An Invitation to Mathematics
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Unit
Functions
Lecture - 26
Affine and Linear Transformations

Today we will talk about Affine Transformations. So, notice that the course of several examples that we studied though, you know during the last few lectures.

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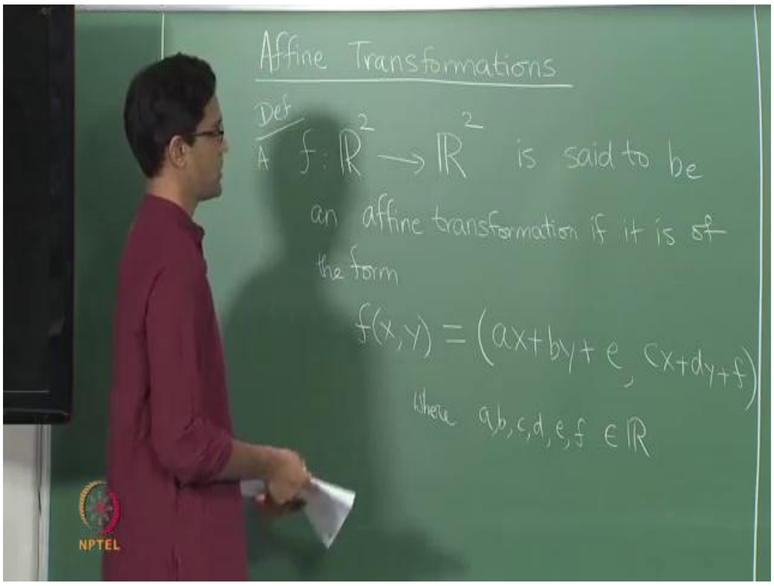

What we really looked at, where functions of certain kinds from R 2 to R 2 and many other functions we studied have the following form. So, here is the definition of an affine transformation. So, definition a function f from R 2 to R 2 is said to be an affine transformation, is said to be transformation if it is at the following form. It is of the form f maps x comma y to, well something of the type a x plus b y plus some constant, so call e c x plus d y plus f, where all these are some constants.

So, some real numbers a, b, c, d, e, f are all some real numbers. So, it basically looks like some constant times x plus some constant times y plus some constant and similarly, the y coordinate also has the same function. So, in such function is said to be an affine transformation, now notice that we have actually seen several examples of these.

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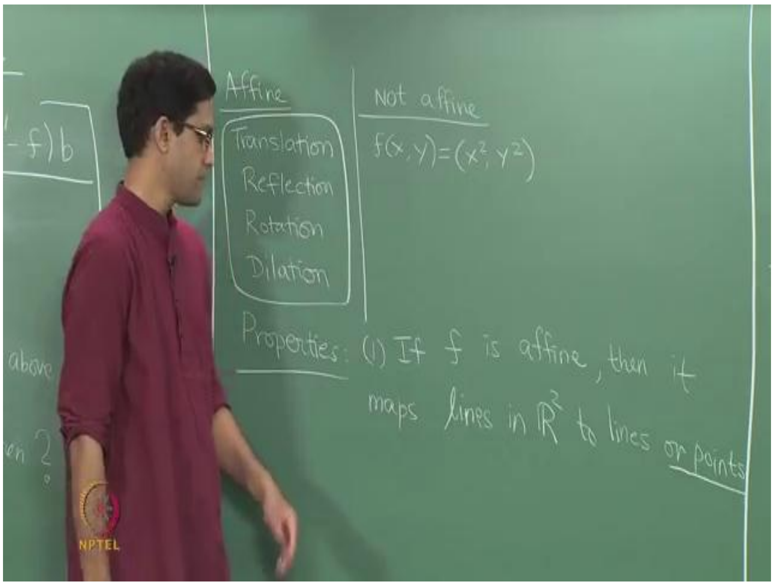

So, what are examples of affine transformations? We looked at all the maps which were say translations or reflection, rotation, dilations, homogeneous dilations or inhomogeneous dilations, and then also we looked at a few other examples of maps like f of x comma y going to x plus y and x minus y and so on, all of them are examples affine transformations. Now, examples of things which are not affine transformations is, what we looked at in the last lecture the very end f of x comma y is something which is a quadratic function of x and y.

