An Invitation to Mathematics
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Unit
Conservation principles
Lecture - 29
Examples – II

Last time we talked about Conservation Principles through two examples, one in which the conserve quantity with the sum of the four positions of the four coins, the other in which the area of the figure form by those points was the conserve quantity. So, we will talk about few more examples this time.

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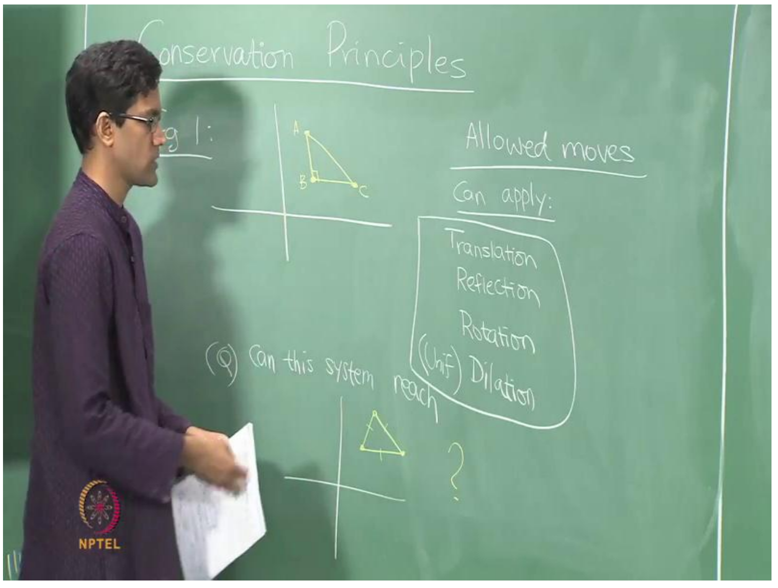

So, let us do the following, let us again talk about examples on the plane, so what is our initial configuration going to be. So, let us now take three points which form let us say a right angle triangle. So, here are three points A, B, C, so those are my initial point, so my configurations here are triples of points like this A, B, C and in this case, the initial configuration is where they form a right triangle. And of course, now we need to give the evaluation rules, how do you know how does the system change with time.

So, how do the positions of these three points change with time, well here is what you allow to do. So, what are the allowed moves at each time, you can apply any one of the following transformations you can apply your translations, you can apply your reflection, a rotation or a dilation. So, these are examples of transformations we talked about, so by dilation I mean uniform dilation, so these are the four operations or allowed moves that you can perform.

So, you can translate this triangle, you can reflect it about any line, you can rotate that triangle or you can dilate the entire triangle. So, and you can at each time you pick any one of these four operations and you perform it that is the idea. Now, here is the question is it possible that at some point of time t that you land up with an equilateral triangle. So, the question is can this system reach the following configuration, one in which the three points lie on the vertices of an equilateral triangle, so imagine this is an equilateral triangle. So, that is the question can somehow reach this configuration.

So, again the attempt to answer such question lies and trying to find something that is conserve during evaluations something that does not change. And in this case it is easy to answer, because we have of course, studied these in depth here is what we know a translation or a reflection or a rotation gives you a congruent triangle, it does not change lengths or angles. So, what you get is just a congruent triangle, whereas dilation is uniform dilation will not preserve lengths, but it will certainly preserve angles.

So, it will give you similar triangle, so that something we looked at. So, another way of stating this is you keep track of the three angles of this triangle, the three angles the list of the three angles is conserved.

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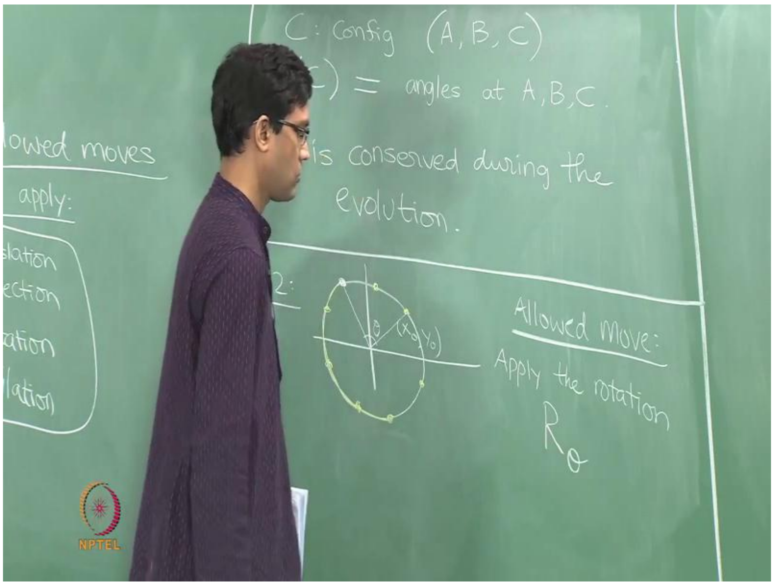

