An Invitation to Mathematics
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Unit
Functions
Lecture - 27
Length and Area dilation, the derivative

Last time we talked a little bit about Linear Transformation. Now, what I want to do this time specifically is focus on Linear Transformations and what they do to areas.

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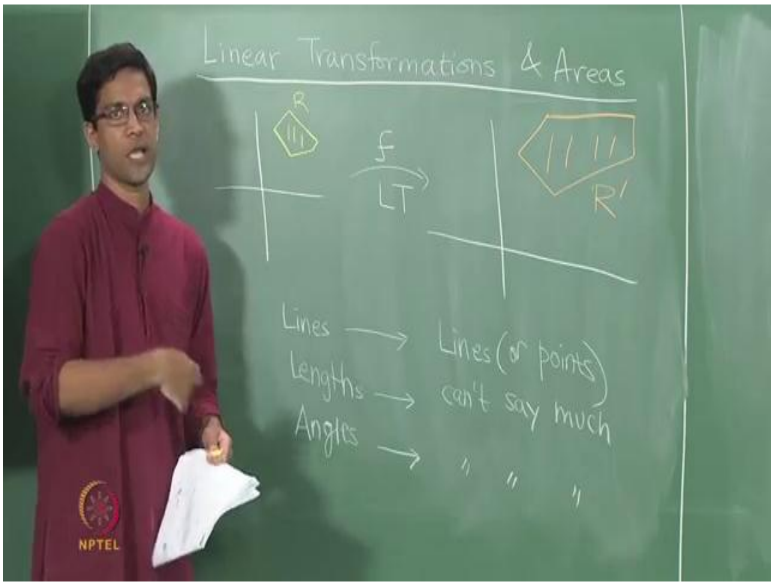

So, for instance, so recall the following features, we said linear transformations being a special case of the affine transformations to the following. They map lines to lines, so this is a… So, what all do we know, linear transformations certainly map lines to lines, well or points as we mentioned last time could maps some lines just to a single point and what else does it do. Well, what it do to lengths? What it do to angles? Now, answers to these questions at least we know through the various examples that we have looked at before.

So, there are instances of linear transformations which preserve lengths and there are others which might you know, uniformly increased length by some factor and yet others which might do neither of this. They may, you know along some directions increased length by some factor, along a different direction, they may increase length by some other factor and so on. So, as far as lengths are concerned, well there is nothing you know uniform that you can really say.

So, cannot say much really about lengths, in general if I do not know any extra information about the linear transformation. Similarly, angles, so again we looked at instances of transformation which preserve angles such as rotations, reflections, dilations and so on. But, then there were instances, a sort of the inhomogeneous dilation, for instance which did not preserve angles. So, again we cannot really say much, so a typical linear transformation might deform the plane in such a way that neither lengths nor angles are preserved.

So, it does train things to these ways, now areas a sort of the third aspect that we studied in our various examples. So, what is that, the difficult thing we want to do? Let us do the following, let us pick some region in the plane. So, for instance I could take, so I just make like simpler let us take a polygonal region by which I mean, let say I take a region bounded by lines. It need not be a regular polygonal or anything like that, so I could take, so something like this.

So, here is the region R that I write down on the plain, I am allowed to choose various possibilities. I could take may be a triangle region or a square, rectangle pretty much any sort of polygonal region that you can think of and what I want to do is to study the effect of applying this function f. So, let me say that I have a function f for linear transformation, so let f be a linear transformation. We could in fact, also allow affine transformations, but let us just strict to linear for now.

I want to look at what f does to this region R. So, the first thing is you know, as we said before it is sort of enough to really figure out what happens to each of these various vertices. So, what f would do is to map these vertices to some points, so there would be, let say 1, 2, 3, 4, 5, so maybe I have again five distinct vertices. So, maybe it deforms set in some other fashion, so it may not have the same form as the original region

So, this is my region R and maybe that is my region R dash and R dash is just the image of R under my, under this map. So, I am just sort of depicting some things schematically here, it is probably not a very good picture in general. So, now, here is the question. I want to compare the area that is enclosed by the region R with the area that is enclosed by the region R dash. So, this is the difficult thing that we have looked at earlier, by what factor does the area increase or decrease.