So, it maps it to x square and y square, observe that an affine transformation can also be alternately described as a map in which both of the components, the x and y components are essentially linear polynomials in x and y, by which we mean it is a polynomial of degree at most 1. So, there are no x squared or x y or y squared or higher powers of x or y involved in either of these two times. So, the quadratic function you know, f of x y going to x squared comma y squared would of course, not be an affine transformation.

Now, so what good is this, so what are the various properties of an affine transformation? So, what we have really means is a definition which generalizes these few examples that we looked at. So, here are some key properties of affine transformations, so the first thing if f is affine, then it maps lines to lines. So, this was one of the key things that we studied in all examples, then it maps lines in R 2 to lines.

So, lines on the plane map back to lines on the plane and let us actually do this computation in general, there is affine I like to specifically focus on. So, let us prove this that it maps to lines to lines.

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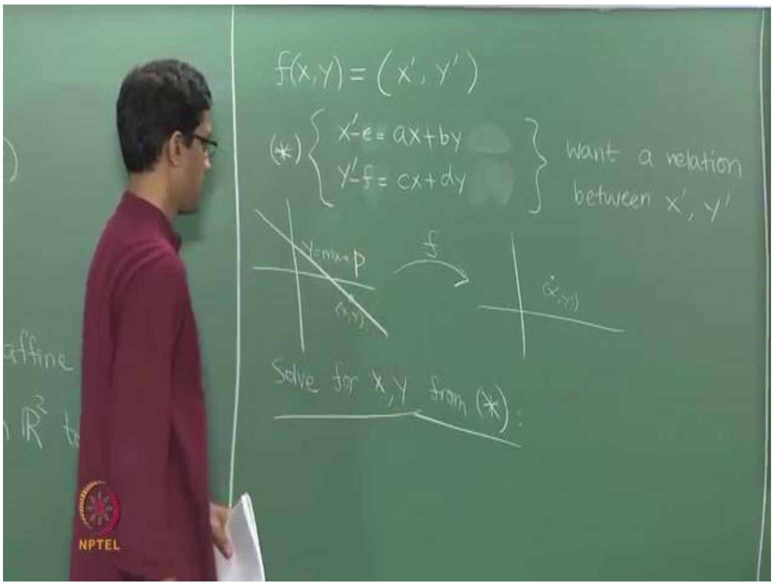

So, let us assume that f of x y has the given form it is affine, so let us call these two components as x dash and y dash, the way we were doing it in all the examples. So, x dash is just a x plus b y plus e and so I just write it as separately x dash is this, the y component is c x plus d y plus e plus f and what we knew is that... So, we need to show that lines map to lines. So, what is that mean? Let us pick a line in the plane, let say a line whose equation is...

So, let us take this easiest form y equals m x plus c that is a line and we want to study the effect of applying the function f to this line is, we want to know what happens to points of this line under the map f. So, again we do the same thing, typical point on this line x comma y has this relation between x and y that y equals m x plus c. We want to know what corresponding relation exists between the coordinates x dash and y dash.

So, these are the components of the image of this point under this function and we want to know, what relation is there between x dash and y dash and that equation will really tell you, if you one can show that it is the equation of a line, then that is precisely the image of this line. So, let us try and find a relation between x dash and y dash that is really the question. So, want to find the relation between x dash and y dash some equation using the known relation that y equals m x plus c.

So, what would we do, the easiest way is the following, from this equation you solve for x and y in terms of x dash and y dash. So, we have two equations, so let us just do it right here, so let us think of this as a x plus b y. So, I am going to do the following, I am going to erase the e push it over to the other side. So, think of it as x dash minus e is a x plus b y and similarly, we could look at y dash minus f equals c x plus d y. And now, there are two equations in the variables x and y which you solve for x and y in terms of x prime and y prime.

