**An Invitation to Mathematics**
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**Unit**
**Conservation principles**
**Lecture - 28**
**Examples – I**

Today, we will talk about Conservation Principles. So, what is the context in which these principles arise, it is when we study systems that evolve with time, so that is the context. So, let me say a little bit more about all these, pretty much by examples. So, we will consider a few examples from which it will be clear what a conservation principle means and what sorts of systems, we will consider and so on.

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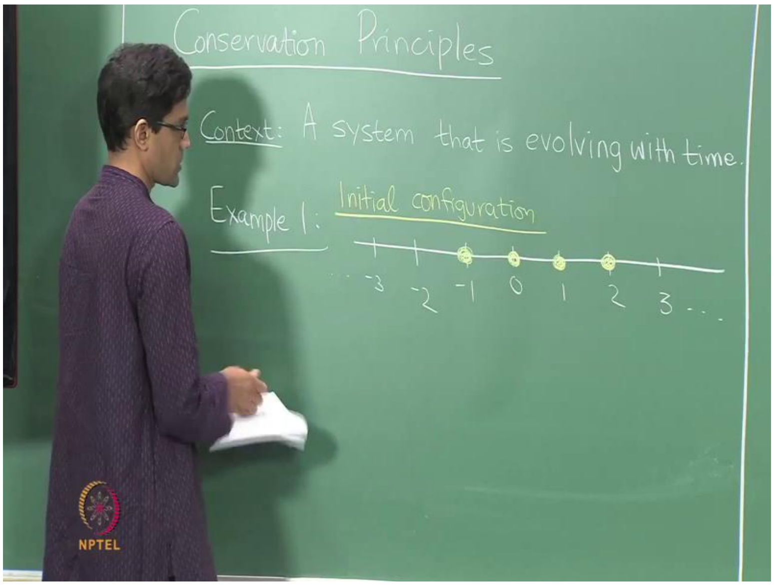


So, here is an example. So, what is my system? So, here is the description of the system, imagine we have all the integers marked of on the number line. So, it is 0, 1, 2, 3 and so on minus 1, minus 2, minus 3, so I have a number line are which all the integer points are marked. And what I do, so what is my system, my system consist of let us call it 4 coins placed at four positions on this set of integers.

So, for a start, here is what I do, I have 4 coins, so one of them placed at minus 1 and other placed at 0, 1 and 2. So, here are my initial positions, I place 4 coins on these four integers, so this is my initial configuration of the system. So, this is what we would call the initial configuration. So, this is what happens at let say time t equal to 0 and my system should now evolve with time. So, then what you have is a rule, which tells you what happens at every instant of time.

So, let us imagine that here time is not quit a continuous variable, but rather a discrete variable, which means, say time t equal to 0 this system look like this. At time t equal to 1, it will change to something else, at time t equal to 2, it will change again and so on. So, the system changes at time steps t equal to 0, 1, 2, 3 etcetera.

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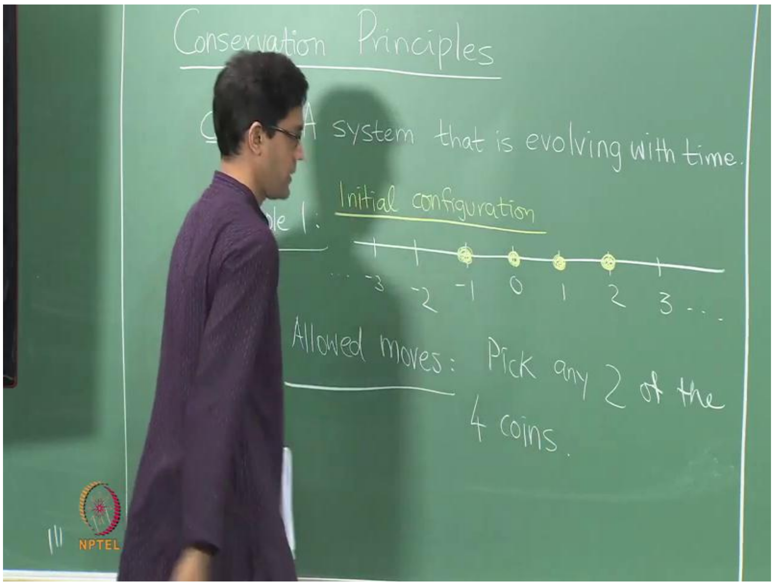


So, what is the rule for evaluation of the system? So, let us call this a move. So, what are the allowed moves or let us how do you we allowed moves or rules for changing the system. So, time t equal to 1, here is what you allowed to do, you pick any 2 of these 4 coins. So, here is the allowed move, pick any 2 of the 4 coins, so choose any 2 and do the following move one of them k steps to the right and the other k steps to the left.