So, how should we, so if C denotes a configuration, so here the configuration is just the three points, the three vertices of the triangle, what is a gamma function if you wish, the gamma function can just leave the list of the three angles. So, this is just the angles at the points A, B, C. So, here the conserve quantity is not just one quantity if you wish, but rather the list of the three angles, but nevertheless observe that the evaluation rule conserves gamma. So, observe that gamma is in fact conserved during the evaluation.

In other words, no matter which of these moves you applying, what you will get will be a triangle which has the same three angles. So, it will continue to remain a right angle triangle throughout in which the top angle of course, is also equal to whatever this is. So, it will, if the three angles are 90 degrees alpha and beta, the same thing will continue to whole for every triangle that you get during the evaluation. So, you cannot get an equilateral triangle that is basically the conclusion here.

But, the key point is really in figuring out what the conserved quantity is, which in this case is for instance, the three angles. Now, here is another example where we will not look at all the transformations we wrote out, but only about rotations. So, let us only look at rotations, so again here the same thing, so what is the usual configuration here, here instead of taking a triangle let me just say initial configuration is just some point.

So, I have a point on the plane that is my... So, the coordinates x and y let call it as x naught y naught, this is the initial configuration, my system consists of just the point and I should tell you how the system evolves. So, how do I transform it at each instant of time, well I apply a rotation to it, so what is the allowed move is you take this point and you rotated through some angle theta. So, rotated meaning you join it to the origin and then you rotate this by angle theta and doing that will give you another point that is the new configuration system.

And similarly, at the next instant of time you again pick some random angle theta and you rotate this by the angle theta, what you get is again a new configuration. So, in this case just from the geometrical description on the system, here is what the system typically goes through. It starts out at this point, maybe you rotated it through a certain angle, and then at the next instant you rotated it by some other angle and so on.

So, it keeps sort of moving around on the circle occupies various positions depending on what angle theta was picked at that time instant, so that is the evaluation of the system. And now, here is the question what is the conserve quantity? So, of course, remember rotations you know they certainly do conserve areas and things like that, but more importantly they preserve lengths. So, here... So, what is an allowed move? So, let me just write that out, an allowed move is you allow to apply the rotation let us give the rotation, it is a linear transformation by the angle theta.

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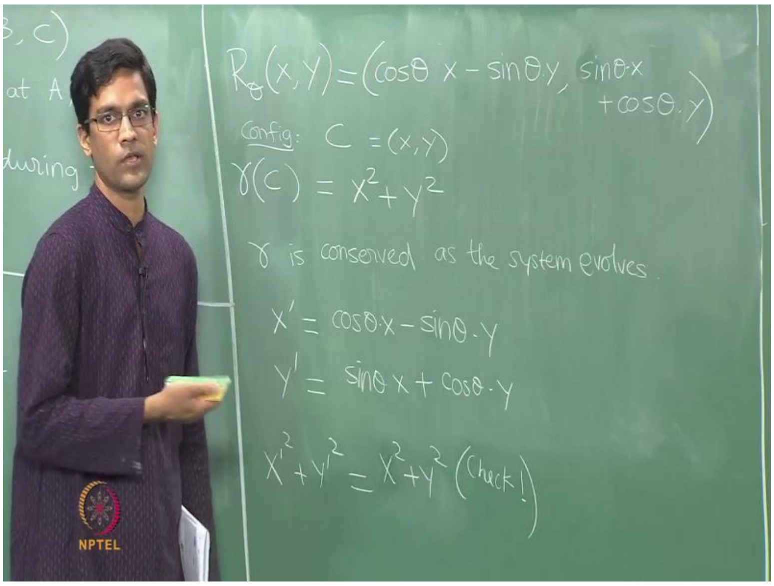

So, what rotations are you allow to apply? R theta has the following formula. So, it is something for you to check, this is just cosine theta times x minus sin theta times y comma sin theta x plus cosine theta y. So, what you are allowed to do at each instant of time is to pick an angle theta and apply this transformation to the point x y, there by obtaining another point x y. Of course, it is much easier without the formula. It is easier just geometrical to see what you are doing it just moving it to a point which is rotated by an angle theta.

