An Invitation to Mathematics
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UNIT I
Polynomials
Lecture – 1C
Applications of Long division

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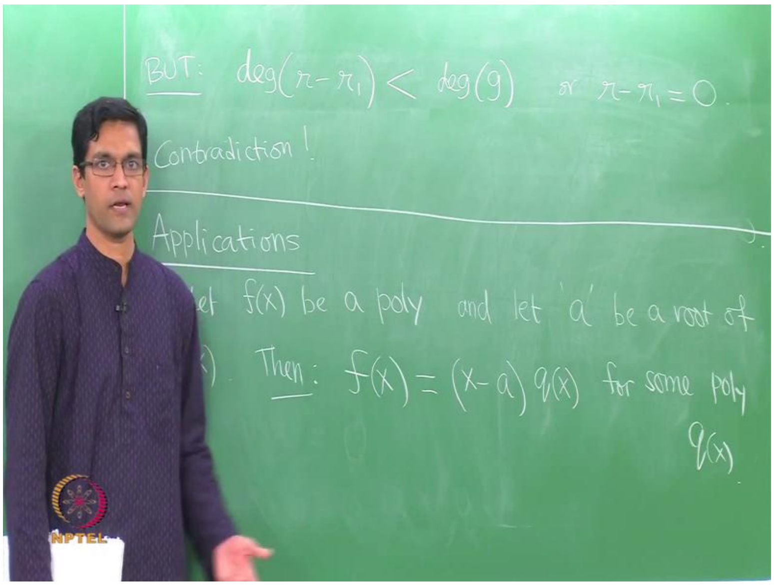

Let we just, you have a couple of Applications of this Long Division business. So, this long division often called the division algorithm is a very important statement regarding polynomials, it has all kinds of very nice consequences. So, for example, if so here is the first application of the division algorithm, let f of X, be a polynomial and let a be a root of f of X. Let a be a root, a is a number, you could assume it is a real number or a complex number and so on, according to the case, a is a root of a f of X. Then, f of X can actually be written as X minus a times some other polynomial times q of X for some polynomial q.

In other words, if f of a is 0, then f can always be written as X minus a times something, it is a multiple of X minus a. So, observe if f has this form of course, if you substitute X equal to a, you will surely get a 0, because there is a term X minus a in front. The key content here is really the converse that, if f of a is 0, then f must be a multiple of X minus a and let us prove it just using the division algorithm.

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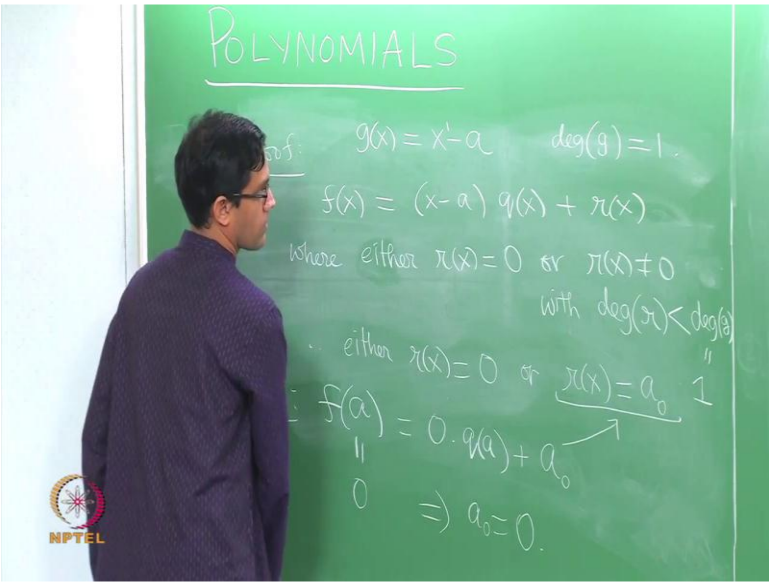

So, let us do the following, let us take, so how do we prove this fact. So, let just take g of X to be X minus a and just apply the long division or the divisional algorithm. It says when you divide f by g, f of X can be written as g of X is X minus a times a quotient plus a remainder, where what is the property of the remainder, the degree of the remainder. So, either where either, the remainder is 0 or it is non-zero and has degree, it is non-zero of with degree of the remainder being strictly smaller than the degree of the polynomial g.

