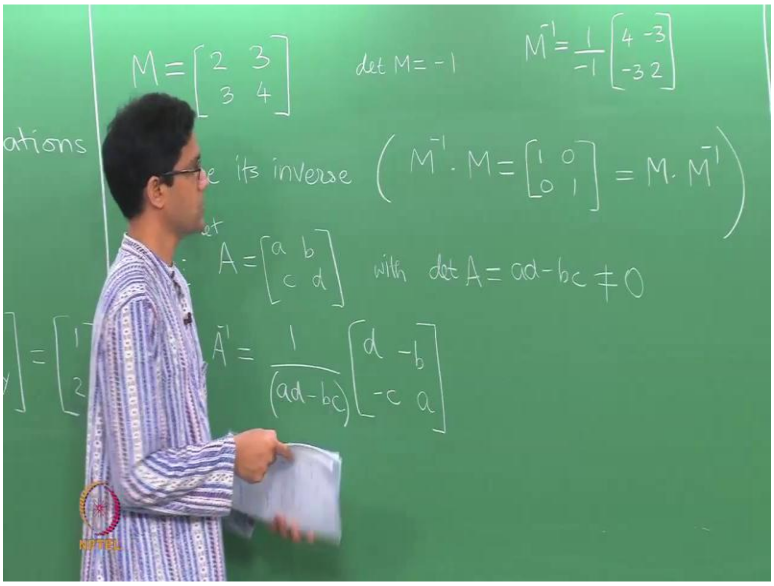

So, here is the thing if you have the matrix of coefficients 2 3, 3 4. So, what is the inverse matrix mean? Well, it is the matrix which when multiplied with the matrix m will give you the identity matrix. So, let M inverse be which it is inverse, in other words it is the matrix which has the property that this into this is what you call the 2 cross 2 identity matrix and it is immaterial which order you multiplied them, you do M into M inverse or M inverse into M gives you the identity matrix.

So, let M inverse denote the inverse of this matrix and so there is a, for a 2 cross 2 matrix there is a nice formula which tells you what the inverse is. So, what is the general formula for the 2 cross 2 matrices? Well, here is what you do. So, given for let say, so here is an aside if I have a 2 cross 2 matrix A, 4 coefficients a b c d, the inverse of this matrix has the following easy description, you do the reciprocal of the determinant of the matrix. So, it is 1 by the determinant remember it is a d minus b c times the entries here are just d a minus b minus c.

So, this just a prescription which works for 2 cross 2 matrices, trying to do this for 3 cross 3 and so on is somewhat harder of course. So, it is the answer and notice of course that the inverse only exists provided the determinant is not zero. So, you cannot invert matrices for whose the determinant is 0, so let a b are 2 cross 2 matrix with determinant non zero. So, determinant of the A which is a d minus b c should be non zero, only then can you find the inverse. So, in this case we are, because the determinant of m is 4 into 2 8 minus 9.

So, the determinant here is the minus 1, the determinant of the M here is a minus 1 and applying this formula here it says that the inverse of M can be computed as follows. The inverse of M is just 1 by minus 1 times d a, and then we do minus 3 minus 3. So, that is the inverse of the matrix M and now, what we do is just multiply. So, now, we are always through.

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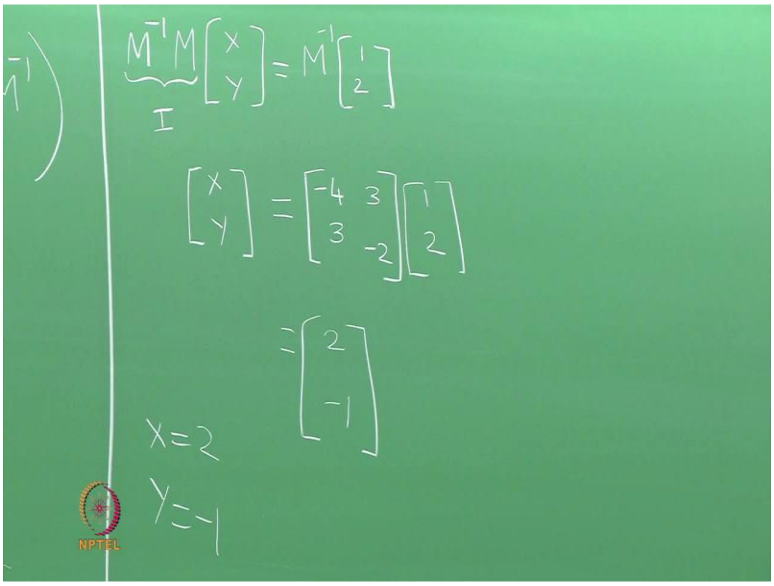