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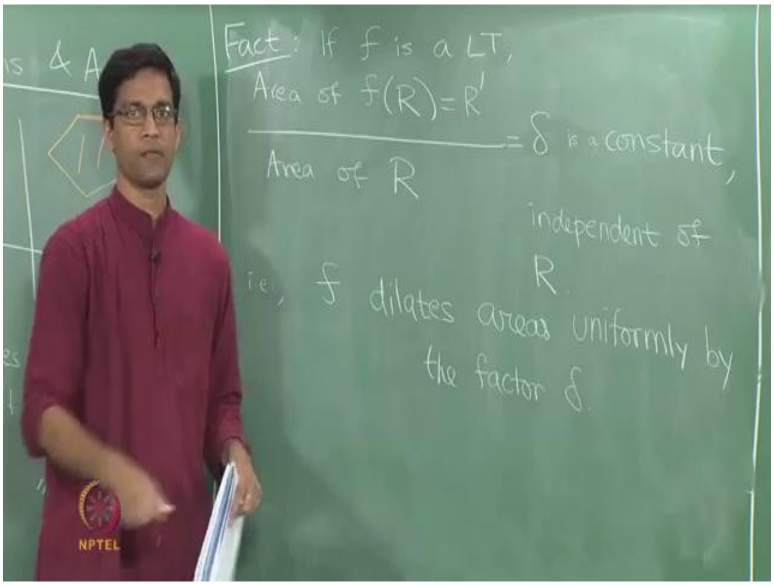

So, here is the question, I want area of the region R dash. So, R dash is what we would called f of R that is the image of R under the function f. So, we call it R dash divided by the area of R. So, this you might want to call the area scaling factor or the area dilation factor, now here is the surprising fact. If f is the linear transformation, then this ratio is a constant independent of what shape R has or where it is located on the plain.

So, surprising fact ((Refer Time: 05:32)) if f is a linear transformation or in fact, also an affine transformation, then this area scaling factor is a constant. By constant, I mean it does not depend on it is independent of the region R. It does not matter, what shape the region has, where it is located and so on and so forth. So, it is absolutely independent of R, the answer will always be the same and so, what is it mean. In other words f is, f dilates area uniformly by this constant.

So, let us give this constant a name, let us call it say delta, this is a, so in other words i e, f dilates areas uniformly by this factor. So, all areas expand by the same amount, that is basically what this means. Even though, so what, why did I call this somewhat surprising, because lengths for instance could suffer, you know different amounts of dilation in different directions and so on. But, when you talk about areas, what you get is the same uniform dilation.

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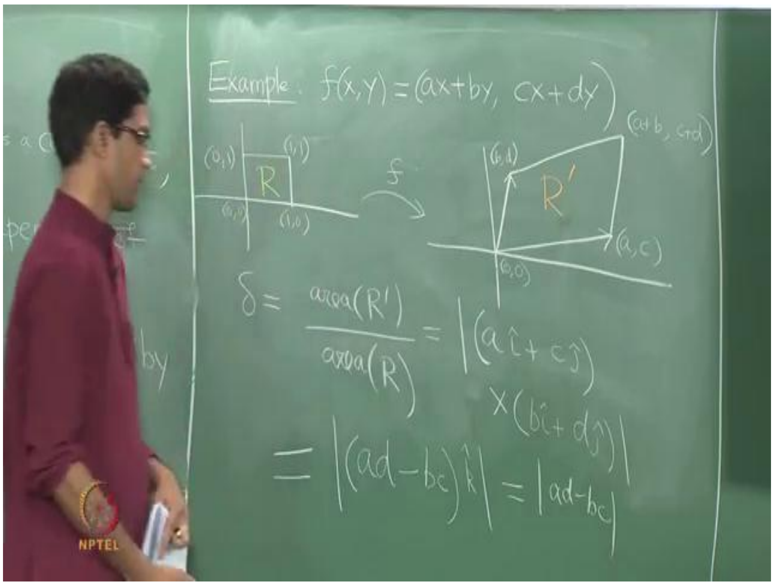