So, you apply a rotation, so now, here is what is the conserve quantity. So, let us define the function gamma for a configuration. So, what is a configuration here? It is just the coordinates of the point. So, the configuration C just denotes the point that we have and we will define the gamma function to be the length or the square of the length. So, let us define the gamma function to just be the length function for your wish and the key thing here is that rotation preserves lengths.

So, observe that gamma is conserved under the evaluation is conserved as the system evolves and why is that. Well, because if you wish you can just actually check this by calling this as x prime and that as y prime. So, let us just write this out one step, I just leave the verification to you, call the x coordinates as x prime. So, this is just sort of a verification from the formula as I said geometrical it is pretty clear that the length is preserved sin theta x plus cosine theta y. So, that is the new position or the new configuration of the system.

And now if you just compute x primes squared plus y primes squared, you will find that the identity cos square plus sin square is 1 will help you prove that x prime square plus y prime square is equal to x square plus y square. So, check this, so here is the conserve quantity, so for instance this in particular proves that if as the system evolves, if you start out with the point at a distance 5 from the origin, the system can never reach a state where that point is now at a distance 10 from the origin, you are constrain to stay within the distance 5 from the origin. So, those are the only possible configurations that you can reach.

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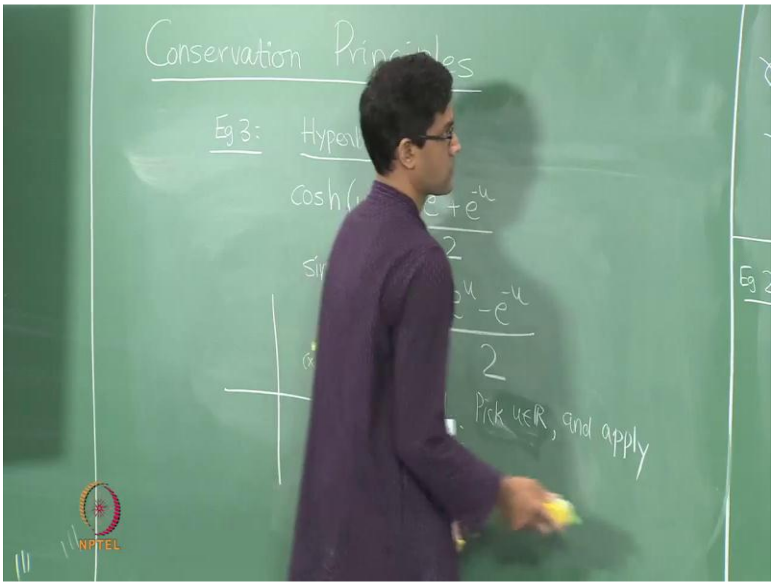

And here is a minor quick of this example that I leave for you to try, so this also involves the so called hyperbolic function. So, let me just recall what is called the hyperbolic cosine the hyperbolic sin. So, in case it is not already familiar, so cosine hyperbolic of u, so u is any real number here is defined to be the following e power u plus e to the minus u divided by 2 and sin hyperbolic. So, hyperbolic sin is defined as e to the u minus e to the minus u divided by 2. So, there is a hyperbolic version of the previous example.

So, there we said the configuration, so now, here is the system I want to study I configuration again is just a single point. So, for instance I start out at some point x naught y naught and now how do I evolve the system. So, what do I allow, so here is the allow move, there allow to apply the following transformation. So, at each time apply a transformation that is called T u for some value of u. So, you pick u, you pick real number just as in the earlier example, you know what is the allow move you pick an angle theta and then you rotate by the angle theta by you applying the transformation just notation by the theta. Similarly, here you pick a real number u and then you apply the transformation T u which I just describe. So, pick a real number u and apply the following transformation.