So, to solve the original system of equations, so recall the original system was re expressed as follows, it is the matrix M times the vector x y is the matrix 1 2. So, now, we multiplied both sides of this equation on the left by M inverse. So, this became M inverse times M times x y is M inverse times 1 2. Now, M inverse times M is the identity matrix and the identity matrix, let us call it I multiplying x y will just give you back x y. So, that is the whole point of doing this, the left hand side is now just become x and y.

Whereas, the right hand side now is easy to compute, so M inverse we just computed is... So, there is a 1 by minus 1 which I will take inside it is minus 2 3 3 times 1 2. So, it is just a question of multiplying these two things out, such easy enough such minus 4 plus 6 is 2, 3 minus 4 is minus 1. So, we conclude that finally, x must be 2 and y must be minus 1 and it is easy you have to check that these two will do the job, you just add plug in back into the system of equations and see that they satisfied.

So, this is the standard place where you would have seen matrices before when you want to solve a system of, say n linear equations in n announced. Now, let us do another example which is sort of interesting, because we have already seen it in an earlier context. So, let me do another example of solving linear equations.

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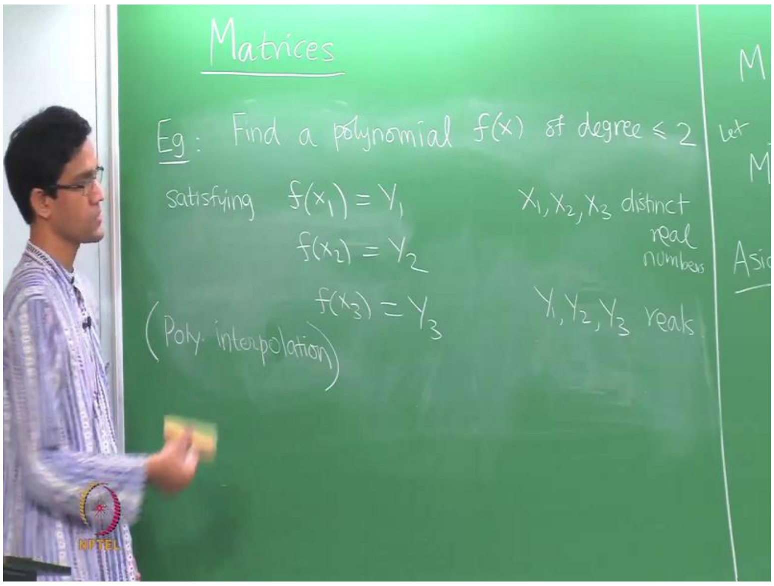

Example, find a polynomial in one variable is polynomial f of x of degree at most 2 satisfying the three conditions f of x 1 is y 1, f of x 2 is y 2 and f of x 3 is y 3, where x 1, x 2, x 3 are three distinct real numbers and the y is just any arbitrary real numbers not necessarily distinct. So, these are just real's and of course, recall this is exactly the problem of interpolation.

So, here I have just taken a special case of interpolation, where you are looking for a polynomial of degree at most 2. And so recall this is just the question of polynomial interpolation that we studied right at the beginning interpolation. So, you might wonder why this is sort of making reappearance now in the context of solving systems of linear equations. So, is in this about finding a polynomial with satisfies a set of constraints.