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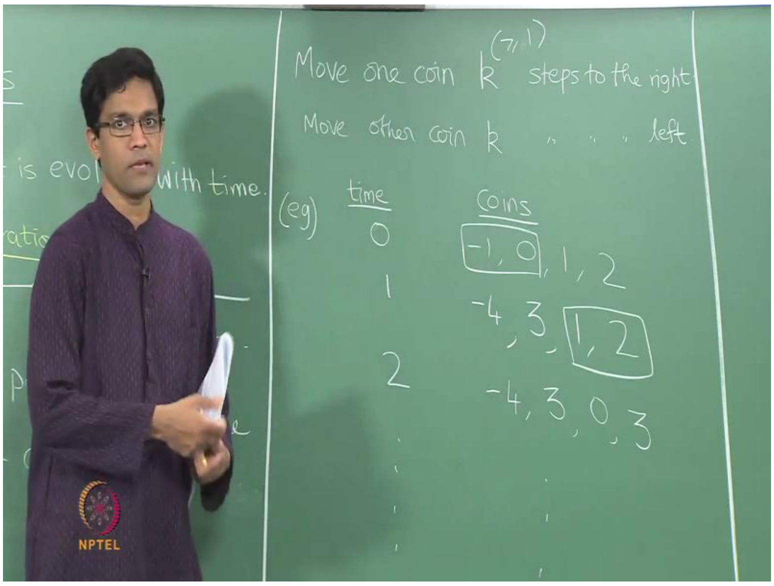


So, move one coin k, where k can be any number, k steps to the right and move the other coin k steps to the left. So, here is the rule for evolving the system. So, let us sort of look at one possible trajectory that the system might have taken. So, let me only note the four positions of the 4 coins. So, at time t equal to 0, so let me just say here is the value of time. So, at time t equal to 0, I have the coins, here are the position of the coins, there minus 1, 0, 1, 2.

Now, time t equal to 1, what I am supposed to do, I am supposed to pick two of these four. So, let say pick minus 1 and 0, suppose I pick those two and what I am supposed to do is move one of them k steps to the right, the other k steps to left. So, imagine I moved say 0 three steps to the right and minus 1 moved three steps to left; so that is a move that I carry out. So, minus 1 moves to minus 4, 0 moves to 3 and these two stay as they are, this could be one thing that happened in the next time steps.

Then, time t equal to 2, I repeat a move meaning, I again pick any two of these four, move one of them k steps to the left and the other k steps to the right and here k can be anything. So, here k I mean just makes sense to be a let say at least 1, if you do not move them, then nothing will happen, let say it is at least 1, it is an integer at least 1. Similarly, so here I can have reputation.

So, for instance, I can choose 1 and 2 in my next step and move 2, one step to the right and 1 one step to the left. So, here is what I could have got turn off the next step, if I chose these two and the first step I chose these two, here I move 1, one step to the left, 2 one step to the right. So, there are 2 coins now placed at the position 3 for instance and so on.

So, you can now imagine how this game placed out, the system evolves at each time steps and say at time n, you know whatever be the configuration at time t equal to n, at time t equal to n plus 1, you should take 2 coins out of that configuration and move them in the described fashion. So, the system evolves with time in this function and as you can imagine, there are many possible trajectory, there are many different things that can happen, depending on what you chose and how much you move it and so on.

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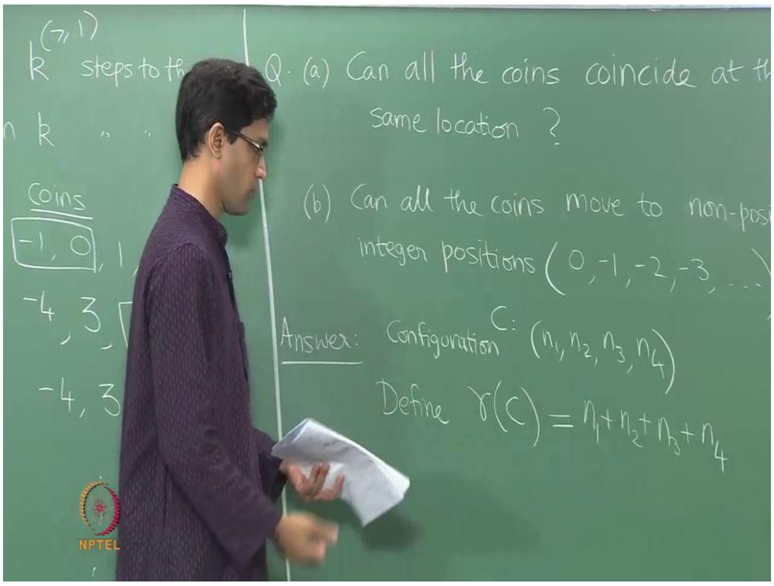