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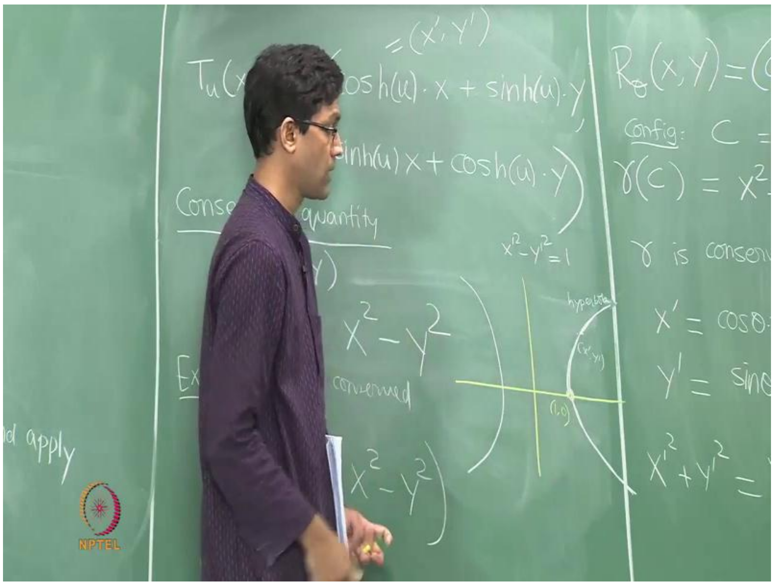

So, what is this just like in the earlier case, here is the definition of T u it takes the point x y to the following point cosine hyperbolic u times x plus sin hyperbolic u times y comma sin hyperbolic u x plus cosine hyperbolic u y, it is almost like the earlier one that we had cosine minus sin, sin and cos here is, it is just cosine hyperbolic sin hyperbolic sin hyperbolic cosine hyperbolic. So, this is allow transformation, so at this point we do not yet have a very geometrical idea what this does, so it is not like a rotation or any such thing somewhat different.

Now, again the question is really the following can you find the conserve quantity for this evaluation. So, here is anything that I like you to check, so here is the conserve quantity. So, what is a configuration to as I said is I just the point coordinates are the point and define gamma of c in this case to be x square minus y square and prove that... So, check, so this part is the exercise check that gamma is conserve and how do you do this, you have to do pretty much the same calculation we do earlier you call this as x prime and y prime the two.

So, these become a new coordinates of the new configuration the system and you will have to check that for this new x prime and y prime, the different x prime square minus y square prime how are remains the same as the whole defines, just like we check for the cosine and sin case that the some other squares remains constants, here after applying to u you get a new x prime y prime who's different of the squares remains at constants.

So, what is this really means, it says the following geometrically that suppose the original system, the initial configuration suppose you were at the point 1 comma 0 then the gamma function corresponding to your initial configuration is just 1 square minus 0 square. So, you are at the gamma function is a one this case and what it says as the system evolves the gamma function cannot change. So, no matter what you are new points is x prime comma y prime, the difference of the square had better be a 1.

So, x prime square minus y prime square in this case must remain a 1, because that is what it was at the initial point of time. So, what this means is, so observe this is the equation of a rectangular hyperbola. So, which sort of looks like this, so in also another branch on the other side. Now, this hyperbola here is the constraint meaning the point the configuration system is constraint such that did can only lie on this particular hyperbola it cannot move anywhere else, you cannot get other arbitrary points on the plane, you are always constrain to only lie on this hyperbolic. So, at each time it can give you hyperbola.

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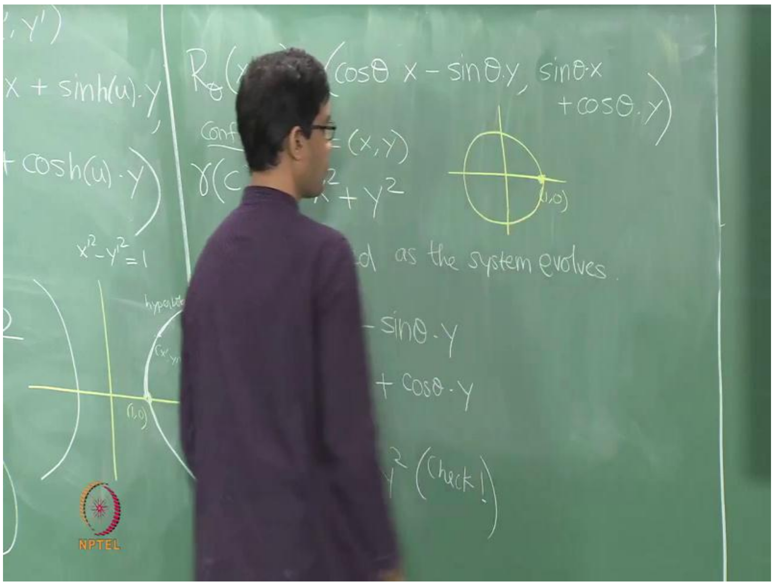

Similarly, in the earlier example what was the corresponding thing in the earlier example that was a circle. So, if you are initial point is 1 comma 0 in this previous example, then at each time t you are constrain to lie on the circle of radios 1, you cannot move out of this circle, similarly in this case you cannot move out of the hyperbolic. So, these are all examples of conservation principles in what you might call describe systems, meaning I am not thinking of time as being continues variable I am only doing something at time t equal to 0 t equal to 1 t equal to 2 and so on.