Now, it turns out that this is in fact, nothing but a system of linear equations, so let see how this can be recast into this from. So, what we are looking for is a polynomial of degree at most 2.

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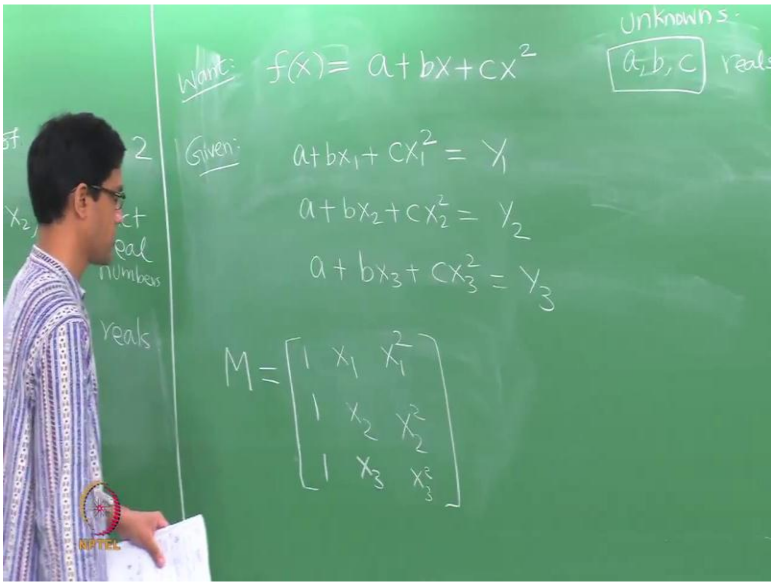

So, what we want is f of x, so let we just write it as follows, let me call it a plus b x plus c x squared, because that is how a polynomial of at most degree 2 looks like, where what are a, b, c. Well a, b, c are some real numbers and those are really our unknowns. So, these are the three numbers that we want to find now, so these are all unknowns. So, the coefficients of this polynomial a, b and c turn out to be a three unknowns we are looking to solve for and what are we given, we are given three conditions.

So, what is given? So, we want to find the a, b, c given the three conditions that f of x 1 is y 1, f of x 2 is y 2 and f of x 3 is y 3. So, we will write those three conditions down, so when we substitute x 1 this polynomial becomes a plus b x plus 1 plus c x 1 squared that is given to be some number y 1. So, x's and y's are the constants here, similarly a plus b x 2 plus c x 2 squared is given to be y 2 and a plus b x 3. So, this is nothing but a system of three linear equations in the unknowns a, b, c with coefficients being 1 x 1 x 1 squared 1 x 2 x 2 squared and so on.

So, let me rewrite this in matrix forms, so let us form the coefficient matrix. So, this is just the following, consider the matrix M 1 x 1, so it is 3 cross 3 matrix now x 1 squared 1 x 2 x 2 squared 1 x 3 x 3 squared that is the matrix M and what is the system now become?

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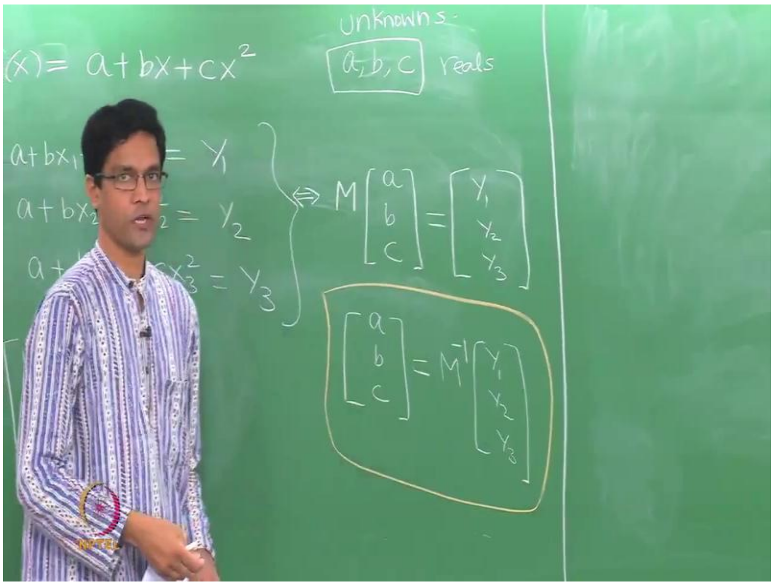