So, let us do that, so we need to do the following, we need to solve for x y from the set of equations. So, I will just call the set of equations as star, to solve for x y from the set of equations star and so I will just write out the answer something that you should check yourself. So, you can do this of course, pretty much by any method that you know for solving systems of linear equations. So, for instance if I eliminate one variable and solve or use any one the other rules.

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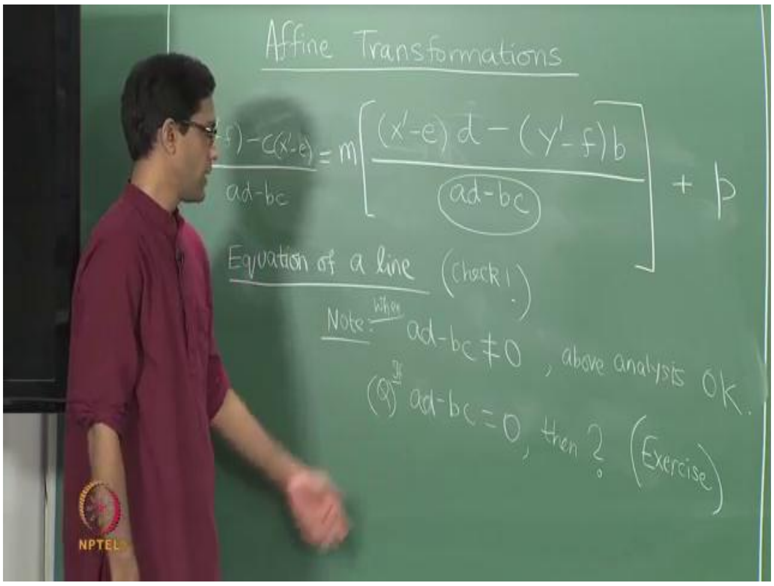

So, here is the answer x becomes x prime minus e times d minus y prime minus f times b divided by a d minus b c, so that is x. So, let me just write it this way, so this is x equals this and similarly, I am going to solve for y, y equals well that gives me a times y prime minus f minus c times x primes minus e divided by a d minus b c. So, what I have now done is to solve for the two variables x and y in terms of the variables x prime and y prime.

So, I find that y is this funny combination x is that funny combination and now, it is more or less trivial to find the relationship between x and y, observe that y is known to be m x plus c. Since, y is known to be m x plus c, the relation that must hold is that, this is y of course, must equal m times x is this plus r. So, I should be a bit careful I use c in two different places. So, let me just make this something else, so I will call this y equals m x plus let us call it p.

So, I should go back and change the constant there, so I have used c already for the, in the definition of function. So, y equals m x plus p, so this is just becomes, so here is the equation. So, I will now get rid of the x and y to obtain a relation purely involving x prime and y prime, so here is the somewhat complicated looking relation between x prime and y prime. But, observe that no matter what this is, if you sort of expand this out, if you write it purely in terms of y prime and x prime and so on, collect coefficients things like that, this is actually an equation of a line.

So, you can sort of observe what this is, this is in fact an equation for line. So, check in fact that this is an equation of a line by cleaning up this equation by just writing this out purely it terms of bring all the y primes to one side and take the x primes to the other side and then again I have the same form. It looks like y prime equals some multiple of x prime plus some constant. So, it is an equation of a line, so that proves what we set out to prove that, if you have an affine transformation f and you have a line, then the image of that line is again a line, it sense lines to lines.

Now, here is a minor point that I sort of gloats over, so in this when I solve for x and y I actually have something the denominator, there is a d minus b c occurring in the denominator. So, here is a minor thing to note, I have a kind of assume when doing this that a d minus b c is not zero. So, this is when a d minus b c is not zero, then whatever I am saying is perfectly okay in a lines will go to lines. So, then above analysis is fine, it is.

Now, question really becomes what about the case when a d minus b c is 0. So, if a d minus b c is 0 then what happens? So, I am going to leave that part as something for you to think about, there is an exercise try working out, you know the similar thing under the assumption that a d minus b c is actually 0, and then see what happens to lines. Is it necessarily true that lines go to lines? It tends out that it is almost right that is a minor to tweak.