So, let us just do this again by example. Let us take f to be 12, I guess I am just taking the most general example, f of x y let us I said it is a linear transformation. So, it something which looks like this, a x plus b y, c x plus d y and let see, what this dilation factor area dilation factor delta could be. So, in order to do this, let us take a simple figure R, let us just take a unit square. So, I will take the origin 1 0, 0 1, 1 1 think of this as being my region R.

So, R of course, has area 1 in this case, it is just a square of side 1 and I want to apply this function f to it. Now, observe that applying this function f does the following, it maps this square to, well in general a parallelogram. So, here is what in square maps to, maps to a parallelogram. So, how do I deduce this? Well, I will just figure out the four end points. So, for instance the origin maps to the origin, I just plug in 0 0, 1 comma 0 maps to the point a comma c by plugging in.

This map to the point b d and the point 1 comma 1 maps to, well a plus b comma c plus d. So, these four are in fact, vertices of a parallelogram and so, now, the question really is in order to figure out this dilation factor. So, let us just work out. So, for the moment let us accept this fact has being true that this dilation factor is uniform that just a single dilation factor delta. So, to figure out what delta is, all we have to do is take this particular choice of R and R dash.

So, I will compute the area of the parallelogram and divide it by the area of those rectangles. So, let us compute this scaling factor delta is just the area of the parallelogram divided by the area of the original region R. Now, of course, the original region has area 1, so I do not need to do anything there, all I have to figure out this the area of the parallelogram. So, of course, the area of the parallelogram there are several different ways of trying to compute the area of a parallelogram, but here is one…

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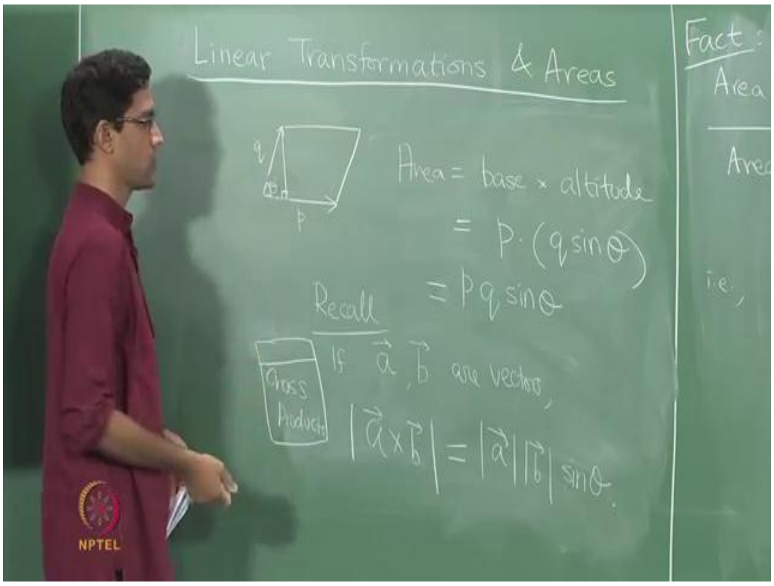

So, if you have a parallelogram in general, so let me just say instead brief due to talk about areas and parallelograms and formulas were those. If I have, say here is theta the angle, now the usual formula says it is base times altitude. So, let us may be call these two sides as something, we will call this p and q are the two sides. The area of a parallelogram is just base times, so p times the altitude, so area equals length of the base times altitude.

So, the base is just p and the altitude, so just by elementary trigonometry is q sin theta. So, q sign theta is this vertical line segment here and so it is p times q times sin theta. Now, of course, sin theta, so the angle here could be, you know it depends on what the angle is, something between 0 and 180 degrees. Now, what do we know about, this quantity here p q sign theta we could try and compute it in various space. But, the easiest way to do this is just by using, you know going back to definition of vector cross products.