So, here is the question, there are two questions is it possible for the system at some time t, so can all the coins coincide at the same location at some time t in as the system evolves is it at all possible for all the coins. So, they are moving in some funny fashion is it possible for all of them to somehow coincide at the same point; so that is question number 1. And here is the other question, can all the coins move to the negative part of the real access, move to non negative integer positions; let us write to move to non positive.

So, what I mean by the non positive integer positions, let us number 0, minus 1 minus 2 minus 3 and so on. So, is it possible for all the coins to somehow move in such a way that the lie on the negative part of the real access. These two questions and what we want to understand is, you know somehow by analyzing the moves that are allowed for the system. Can it somehow happen that at sometime t that configuration a happens or configuration b happens, so those are the question that is we want to answer.

And here is the answer, which is based on the following important observation, notice that if you know the rule, let us define a functions. So, let me say the configuration, so suppose I given a configuration, so what is a configuration is just the four positions of the 1st coin, 2nd coin, 3rd coin and the 4th coin, so I have these four positions.

Now, let us define a function. So, let me call this configuration as C, let us define a quantity, let us call it gamma of C, it is a quantity which depends on this configuration C. Let us define a just be the sum of these four numbers n 1 plus n 2 plus n 3 plus n 4. So, to each configuration we associate this number, it is just the sum of four positions.

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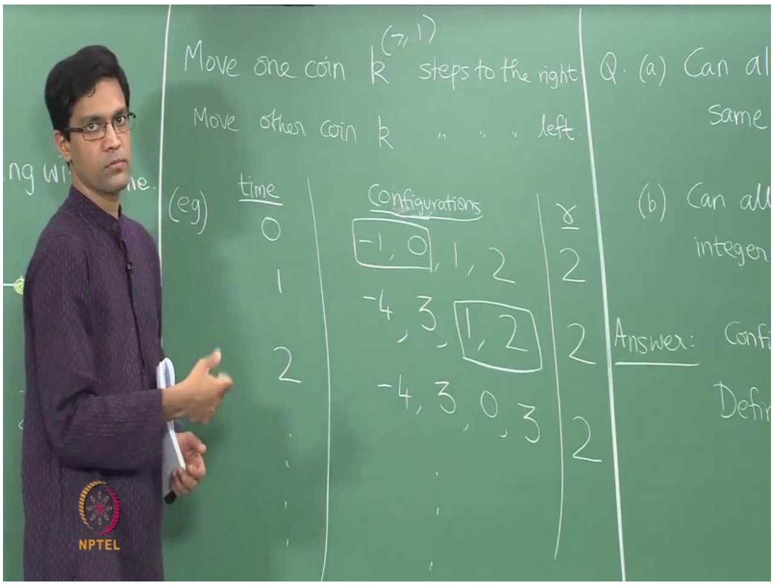


Now, observe what happens as the configuration evolves, so let us come back here and write down these numbers. So, here are my configurations if you wish, so let me now call it configurations and let us compute the gamma function on each of these configurations. So, now observe that the very first guy has let see 2 plus 1 3 minus 1 is 2. Now, the second fellow here is 3 plus 3 6 minus 4, which is 2 again, this is 6 minus 4 which is 2 again and so on.

And here is what you will observe, pretty much any configuration that you can obtain by applying the rules will always give you the same value of gamma, gamma will always be a 2. And this of course, not at all surprising, if you really analyze the rule properly, so what is the rule say, well you pick any two of these four numbers n 1, n 2, n 3 and n 4. You pick two of them, you increase one of them by k and you decrease the other by k; that is what the rule says.

And of course, doing that will not change the sum of the four numbers. So, one of them will increase by k, the other goes down by k, the total sum remains the constant. So, as the system evolves the function gamma does not change; that is the key observation here.