Now, the conservation loss that you are lightly to hard before in the context of physics, classical mechanics specially are there of course, time is a continues variable. So, you imagine that time varies continuously and as t varies there is a system with varies continuously with time. But, no matter how it various there are some quantities which are conserve. So, let me just make some brief remarks about conservation principles in the context of classical mechanics.

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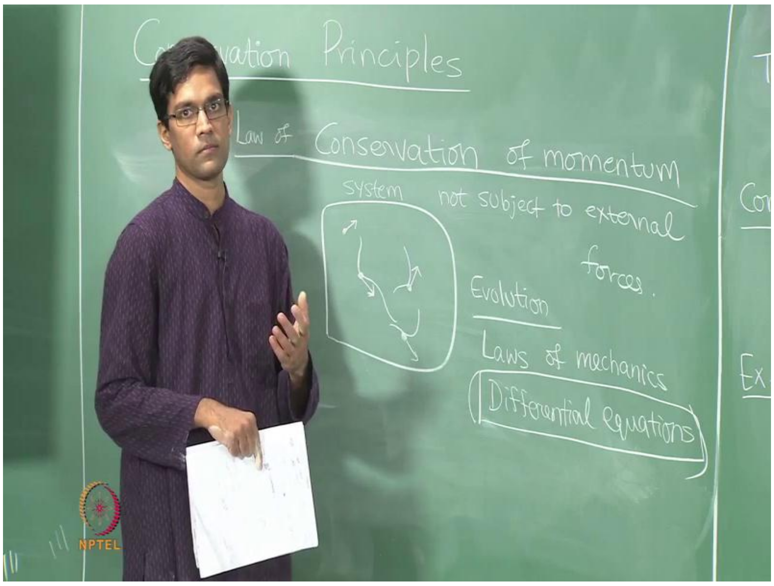

So, here is the various conservation loss that you might have seem especially for instance law the conservation of momentum. So, this is an in fact and instance of the sort of conservation principles it is we are talking about. So, we often called the law of conservation of momentum, so again what is the set up there you have a system. So, it is a system classical mechanics it could have a bunch of particles and so on. But, there is of course, one important restriction here that the system is not being at add upon by external forces.

So, system not subject to external forces and system is probably some initial configuration. So, there are you know various particles moving at various play, you know along various directions. So, they have various momentum of instants, so this could be the picture of the system, now the particle themselves are interacting. So, there are... So, how does the system evolve with time, the evaluation is according to well the laws of classical mechanics if you wish. So, the evaluation is according to... So, what are the rules for evaluation.

So, in our case we wrote them out as certain allow moves, we said these are the things you can do to change the state of the system. Now, in the case of classical mechanics, the laws of mechanics or for instance we wish Newton laws which are essentially some differentially equations. So, the evaluation is by certain laws of machines for instance Newton laws, but what are these you know just written out mathematically well these are actually certain differentially equations.

So, these differentially equations tell you how the system changes with time. So, they will be given by say the derivative of the momentum is therefore and so on. So, there are some rules which tell you what the trajectories are that these particles take and there are rules for how they interact and so on. So, these differentially equations govern the evaluation of the system. Now, here the conserve quantity is something which has the following property that you know no matter what the differentially equation tells you happens to the system, the total momentum of the system can never change.

So, the form of the differential equations is such that even though the system changes with time t, the total momentum cannot change. So, at the end of the day the conserve quantity very much depends on what the rules for evaluation r that is a thing that we have seen throughout in the very first example. The evaluation rule was something special you had you know what we looked at last time a couple of coins picked out and move equally and you know move by the same amount in a opposite directions that very special form of the evaluation rule guarantee that the total some of the positions would remain constants.

Similarly, if in the case where we had transformations certain allow transformations of the plane, the only allow transformations where all things which had determinant one or at least absolute value determinant one. So, the total area was conserve, so the fact that you had a very special restricted set of evaluation rules is what guaranties or is what allowed as to find certain nice conserve quantities. A similarly here the rules of classical mechanics are the differentially equations have this very nice form, assuming there are no external forces, this short of guaranties that no things like momentum or in other contacts the laws conservation of energy and so on. So, these rules guaranty they exist of certain conserve quantities that is sort of the broad model in the story. So, we will talk about matrices and other things next time.