So, let me tell you the answer, so that you can check it against this when you perform your analysis. So, here is the thing I wrote out, if f is affine, then it maps lines in R 2 to lines, now that is provided a d minus b c is not zero, if a d minus b c is not zero some lines could get map just to a point, it maps lines to lines or points. So, that is the only addition that I need to make. So, it could very well happen that lines map to points that is the only change. So, let just look at one example, you see how this might happen and this only happen in the case when a d minus b c is 0.

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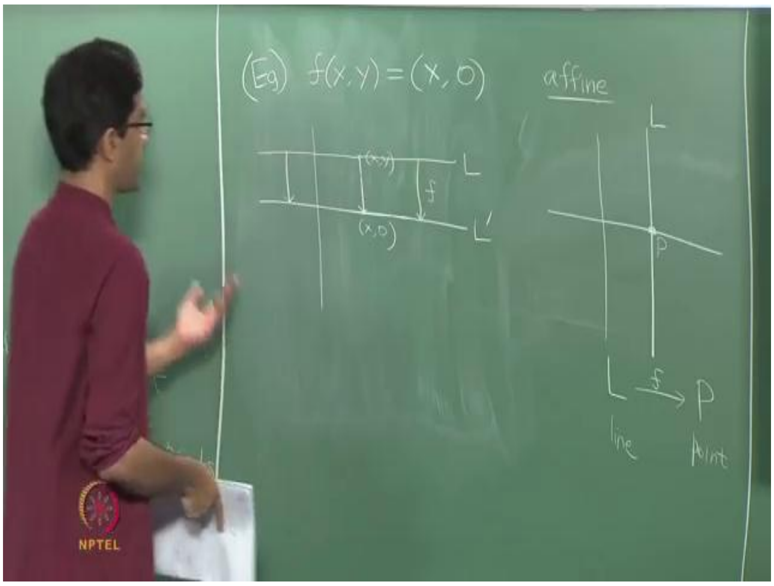

So, here is an example if you just take the function f of x y to just be x comma 0 for instance. So, this is an affine transformation, because of course, both components are just some linear functions involving x and y, if does no y in both components. Now, this function has a following property, so what is really is this function, it is what you would called the projection to the x axis. So, if I give a point x comma y all it is doing is, it is just projecting it down to the x axis, so point x comma 0.

Now, this projection map has the following feature, if I take a line say for instance, if I took a line like this say parallel to the x axis that is my line, it is image under this function would just meets projection. So, it would be the x axis this would be the image of this line under this projection, so this line L would of course, map to the line L dash under the function f that is one case.

But, it could very well happen that you chose a line which is say parallel to the y axis. So, if you took the line L to be a line parallel to the y axis instead and you apply this function f to that line and all it is going do is projected down to the x axis and when you do this, all you will get is just that the single point p, where that line meets the x axis. So, in this case what you get is just a single point, so the image of L, L the line maps to p just the point under the transformation.

So, it just turned out that in the examples we looked at last time this never happen, because all of those examples has this a d minus b c being non zero and so lines always actually map to lines and not to points. But, when we expand the definition of an affine transformation and allow for all possible constants a b c d and e f, then you have to short of be prepared for things like this happening. But, every now then some lines would get map to points, but it is not two series of problem. Now, let us look at property number 2 of an affine transformation.