Now, observe g is just a polynomial X minus a. So, it has a degree 1, because that is the highest power of X, which appears, degree of g is 1. So, of course, this must be a 1. So, 1 is the tell us about the degree of the remainder r, it can be only be a 0, because the degree strictly smaller than the degree of g. So, there is only one possibility for the degree, the degree is always 0 or higher, so therefore, we conclude that either r of X is 0 or r of X has degree 0.

Now, what is the degree 0 polynomial look like? It is only got a constant term, remember it looks like a naught plus a 1, a 1 X plus a 2 X squared and so on, but if it has degree 0, it means it only has the constant term a naught. So, it is basically a constant, what we are concluding from here is that the remainder is a constant, it is a either a 0 constant or possibly a non-zero constant.

We claim is, it is in fact the 0 constant, it cannot be a non-zero constant, why because we have some fact about f, we know that, if you plug in X equal to a, f become 0. So, now, observe that if I plug in X equal to a, the left hand side is f of a, which is a 0. So, what is given about f, whereas the right hand side becomes the following, if I plug in X equal to a, of course, this term becomes a 0. So, it become 0 times q of a, which is a 0 plus r of a, well r is just you know, it is 0 or a naught.

So, let me take this second case, I claimed that this case cannot really appear, so plus this remainder a naught. Now, we just look at what both sides are, the left hand side is 0, the right hand side has a 0 in the first term plus an a naught, so a naught had better be a 0. So, we conclude from here that a naught is actually a 0. So, what does that mean? It says that the remainder cannot really be a non-zero constant, we conclude that the remainder is 0. So, that tells you that f of X is just X minus a times q of X, you do not have any remainder term there, so that is exactly what we wanted to conclude.

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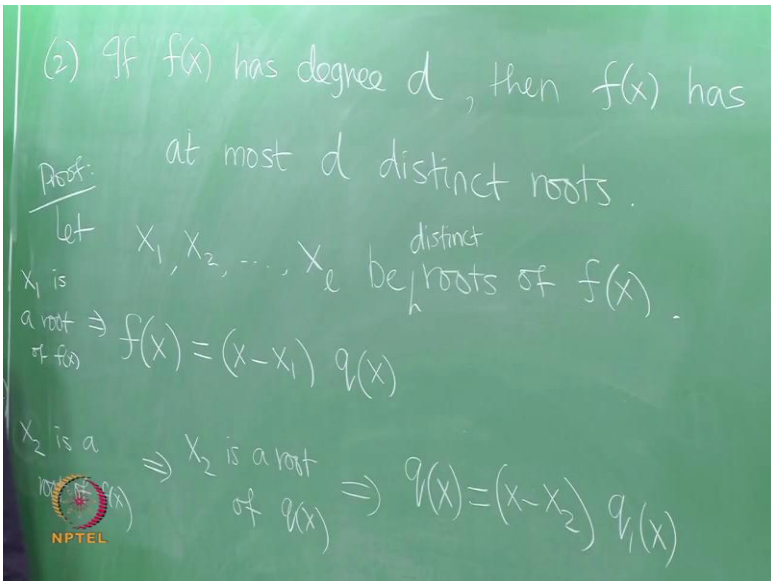

So, that is the first application, so we will just complete the proof of that. The second thing here says if f has degree d, if f of X has degree d, then it has at most d distinct roots then. So, our polynomial of degree 10 for instance can have at most ten different roots, it cannot have 11 or 12 or anything higher than 10. So, let us see why this must be true, so here is a proof of this fact, so let us write down all the roots.