The system becomes the following, it is the matrix M, so this is the same thing as equivalent to M times the vector a b c that is the unknowns becomes y 1 y 2 y 3. So, we have re written our system of equations, and now all we required to do is the try doing the same thing as before which is to sort of multiply both sides by the inverse of M. So, what we want is now therefore, well this equation therefore can be solved as follows.

So, let us do it here the solution a b c is therefore, case the matrix M inverse multiplied by y 1 y 2 y 3. So, let us keep this in mind that is the solution that we finally want for the three coefficients a b and c. So, what is that really mean? We need to compute the inverse of this matrix M in order to figure out, how this is to be solved. Now finally, inverse of a 3 cross 3 matrix is somewhat trickier than finding the inverse of just a 2 cross 2 matrix.

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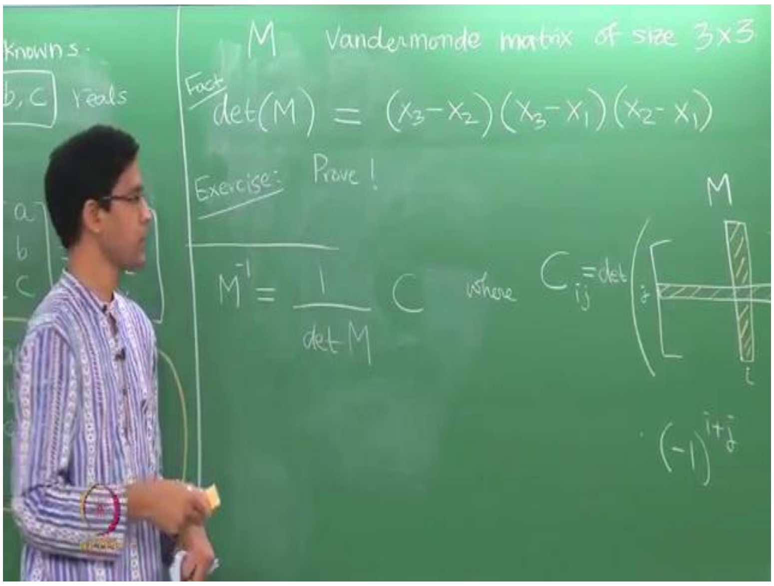

So, for a start what we will need as before is the determinant of this matrix M. Now, this particular matrix M which has this very particular form 1 x 1 x 1 squared and so on. So, such a thing is it has a name, it is a very special matrix it is called the Vandermonde matrix of size 3, if you wish of size 3 cross 3. So, as a Vandermonde of any size n cross n, so here is the 3 cross 3 Vandermonde matrix that is M and the corresponding determinant the Vandermonde determinant has rather well known explicit expression, it is rather surprising at first if we have seen it before.

So, let me tell you what the answer is here. So, here is the fact the determinant of the 3 cross 3 Vandermonde matrix has an exceptionally nice form, it is just the following product, just the product of three terms x 3 minus x 2 x 3 minus x 1 and x 2 minus x 1. So, here is the first exercise try this out compute the determinant of this matrix using you know what our properties you know how determinants are computed. So, exercise prove this calculate and prove that this is in fact.

So, the determinant is this nice product and now what we need to do next is compute the inverse of this matrix. So, the prescription for the inverse is the following, so how do we compute M inverse. So, M inverse is as before it is 1 by the determinant times another matrix here, now this matrix here sometimes call the cofactor matrix. So, let me maybe just call it or the ad joint.