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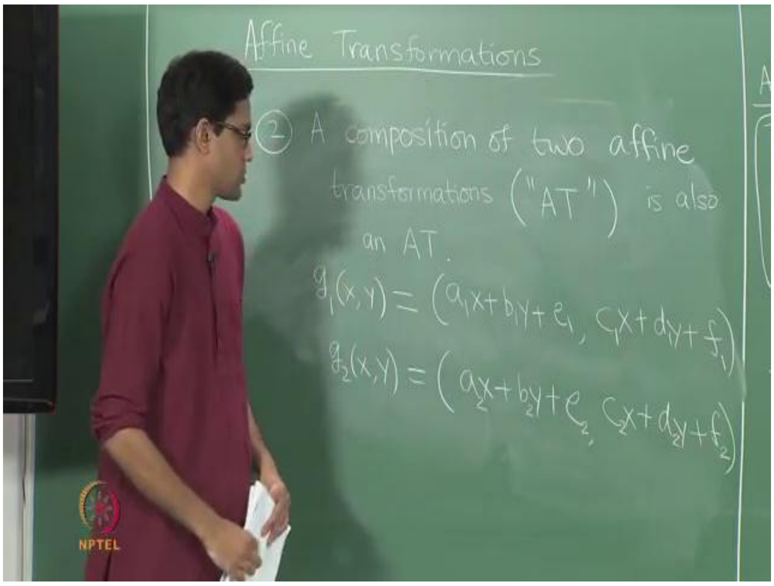

So, property 2 a composition of an affine transformation is again affine, a composition of two affine transformations. So, I am just going to abbreviate affine transformation to a t is to avoid having to write it out in full each time. So, if have to two affine transformations there composition is also an affine transformation. So, let us verify this please I will do part of the combination, so suppose I have two transformations one are them is called g 1. So, it is looks like some a 1 x plus b 1 y plus e 1. So, I am going to put 1's there, so again I have c x plus d y plus f now I will put 1's on the constants.

And similarly I have g 2 x y another affine transformation which again is a x plus b y plus e, c x plus d y plus f. But, now I think of the constants as being different constants now. So, I have I do not know 6 2’s are 12 constants like there are a, b, c, d, e, f 1's and a, b, c, d, e, f 2's. So, there are 12 different constants involve and g 1 is the think on top g 2 is the affine transformation the bottom. Now, remember the composition this we consider last time is just defined to be... So, if I want g 1 composition g 2 it is just the map from R 2 to R 2.

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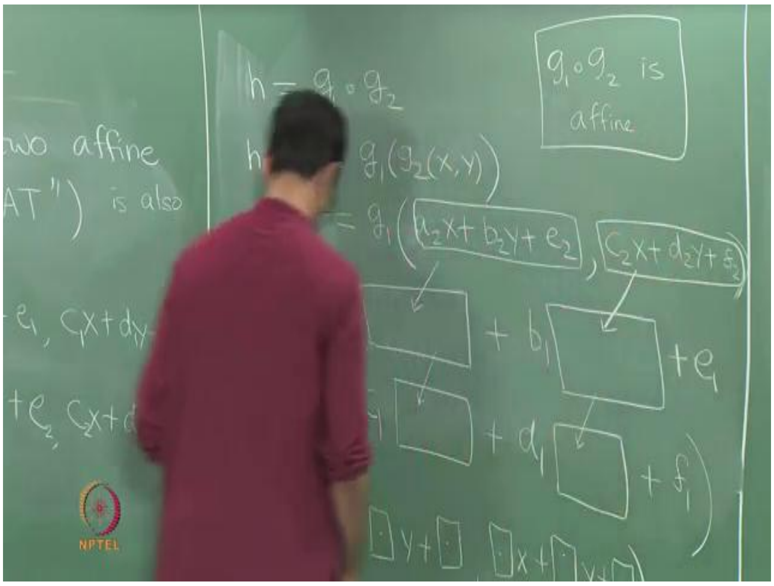

So, let us define the composition h that is the definition of the composition h of x y is just g 1 is acting on g 2 of x y. So, let us just compute this, this is just g 1 acting on, so what is g 2 of x y it is just a 2. So, it is a first component and the second component is just as 2, so all I introduced act g 1 on this new x coordinate and this new y coordinate. Now, what is the definition for g 1, the g 1 is just the following you need to do a 1 times the x coordinate.

So, a 1 time what is the x coordinate it is just thing here plus... So, this is the x coordinate of g 1, so a 1 times the new x coordinate plus b 1 times the new y coordinate or whatever the y coordinate is here. So, we need to plug in this here plus e 1, so this is the definition of g 1 for the first component and the second component. So, remember g 1 has a second component as well that is just c 1 times this same x coordinate plus d 1 time the same y coordinate here plus f.