So, are you know let us say suppose we know some number of roots, let X 1, X 2 till X l be roots of f, let them all be distinct. So, I am looking at different numbers be distinct roots of f. So, now, we use the first application, where it said any time I have a root that if a is a root, then X minus a divides the polynomial f. So, f can be written as X minus a times something. So, we just do the following, we write f of X.

So, since X 1 is a root, we conclude f of X is divisible by X minus 1, X minus X 1 times q of X, so because X 1 is a root. So, let me write this on X 1 is a root of f of X, implies f can be written in this way, so that is what we just conclude it. Now, let us plug in X 2 into this equation, if you plug in X 2 for X, then f of X 2 is 0, X 2 on the right hand side, you will get X 2 minus X 1 times q of X 2 and X 2 is not equal to X 1, so observe.

So, second conclusion of following, since X 2 is a root of f of X, it actually implies that it must be a root of q of X as well; X 2 is a root of q of X. Because, X 2 minus X 1, the term in front is non-zero, which means again you use the same thing which says, since you know you have a root of q of X, X minus X 2 divides q of X. So, let us called the quotient as something else, let us call it q 1 of x.

Now, we what we get, putting these two things together, f of X is X minus X 1 times q, q is X minus X 2 times q 1. So, putting these together you conclude that f is actually X minus X 1 times X minus X 2 times this polynomial q 1 of X. Now, again the same thing, I have a third root X 3, I plug in the third root here, I get a f of X 3 is 0, I plug it into the right hand side, I get X 3 minus X 1 times X 3 minus X 2. Both those are non-zero terms, because X 3 is not equal to X 1, X 3 is not equal to X 2. So, these two are non-zero, so therefore, it must be q 1 of X 3 which is 0.

So, again we conclude the same thing, so X 3 being a root of f, so f of X 3 is 0, therefore, implies that this quotient q 1 must have X 3 as a root. So, again we do a same thing, therefore, X minus X 3 must appear in q 1, you can write it as a X minus X 3 times something let us call it q 2. Now, you keep proceeding and at every step you conclude that, the term of the form X minus X 3 will divide the next follow, X minus X 4 will divide the next one and so on. So, you keep going and conclude you know, as at the very end of the process that…

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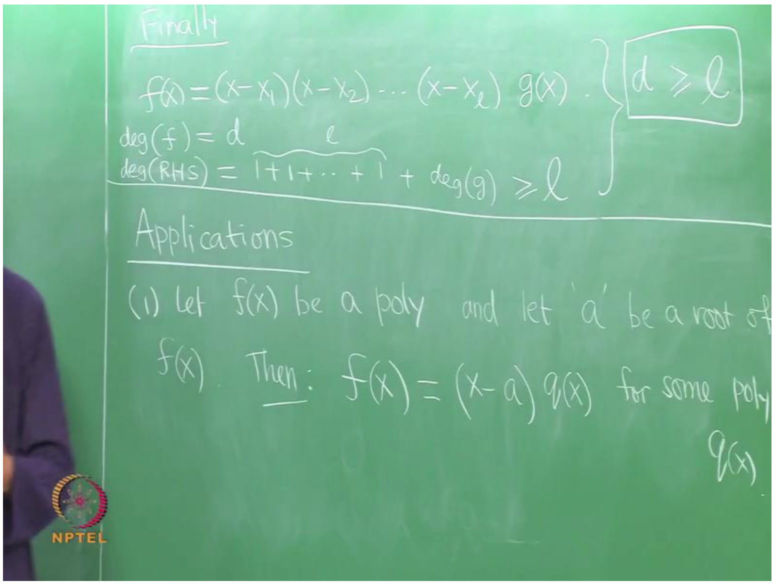

So, keep going, finally what you conclude, keep pulling out various factors like this, all the way to X l times some you know some quotient, let us call it let me call it g of X that is the final quotient, the quotient get the last step. So, finally, this is my conclusion. Now, observe that again it is the same degree argument; on the left hand side f of X has degree. So, what is the degree of f of X? So, the degree of f of X is d; that was the assumption.