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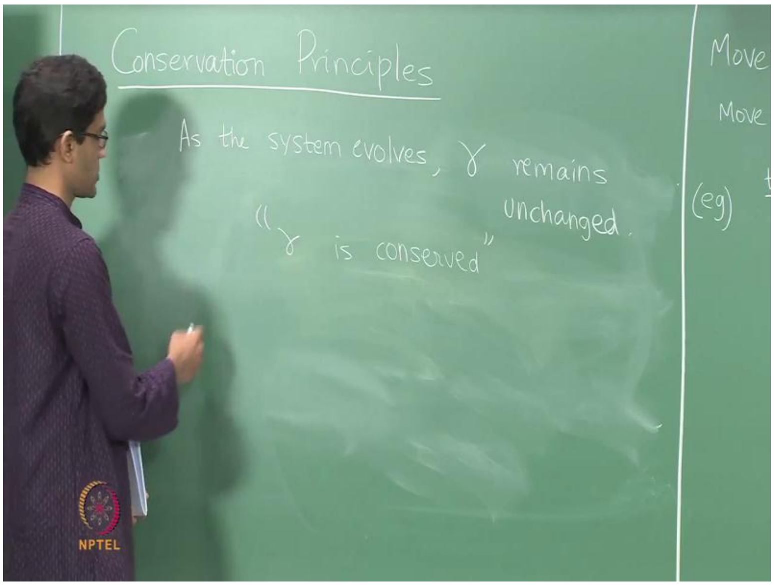


So, observe that has the system evolves, the gamma function remains the same. So, this is often expressed by saying that the gamma is conserved. So, of course, you know what finding a quantity, which is conserves, may not be very easy in any given situation. But, if the evaluation the rules for evaluation admit a conserved quantity, it often makes answering questions like what we post, somewhat easier at least in some cases. So, let us now use this to answer our first question.

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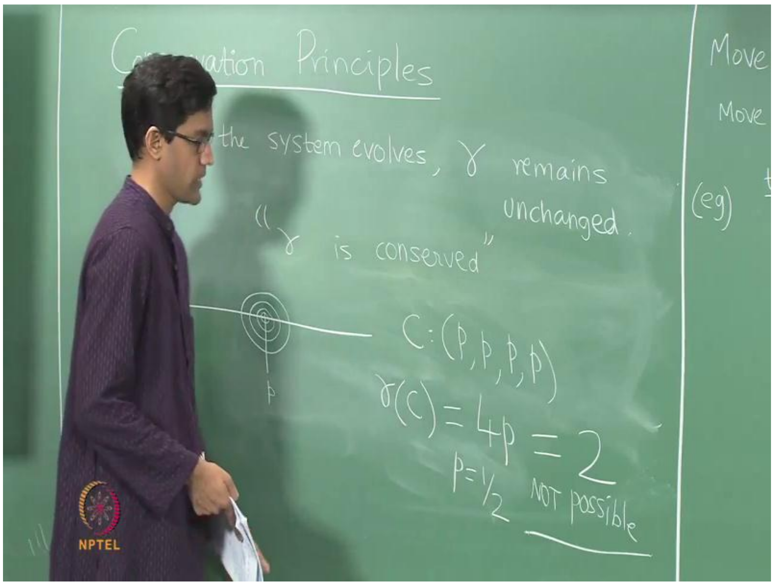


So, the first question says can it happen somehow that all the 4 coins are located at the same point. So, I want to know, let us call this integer as p, can it happen that my configuration, so can I land up with a configuration that looks like p p p p; that is a question as time evolves. Now, let us see this is possible. So, what is gamma? The gamma function for this configuration is just the sum of the four numbers; that is 4 p and what we know the gamma function does not change, so the starting configuration had gamma function as 2.

So, of course, whatever configuration you can get at some time t had better also have the same gamma function. So, if at all it is possible to get p p p p, then this equation must be true, 4 p must be equal to 2, on other words, p is half, but observe that is not possible, because p was supposed to be an integer, p is position of the coins. So, p is an integer, so this is not possible. So, it is actually not possible to get all 4 coins to coincide of the same point.

Now, I leave the second question as an exercise, try using the same argument just the same, apply the same gamma function. To conclude that, it is not possible for all 4 coins to somehow move to negative part of the access. So, the second question is an exercise that you have to try.