So, I did bother writing it out in full, but just grammatically what g 1 does is just wherever you see a 1 x you have to pluck in the new x component whatever you applying g 1 2 and wherever you see a y you need to plug in the whatever be the y value. Now, doing this of course, now involve expending in all this stuff a 1 times whatever all of these, b 1 times this and so on. But, it is already clear that if you multiply a 1 by this linear polynomial in x and y what you are going to get this again another linear polynomial x and y.

So, the first terms is just a 1 a 2 x plus a 1 b 2 y plus a 1 e 2, similarly the second term involves multiplying a constant by what is already a linear polynomial and x and y and by doing this you still produce just a linear polynomial x and y. So, it is clear that the final answer is just again going to have the form of an affine transformation, it is going to look like some constant times x plus some constant in y plus some constant and then again some constant plus x some constant plus y plus some constant.

So, all my boxes are just place holder for constant, so thing of each of these as containing some constants. So, if you actually do the calculation yourself we just going to find that it is again of the same form and so the composition is in fact affine. So, therefore, are conclusion is that is composition of g 1 and g 2 is again affine, so among affine transformation.

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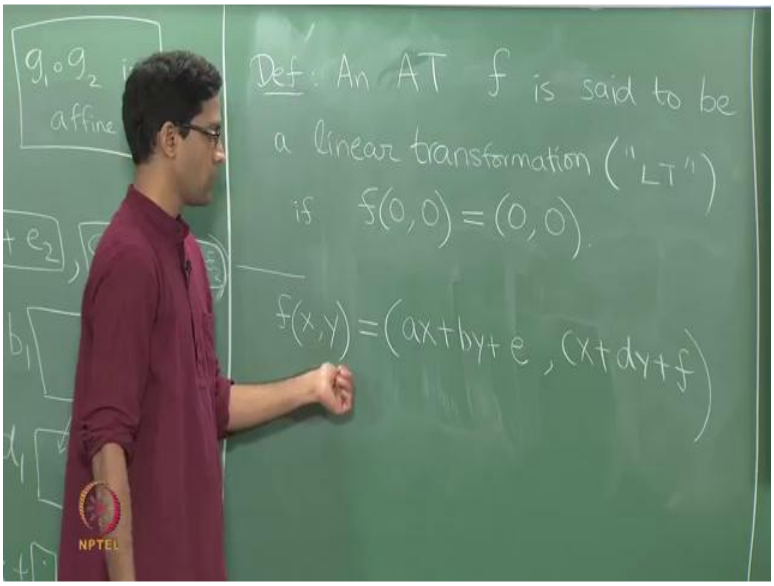

So, here is a further definition a special class of the affine transformation. So, an affine transformation f is said to be a linear transformation, so let us abbreviate that to LT if it has the following further property that f of 0 comma 0 it maps the origin to the origin. So, here is an addition property that you demand of a linear transformation, so what is that mean in other words if you sort of just unravel what this definition implies such a linear transformation has to have the following form.

So, because it is affine it must look like a x plus b y plus some constant c x plus d y plus some constant a it is although constant is f. But, now if you take x and y both 0 and you demand that the right hand side is again 0 what; that means, is that these two constant e and f had better be a 0.