So, recall, so we have already talked about cross products in one of the earlier lectures. So, recall if a and so, I will just use notation for vectors. If vector a, vector b are two vectors, then their cross product a cross b has the following magnitude. The length of the cross product is exactly the length of a times the length of b times sin of the angle between them. So, I am actually a sort of appealing to some previous knowledge of cross products, so this just makes it much easier.

So, if you are familiar with this, it is quicker to get to the area formula. If not, there are sort of other ways that you must play around with in order to, to get to the same answer that I will get to in a moment. So, what this says is to find the area of a parallelogram, you just think of the two sides has being vectors. So, you think of these as a vector, this as a vector and the area is just the magnitude of the cross product, so that is a very nice description of the area.

So, let just apply this to this situation, so let us come back here, we will trying to figure out the area of this parallelogram or prime ((Refer Time: 12:33)). So, what we want to do is to think of one side and the other side as both being vectors. So, this is nothing but, the absolute value of the cross product of these two vectors and now, I will use vector notation again. So, recall that i cap and j cap are often the standard notations for the unit vectors along the x direction and the y direction.

So, I have this vector is a i cap plus c j cap cross product with b i cap plus d j cap, so that is a cross product of these two vectors and I need to find the magnitude of the cross product randomly. And, so here what we need is the definition of the cross product. So, if you actually just compute this cross product, here is what you will get. It is going to be, so this is the magnitude of… So, for instance i cross i is 0, so you should just think of this as, you need to expand this out completely distributing it using the Distributive law, i cross i is 0, i cross j is k.

So, this just gives you a d times k and then, the other term there will just give you minus b c times k and k cap is just the unit vector along the z axis. And so, this final answer here is just the absolute value of a d minus b c. So, what is this mean? Well, this just says that this dilation factor.

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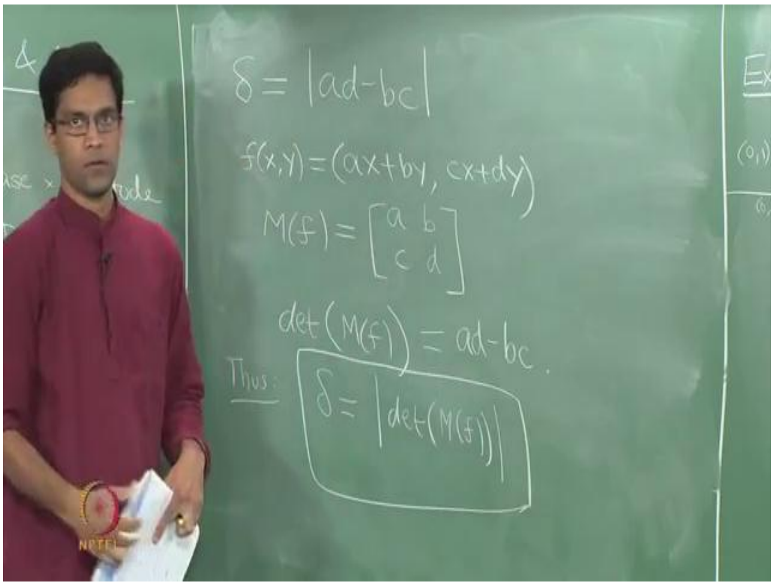

So, here is the conclusion, area dilation factor delta is nothing but, the absolute value of a d minus b c. And now, what are a b c and d? Recall, those were just what appeared in the definition of the function. The linear transformation is exactly a x plus b y, c x plus d y, so the a b c and d are these four numbers and recall again that last we said these are best encoded in the form of a 2 cross 2 matrix. So, you should really think of this linear transformation as being encoded by this matrix a b c d.