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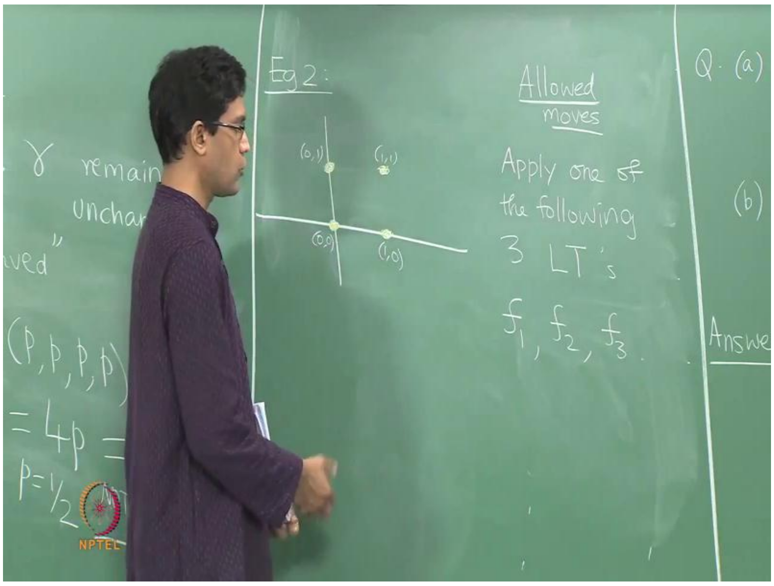


Now, let us do some more examples, so here is an example 2. So, here is again you know we need to describe the system. So, now, my system is following, it is a configuration of 4 coins on the plane. So, what are my 4 coins? So, let us just say I have one of them is, so they are the four corners of the unit square. So, this is my initial configurations, so what are these points, here is 1, here is 1. So, these are 0 0, 1 0, 1 1 and 0 1.

So, I place 4 coins on the plane and now, what I do, I need to tell how the evolution takes place. So, what are the rules for evolving the system, so here are the allowed moves and each time t, we apply your allow to do the following apply one of the following three linear transformation, what are the three, I call them f 1, f 2, f 3. So, there are three function, there are all linear transformation, what you do can apply that function in all four points at once.

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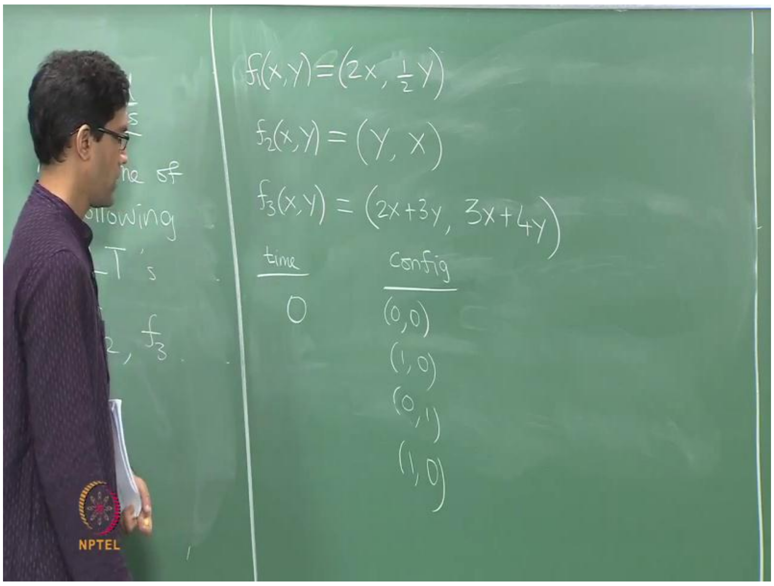


So, let us first write out, what are these linear transformations. So, f 1 is define as follows f 1 of x y is 2 x comma half y, that is one function, second function f 2 x y is y comma x and f 3 of x y is given by it is 2 x plus 3 y, 3 x plus 4 y. So, here is what we do, we explicitly give three linear transformation, three transformation of the plane and a move consist of the following at each time t, you pick one of these things possible linear transformation and you apply it to the four points, so let us given example.

So, here is the configuration, so here is time. So, what are the configurations, well at time t equal to 0, the initial configuration consist of the four points 0 0, 1 0, 0 1 and 1 0, so that is where my 4 coins are placed.