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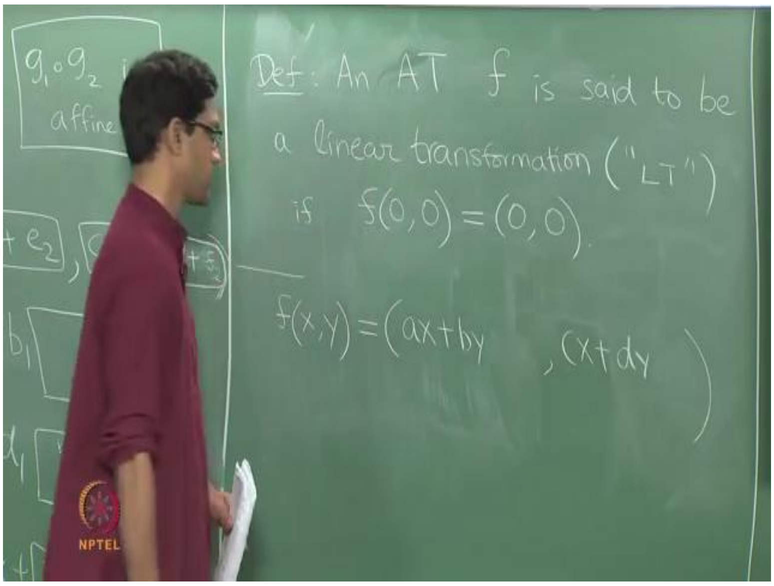

So, a linear transformation is simply an affine transformation in which there is no e f it is just something that is look like a x plus b y comma c x plus d y. So, that is a form of a linear transformation and affine transformations or linear transformations are really very close cousins it is often enough to just study linear transformations and that is what usually done and that is because of the following simple observation that if you have an affine transformation.

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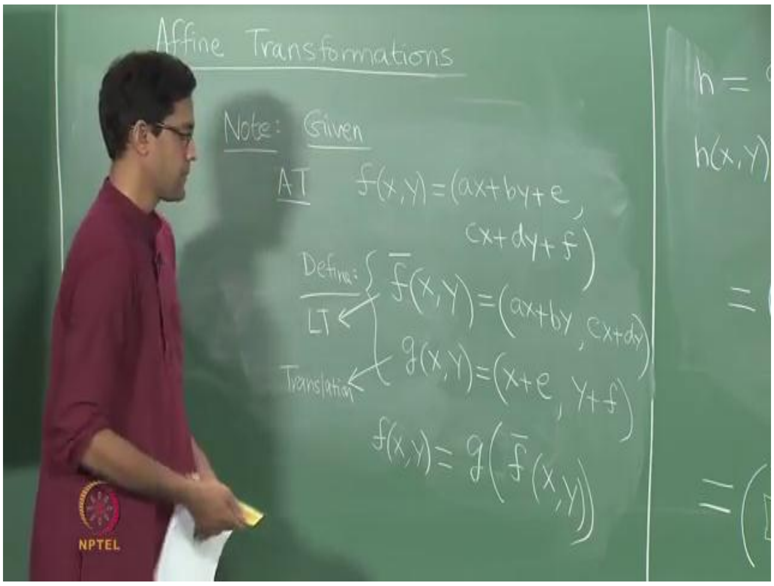

So, note, given an affine transformation f that is a form what you can do is to express f as composition of two things. So, let us define let us maybe call it f bar x y to just be the same thing without the e and the f. So, this is sort of the underline linear transformation as one might call it, you define given the affine transformation f you define a new linear transformation called f bar which is just a x plus b y c x plus d y.

And let us define translation map, so it is call it g it is just a pure translation it just let say translation x by e and y by f. So, I define these two maps, one is a linear transformation, so remember f bar is a linear transformation, g as we have studied earlier is just a translation moving everything by e comma f. And now observe that the given original affine transformation f is simply the following, you can think of it as first apply f bar which is the linear transformation and then you perform a translation.

So, this is just... So, let us write to the like this g of f bar of x y, so the transformation f is nothing but the composition of g and f bar. So, you can always realize an affine transformation as a composition of a linear transformation with a translation. So, since translations are very, very easy to understand there is do a very simple thing one often just study is linear transformations, if you understand linear transformations well then all you must do in order to get an affine transformations is further composer with some translation.

So, we often just try to understand linear transformations and again linear transformations have well at least the preliminary advantages you do not have those two constants e and f to worry about.