And now, that we do this, the number a d minus b c again has another interpretation. So, a d minus b c, recall it is just what we will call the determinant of this matrix. So, recall that the determinant of this 2 cross 2 matrix, here is exactly a d minus b c. So, what this means is that the scaling factor has a natural interpretation in terms of the matrix of the linear transformation. So, thus conclude that the scaling factor delta is nothing but, the absolute value of the determinant of the matrix in the linear transformation, this is the final conclusion.

So, in general for an arbitrary linear transformation for given the transformation f itself and you want to know, by what factor does it is scale areas. All you want to do is to just write out the matrix corresponding to the linear transformation and then, compute it is determinant and the absolute value of the determinant is exactly the thing that we want the scaling factor. So, returning in some sense to something we started out with, let just do the something for functions from R to R.

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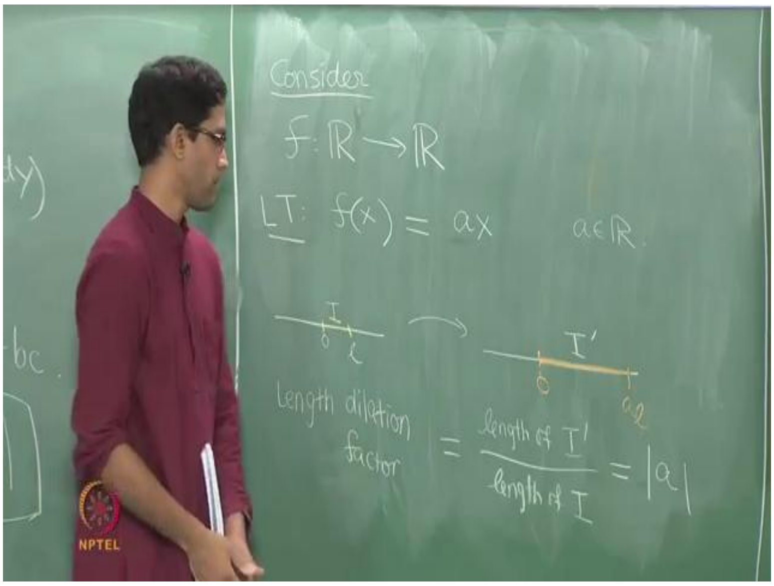

So, let us do the following, let us consider, let us go one dimension lower instead of considering functions from R 2 to R 2. Let us consider functions from R 1 to R 1. So, by which we mean just functions from the real line to itself and here, what would a linear transformation mean. Well, a linear transformation from R 1 to R 1, just copying the definition that we used for two variables. A linear transformation now can only act on one variable and only give you one real answer.

So, I mean there are no two components, well it is just something of the following form, it just a x. If I have two variables, I can do a x plus b y, c x plus d y and so on. If I only have one variable x, all I can do is just multiply it by some constant and such a thing is what you would call a linear transformation in one variable. So, a here is some constant, so such a thing is linear transformation. Now, well, so again you know, you could try and do the same notions that we had earlier.

For instance, we can now talk not about area dilations, but of length dilations. So, you could now ask the following things, since we are now in just one dimension, the notion of area is most naturally replaced by lengths. So, for instance you can ask, suppose I took the line segment of some length l and I apply the function f to it, what will be the new length of my line segment. So, for instances, let say I start imagine the line segments starts at 0 and goes to l and now, we apply this function f to it and ask what happens to this line segments.

So, let me call this interval, let me call it I is my interval, I apply my function f of 0 is 0 and of course, by definition f of l is just the point a l. So, this interval I, when I apply the function f just becomes potentially a lager interval. So, if a is for instance 2, it is an interval of size 2, so this is my new interval I dash. And so, now, you can ask, what is the length dilation factor in this case? So, here the length dilation factor for the map f is the following. It is just the length I dash divided by the length of I and observe here that I dash has a times the length of I.

This is a l and this is a, so this is just going to be a, so this is of course, if a is positive. If a turned out to be a negative number, then what you would get would really be, you know the same interval, but in the opposite direction. In case of the length in that case should really be the modulus of a l. So, actually speaking, if a is negative since you know the both top and bottom are positive numbers, what I should get out as an answer is a positive number.