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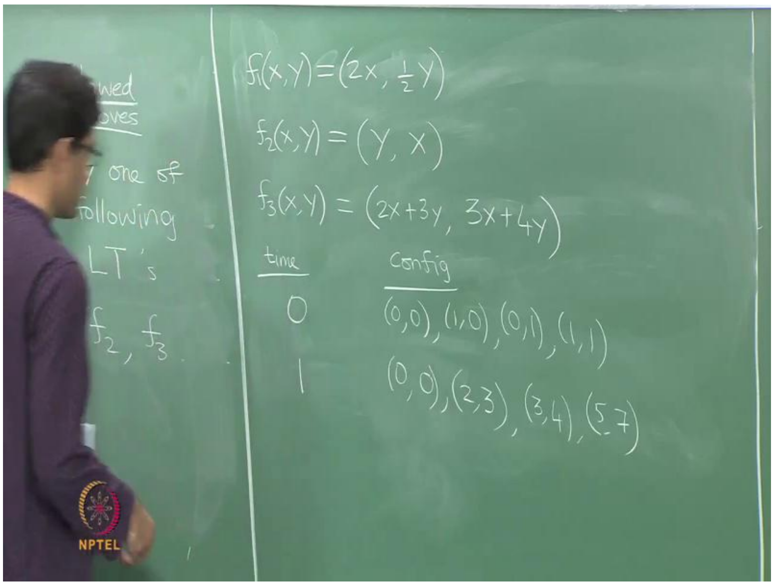


Now, what happens at time t equal to 1, so maybe we will this; that is the position at time equal to 1, I pick one of these three linear transformations. So, let us imagine, I pick the transformation f 3. Now, what I will do is, I apply f 3 to all these four points, after all it is a transformation, so what is f 3 do to 0 0, well we have to calculate, but it is clear it maps to 0 0 again, if you take 1 comma 0, it maps it to 2 comma 3, take 0 1, it maps it to 3 comma 4 and if you take 1 1, it maps a to 5 comma 7. So, these four become a new position of the 4 coins.

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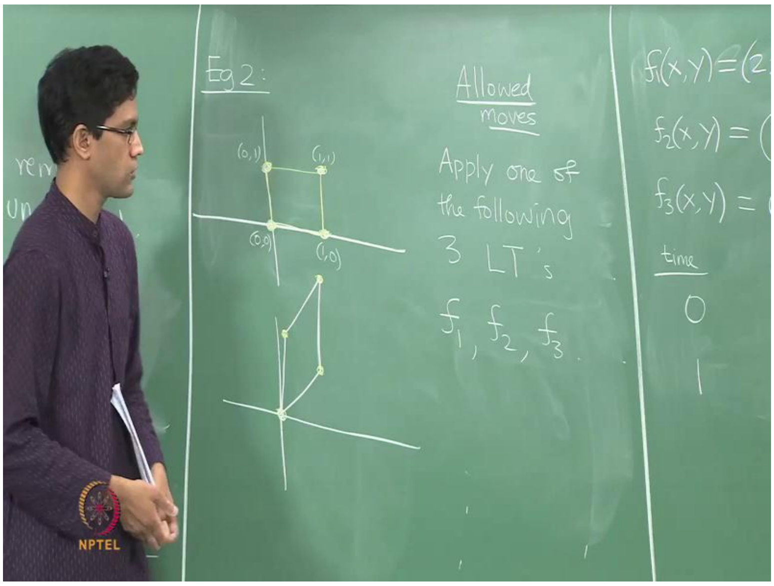

So, observe those are actually the four corners of parallelogram, so we are sort of looked at this things only look at the transformations. So, 2 comma 3, 3 comma 4, so it some parallelogram like this. So, those corners of parallelogram are now the new position of the coin. So, originally they were at the corner of squared, after at time t equal to 1, may be become parallelogram and time t equal to 2, you again pick one of these, you pick f 1, or f 2 or f 3. You apply it to this parallelogram, it will transform to something else, those become the new positions and so on and so forth.

So, the evaluation is given by the following rule, you pick any one the three, f 1, f 2 or f 3 and you apply it to all four points all 4 coins. So, now, here is the question, the system now of course evolves with time and we ask the similar thing at some point of time is it possible for the system to be in the following configuration.

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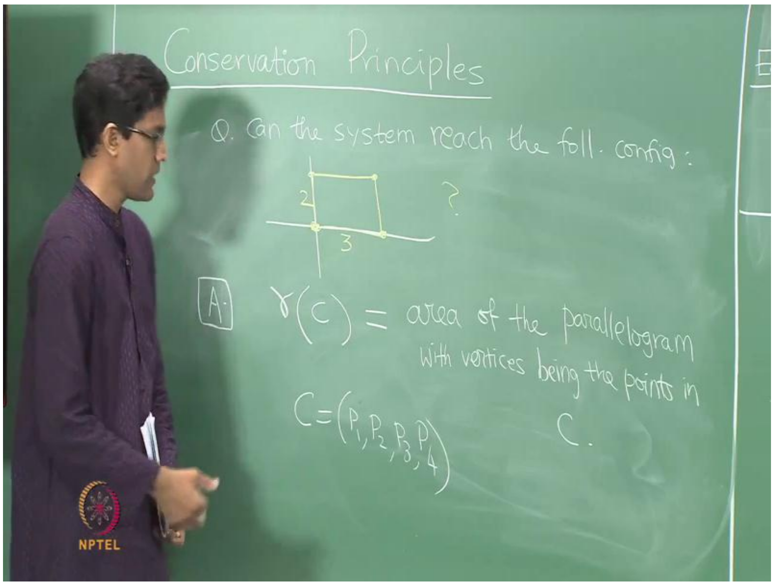