So, let us give it is and name let us call it C for now it is 1 by determinant times a matrix

C, where what is this matrix C, C has the following description you want to figure out what is the i j'th entry of C, C is again a matrix n cross n in this case 3 cross 3 and you want find the i'th row j'th column entry of C it is obtained as follows. So, here is the procedure, so let me describe the procedure you take the matrix M. So, here is the matrix M itself and you do the following, you look at the i'th column of M. So, I take the i'th column of M and I take the j'th row of M.

So, you take the i'th column and the j'th row of M and you remove these two things from M. You will you remove both of these from M what you left with now is a matrix of size 1 lower. So, in general if it is an n cross n matrix, you will be left with something of size n minus 1 cross n minus 1. In this case, since m is 3 cross 3 you get a 2 cross 2 matrix and you take the determinant of that 2 cross 2 matrix. So, here is the deleted matrix of course, determinant you take and C i j is that determinant multiplied by a certain sign.

So, it is this determinant times plus 1 or a minus 1 and the sign is given by this explicit formula it is minus 1 to the i plus j. So, such an expression is sometimes called the minor of you know the corresponding element whatever j i'th element of M and so on. So, in n case you will not seen this before it is not a big deal I am not going to worry too much about this. So, ((Refer Time: 17:04)) it I mean for our purposes it suffers to know that there is an explicit expression for the inverse of any n cross n matrix given in terms of you know determinants like this, one lowers are determinants.

Now, M inverse is therefore, has this very explicit descriptions. So, in our case we can just use you know since we know M we just delete the appropriate row and column compute the determinant multiplied by a sign things like that. So, the inverse can be computed, so let me just tell you what the inverse of M looks like. So, I want to write all nine entries maybe just the entries in the first column.

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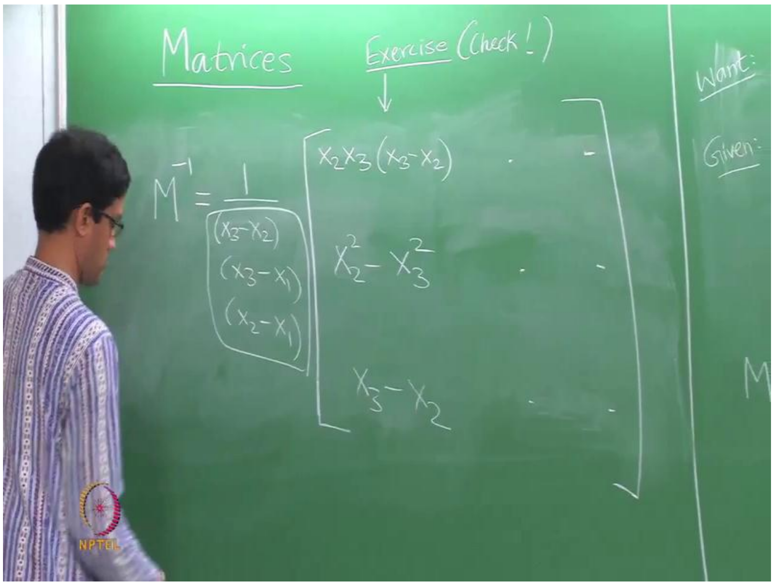

So, claim is the inverse of M is 1 by determinant of M times may so times the matrix C that I wrote out and here are the entries of the first column of C. So, it is x 2 x 3 times x 3 minus x 2, this is x 2 square minus x 3 square, this is x 3 minus x 2, and then of course there are another six entries here second column third column which I am not writing out at this point. So, exercise number 2 check that these entries are correct. So, exercise check these using the description that I just give you, so exercise check.

So, those are the entries of C at least the first column entries of c. Now, one can of course, simplifies this because the determinant of M again has the nice form, it just the product of 3 terms, this is x 3 minus x 2, x 3 minus x 1, x 2 minus x 1. So, this product is sitting on the denominator, so you can sort of cancel some terms out from the first column. Now, the second something that if you do, but I want to just complete this calculation here recall what we wanted to find the first place was the polynomial f.