So, here is the general answer. The length dilation factor for max from R 1 to R 1 is just the absolute value of a, just the absolute value of the constant l. Now, observe again there is just like linear transformations from R 2 to R 2, here the length dilation is uniform. Pretty much no matter where you keep this interval of length l, no matter where you place it on the real line, it will always be expanded by the same ratio a, by the same factor a. So, this is what we mean by uniform length dilation.

It always expands by the same factor independent of where it is placed. Similarly, in the case R 2, no matter what the shape of your region and no matter where it is placed, it always expands by the same factor delta which is the determinant.

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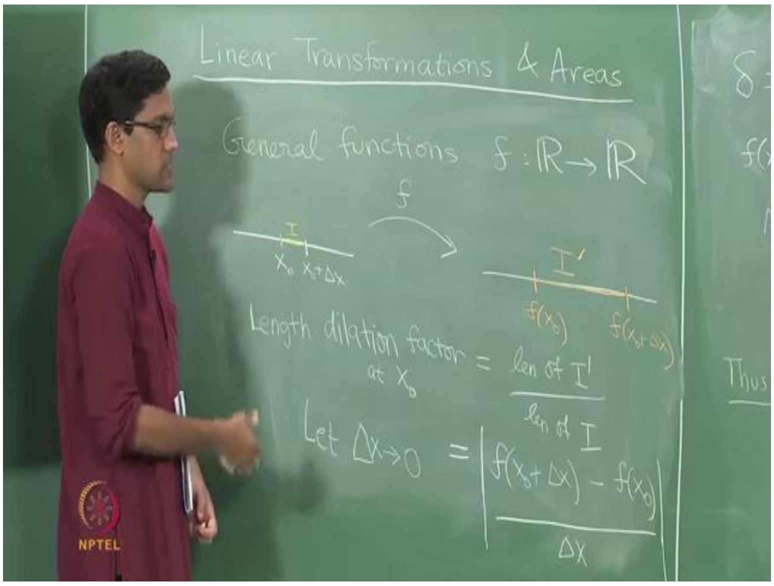

So, now, let us just makes this a little bit more general, so here is one final point about general functions from not linear transformations. We just take an arbitrary function from R to R. So, let us say it is a nice enough functions, it maybe it is a continuous function, it is a differentiable function, things like that. So, imagine a nice smooth graph of this function. So, if I have a same arbitrary function, sufficiently nice an arbitrary for now, let us ask for the same dilation factor designs.

So, here is what it means. So, I have this function f, it is not linear necessarily, what I want to do is the following. I want to take a point x naught on the real line, I want to take an interval, so I want to take say an interval I. One of whose points, one of whose end point is x naught, so I take x naught and x naught plus. So, the other end point let us call it delta x. For now, delta x is just any real number, but eventually we will think of it as being a very small real number, so other these are very small interval.

So, I pick some interval around x naught or with x naught is one of it is end points and ask, well what happens to this interval when I map it under the function f. So, what happens to the two end points? For instance X naught will map to the point f of x naught, the right end point will map to, say this right end point x naught plus delta x. So, the new interval I get is exactly this guy I dash, so now I can ask for the same question, what is the dilation factor, what is the length dilation.

From comparing I with I dash, well this is the length of I dash. The length of I dash divided by the length of I and that is exactly… Well, what is the length of I dash? It is f of x naught plus delta x minus f of x naught. Well, actually I should put modulus, because I do not know f of x naught plus delta x could be to the left of f of x naught. In my diagram I drawn it to the right, but I do not know which way it lies. So, it is the absolute value of this difference divided by the length of I, the length of I is exactly delta x.