So, question can the system reach the following configuration. So, well what configuration do you want, we want the four points to be the corners of the rectangle of side 3 and 2. So, is it possible as a system evolves for it is somehow get these four points to lie on the corner of rectangle of sides 3 and 2, so that is know the question. So, again if here is the answer, which again follows the same prototype as we did for a earlier case, what will try and do is the find the conserve quantity, if try and find the function which does not change with time.

So, as a configuration changes the functions, how to remain the same and what is the function here, well it is rather easy the gamma function. So, let us define gamma of a configuration, let see the some configuration from four points in we define the gamma function is follows gamma of a configuration is just the area of the parallelogram. So, in general always be a parallelogram. So, it is area of parallelogram with vertices being four points of C with vertices being the points in the configuration C. So, remember now C, now denotes, it is a set of four points p 1, p 2, p 3, p 4. Now, the function here, which is the area functions? And again, observe that, so this is something that we did last time.

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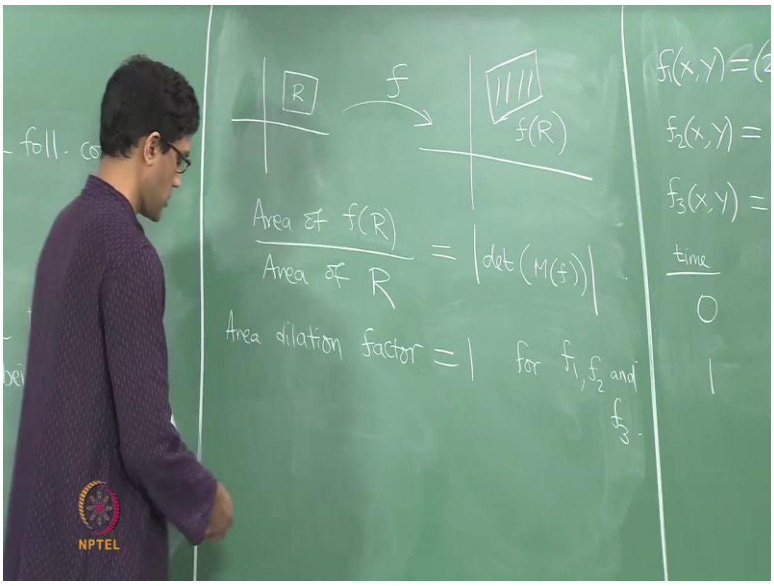


The area is just given by, so if I apply, so now observe, but last time we had the following fact that if I have a region R. So, let us recall the following, if I have a region R in the plane, let says square or pretty much any polygon and I apply a linear transformation to it. So, I apply a linear transformation and this might become some parallel here, this is the image f of R.

So, here is what we said the area of f of R divided by the area of R. So, this was what we call the area dilation factor can be simply computed by just taking the determinant of the matrix representation of f. So, this we said is just the absolute value of the determinant of matrix encoding of f, this is the area dilation that the transformation f in the uses. So, let us do it an each of these cases. So, I have my function is f 1 is just give n by 2 x and half y.

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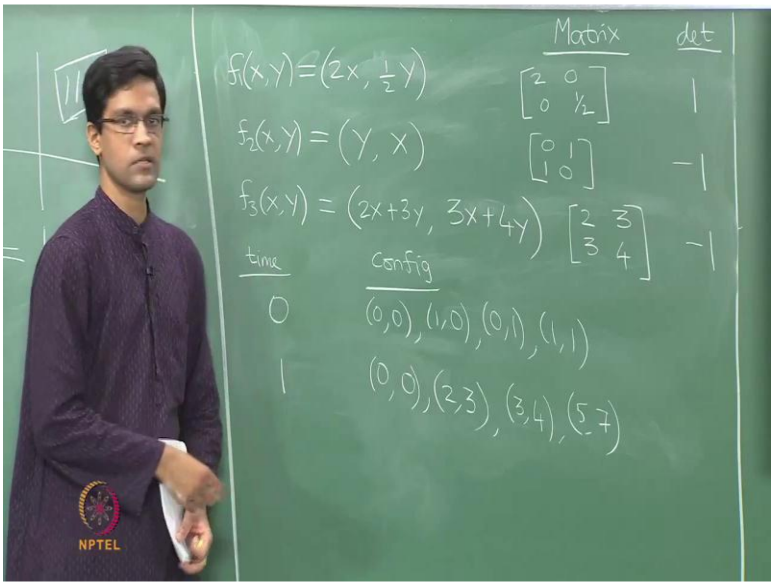