So, the polynomial f was actually a plus b x plus c x squared, so it is of course composed of I mean the 3 real unknowns are a, b and c. But, suppose I wanted my polynomial f of x you know the full polynomial not just the 3 you know coefficients separately. So, then how do we go about finding f. So, assume for the moment that I figured out what a b c are from my description of M inverse and so on and so forth by using that this property here it will tell me what a, b and c are, but let me recover f from a b c.

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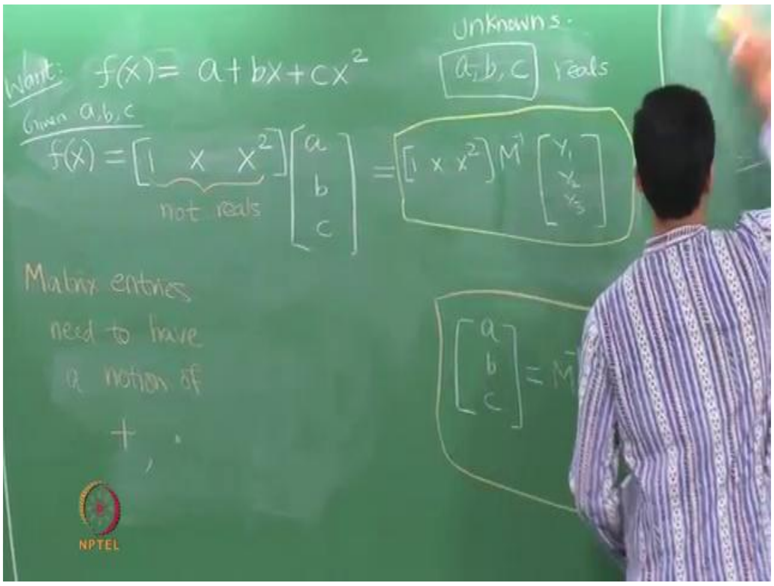

So, how do I figure out what f of x is given a b c how do I find f observe this is just the following, again it has a description in terms of matrices. So, here is what I can do, I know the column vector a b c that is the thing that I am determining form this procedure. But, if I want to put them back stitch them back to together to get the function f all I do is multiply that column vector by the row vector which has the entries 1 x and x square why, well you all you have do just multiply them out one times a plus x times b plus x square times c that is exactly how do you do the multiplication of these two things.

Now, observe the following fact that here is already something slightly different from what we have seen so for the entries of this matrix are these vectors or not numbers anymore. So, observe there are x is now appearing, so these are not no they not real numbers for instance, instead there actually elements well there variables here, but that is perfectly.

So, whenever you encounter matrices it is not necessary that the only matrices that you know it is to have matrices which do not have real numbers or constants in them, it is to have say other kinds of objects it is fine if you have matrices which have polynomials in them for instants are more generally it is to have matrix entries be anything which you can perform addition on or multiplication on and so on. So, the kinds of things that you put into matrices or into vectors, those entries really only need to have some sensible notion of addition and multiplication define.

So, matrix entries are.. So, this is another point note here, matrix entries need to have notion of addition and multiplication between them that is pretty much all you need, we do not need anything else. So, it is to have 1 x x square d matrix entries, because we do know how to multiply you know... So, there polynomials in x and we know how to multiply to polynomials together for instance. So, are we know how to multiply or add polynomials, etcetera. Because, this is all that will really come up when you try and multiply or odd matrices. So, let us get back to this we know that the polynomial is just 1 x x square times the vector a b c.

So, we could go ahead and say this is just the vector 1 x x squares times, well the matrix M inverse. So, remember a b c is just M inverse times y 1 y 2 y 3. So, we have the polynomial f of x can be obtain in the following fashion, you take the matrix m inverse you sort of sandwich it between a row vector and column vector, the row vector being the entries 1 x x square, the column vector having entries y 1 y 2 y 3. So, this is the sort of the final expression here is the polynomial f of x that one want. But, let us just do since we wrote out the first column of M inverse let just to one step of this.