So, this is just, so my delta x is any way positive, so I can just put it within the modulus. So, here is the answer, the length dilation factor comparing I with I dash, so this is what we would sometimes call it is the length dilation factor act to the point x naught. Observe, if I change I to lie somewhere else, then of course, the length dilation factor we had to take those two end points in the curve. So, the length dilation factor, it is x naught is really this quotient here and now, I am think of delta x as being smaller and smaller number.

So, we now let delta x also approach 0, let it go to 0. So, when you do this, what you are doing really is taking smaller and smaller and smaller intervals around x naught and asking by what factor is their length dilated by this function f. And so, we want to really consider the limit of the right hand side as delta x goes to 0 and that limit is really, well it should be familiar. If you have seen this before, it is just the derivative.

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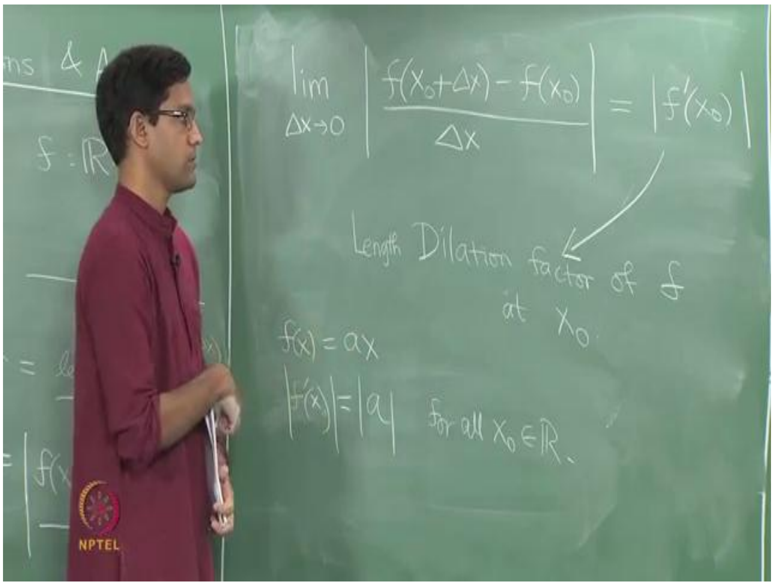

The limit has delta x goes to 0 of this quotient, so this quotient here is of course, just the derivative. So, assuming the derivative at x naught exist, at this limit exist, what you have is just the absolute value of the derivative at that point. So, what this analysis tells us is the following geometrical interpretation of the derivative. The absolute value of the derivative essentially tells you the dilation factor at that point. So, what is this?

This is exactly the dilation, length dilation factor of the function f at the point x naught. So, all it is doing is just keeping track of get the amount by which intervals gets scaled, when those intervals are in a small neighborhood of x naught. But, observe that unlike the case of a linear transformation, this is not uniform. The derivative, the value of the derivative may not be a constant at all points, it is of course, a constant if f is a linear transformation.

So, observe if f of x is just the function a x, then we said that this, the length dilation factor… So, if you compute f prime are at x naught, for any point x naught, the answer is always a. So, the absolute value is just the absolute value of a, this is independent of x naught, this is for all x naught, that is the reason why if it is a linear transformation, you always get a uniform dilation. The amount of dilation is always modulus of a, no matter what point x naught you are talking about.

But, for the general function, for the general differentiable function this may not happen. For different points x naught the value of the derivative might be different and so, what the function really does is to dilate intervals differently, depending on where the point is. So, this is really a known uniform dilation in general, but nevertheless this particular geometrical interpretation is useful to keep in mind and this is sparkly, why we talked about area dilations for example, in the case of maps from R 2 to R 2.

The more general thing there… So, the natural question here is, what if we were not considering linear transformations from R 2 to R 2. What if we had a more general function from R 2 to R 2, say it differentiable lines and so on. Then, it turns out that you can still try and figure out, what the area dilation factor would be and that would now involve the notion of partial derivatives. So, that is the reason or that is one natural way of thinking about what partial derivatives do. They really give you a way of trying to compute the area dilation factor at each time. So, we will talk a little bit more about all this next time.