So, let us write the matrix corresponding f. So, it is just 2 and half 0's here; that is the matrix corresponding to x 1, the matrix corresponding to f 2 is just well at 0 1 and x needs to 1 0. So, you should just check that what I am doing is write here and this is just 2 x plus 3 y, so 2 3, 3 x plus 4 y, so 3 4. So, this was a recipe for the constricting the matrices from the formulas for the functions and now, let us do the following is compute the determinacy of these matrices.

So, if you compute the determinant is just sort of the a d minus b c formula, here it is a 1, this is a minus 1, 4 2's are 8 minus 9 minus 1 observe that the first transformation has determinant 1, the remaining 2 have determinant minus 1. But, in any case the absolute value, the modules the determinant is 1 in all three cases. So, what is that imply, it says that, well if you take f 1 or f 2 or f 3, they all the scale the area by a factor of 1, so the area dilation factor is 1 for all three of our functions for f 1, f 2 and f 3.

So, all three have the same area dilation factor means that, if you start with since our initial configuration was as square of side one. So, what is this imply, I had square of unit square of area of 1 to start with, no matter of how it evolves.

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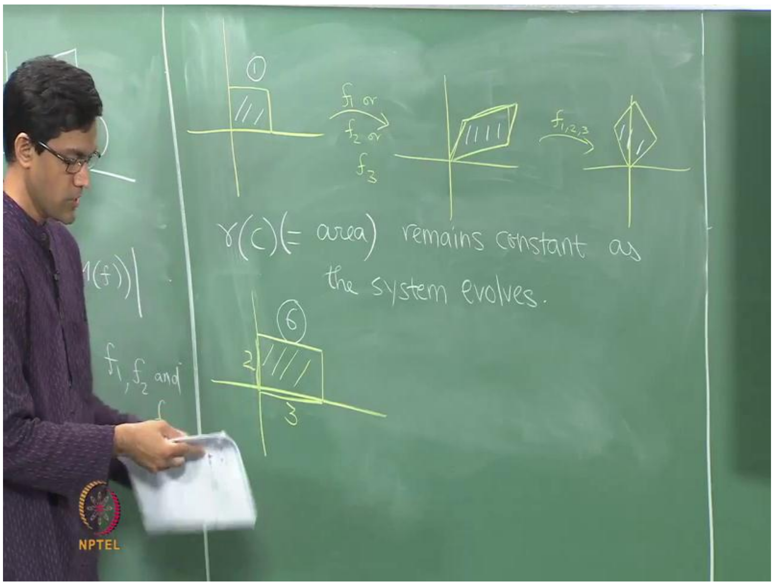


So, no matter, so this is our initial configuration and now, how does a system in evolve with time, you apply either f 1 or f 2 or f 3 in order to get the next step. So, in the next step is something. So, result of applying if f 1, 2 or 3 to this, and now what these and now what do we do in next step you again apply f 1 or f 2 or f 3, what is at make it well it make it something else and so on.

So, this is how the configuration evolves with time, but what we know is that at each step the area dilation factor is 1, what is that mean, the area of this figure, whatever it may be, will always remain the same, because the dilation is just may for factor 1. So, these figure whatever they are all have the same area, so observe that gamma of C, we just a area of the parallelogram.

So, gamma of C, which is the area this case remains and changed, then constant as a system evolves. So, let us use this to answer the question again. So, here is the conserve quantity for this evaluation rule, as this system evolves the gamma remains the area remains constant. So, the question we ask is it possible at some point of time for these four points to form the vertices of a rectangle of area 6 is of course, not possible.

So, because area would be a 6 here, whereas the initial configuration that we started out with only had area 1, you can never get the this four points make a figure whose area 6, they were always only make a parallelogram of area 1. So, these are two examples of conserved quantities and where in the factor there is a conserve quantity allows you to make at least answer some question rather quickly. The impossibility of attaining certain configurations can be quickly answered from this. So, next time, we look at few more examples of conservation principles.