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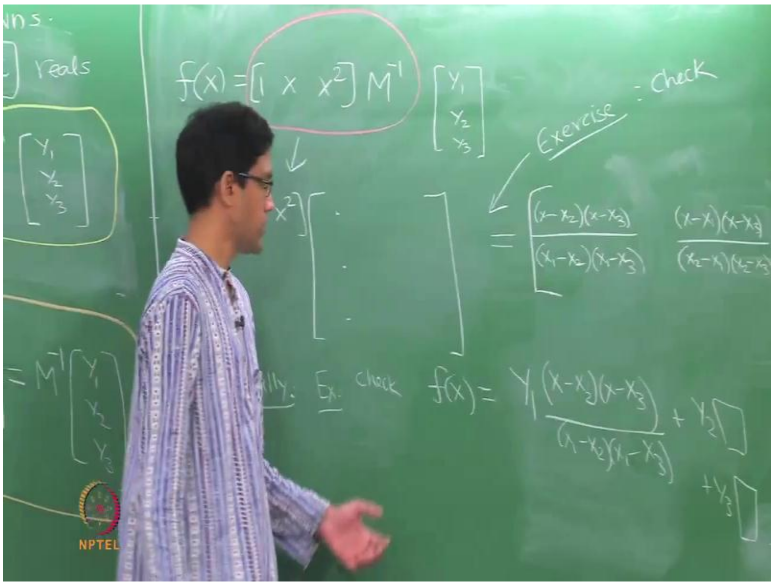

So, you take f of x as I said is just 1 x x squared times M inverse times well this column vector. Let us do the following, let us first multiply these two things out this row vector 1 x x square with the column vector M inverse with n cross n matrix 3 cross 3 matrix M inverse. So, this calculation is the following it is 1 x x square times well I just wrote out the entries of M inverse. So, after simplifying after dividing by determinant and so on, so you sort of get three entries here.

Now, when you multiply these two things out here is what you will find, so there are three entries. So, let me just tell you what the first entry looks like, it is just x minus x 2 times x minus x 3 divided by x 1 minus x 2 x 1 minus x 3 and similarly the other two entries are going to be such write them out as well x minus x 1 x minus x 3 x 2 minus x 1 x 2 minus x 3. So, I sort of wrote them out, so this should have been row vector rather than the column vector.

So, these three things should have been written really across a row, so it is just to add So, it is first entry is this, the second entry of this is x minus x 1 x 3 and the third entry would similarly be the corresponding. So, let me just skip the third entry anyway this is something for you to check. So, check again, so exercise check that this is correct, check this equality and you should already figure out that these are things that we have seen before each of these three things has appeared when we talked about Lagrange interpolation.

In fact, these where exactly those 0 1 polynomials, this polynomial that is in the first entry here is precise to the polynomial which takes the value 0 on x 2 and x 3 and takes the value 1 at x 1. Similarly, entries 2 and 3 will be the ones which takes 1 at x 2 or x 3 and 0 on the other two variables. So, finally, what is the final answer function f of x that we were looking for is just this row vector multiplied by the column vector y 1 y 2 y 3 and if you complete the calculation you will see that is exactly the formula for Lagrange interpolation.

So, the final step, so I will again leave this as an exercise. So, finally, check size check that the formula that you get for f of x using this approach in terms of solving systems of linear equations is exactly whatever you would have gotten by ((Refer Time: 27:13)) Lagrange interpolation x 2 x 1 minus x 3 plus two more terms y 2 inter something plus y 2 inter something. So, what this ((Refer Time: 27:38)) demonstrate is that the same problem can often be approach in more than one way, what we did earlier with Lagrange interpolation is sort of start with the idea of this whole 0 1 idea of trying to find polynomials which did the 0 or 1 thing and using that to quickly get the answer.

So, that allowed us to pretty much instantly write out the candidate polynomial, here it is s a somewhat more systematic procedure we do not really start with that brilliant idea. But, just systematic way of solving for the three coefficients that occur in the polynomial and thinking of the three conditions are just being three linear equations for those variables. So, this is somewhat more systematic procedure, but the end of the day it pretty much, you know what pops out is exactly the same 0 1 idle. So, we will talk about other applications of matrices next time.