Whereas, the degree of the right hand side, observe the right hand side has well, how many times as it have, there are l terms here and then one more which is in one other g of X. The degree of the right hand side abbreviate to adjust is nothing but you know remember the property, if have the product the degree just adds up. So, I have each of these factors has degree 1, so it will be 1 plus 1 plus 1, you have l such 1's plus the degree of g, so whatever this is, this is at least l.

So, the degree of right hand side is at least l, whereas, the degree of the left hand side is exactly a d. So, what you conclude from here, hence since a left hand side and the right hand side are both equal, you conclude that d must be, d is the degree of the left hand side and so that at least, because the right hand side is the same as the left hand side. So, degree of the RHS as same as the degree of the LHS which is d.

So, d is at least l; that is the conclusion that is exactly what we wanted to conclude. So, this was claim that you can have at most l distinct roots. So, this is just one final thing before we end this small lecture. This has one very nice consequence a different way of rewriting the same thing, it says polynomial of degree d is determined by it is values at d plus 1 points.

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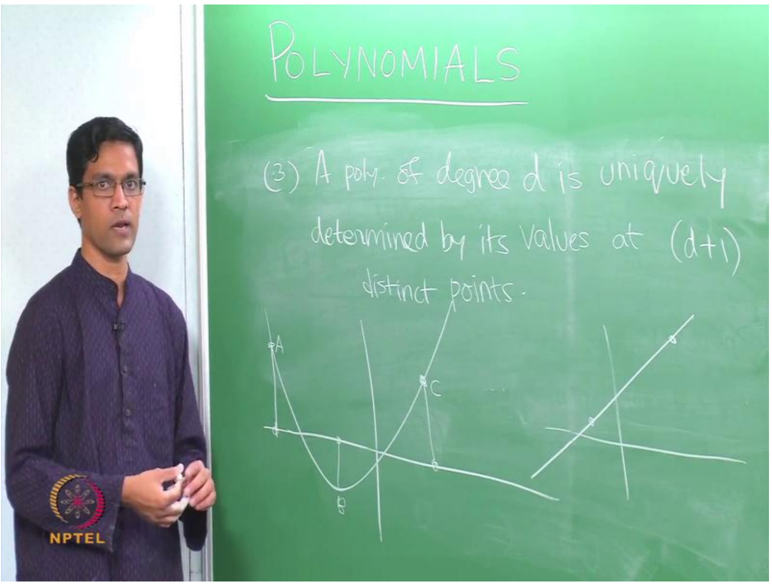

So, since we would has other applications in the same thing, a polynomial of degree d is uniquely determined by it is values at d plus 1 distinct points. So, it is like saying, if I know that my polynomial has degree 3, then if I know the values of the polynomial at four points, any four points distinct points, then if the polynomial is uniquely determined, you cannot have two different polynomials which have the same value at four points.

Similarly, you know we already proven an example of a polynomial of degree 2, if we do a graph of the polynomial, it saw it was a parabola. So, in general quadratic polynomial will have a graph as a parabola. So, one instance of this is like, saying the parabola is uniquely determined, if you just tell me, what it is the values are at three given points. So, if I just tell you the value at this point, the value at this point and say the value of this point, so through these three points here, so let us call them A, B and C, there is a unique parabola passing through the points A, B and C.

So, sort of a 1 simpler version of this is, if I have a polynomial of degree 1, it is graph is a straight line and a straight line here according to the polynomial of degree 1 is uniquely determined as soon as you tell me it is values have two points. So, just like saying there is a unique straight line which passes through any two given points. For a parabola, you need three points that you will determine in the parabola, if you have a cubic curve, a curve of a degree 3 polynomial, if you give it is values at four points, it uniquely tells you that what the curve is. So, this something that, we will start out looking at in the next module.