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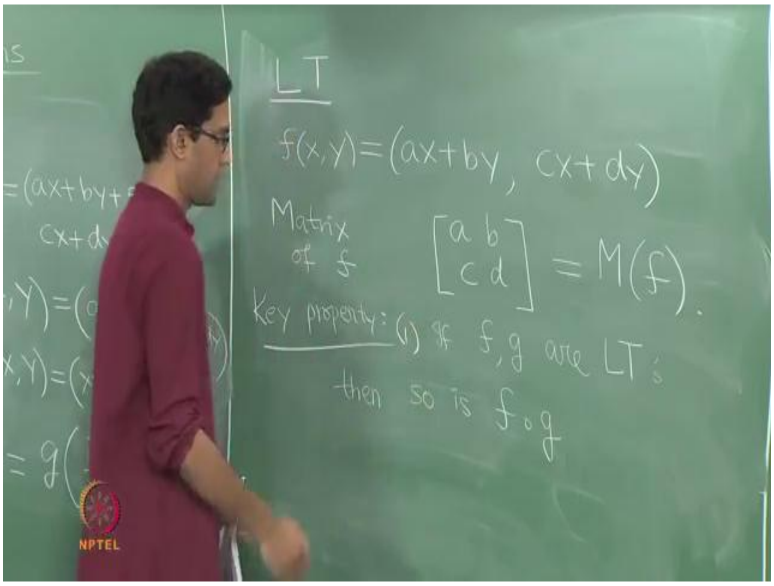

So, let us now consider linear transformations, so if I give you a linear transformation f I only have four constants a, b, c, d which define my linear transformation it is just purely define in terms of these four constant and what one does is to encode this linear transformation. So, we often define an encoding the way of keeping track of these four constants, what is called the matrix of f we just write down a, b, c, d in the form of a 2 cross 2 matrix.

So, recall that the matrix is nothing but an array of numbers in this case we have these four numbers which we arrange in a form of a 2 cross 2 array and we call this matrix here as the matrix of f. So, it is a same information if you wish as the linear transformation itself, because the linear transformation is in fact determine by these four numbers a, b, c and d, so this matrix of f is often called m of f this was notation.

And so at the moment it just look like you know it just some book keep to device instead of writing out the function in full all we are doing is just writing out the same thing in the form of a matrix, but it goes beyond that. So, here is the key reason for the usefulness of this matrix notation. So, here is a key property, if you take two linear transformation, so if f g are two linear transformation then so is there composition. So, that is the first property then so is a composition f composed to g.

So, remember we have already said this for affine transformations that if you compose to affine transformations what you get is again affine. But, same things to for linear transformations, if you compose to linear transformation you still get something linear. So, that is the first key property, but here is what makes matrices an especially important tool here.

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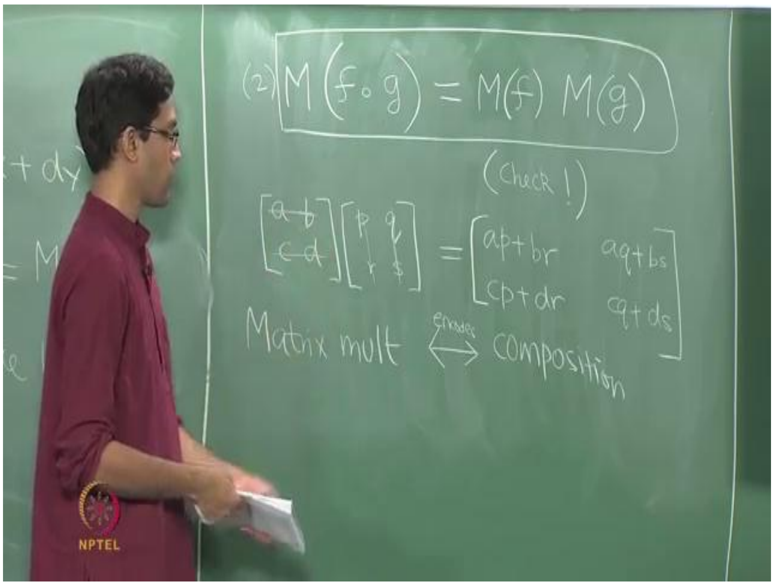

Property number 2, so if f compose g is again linear then of course, there is it is matrix, there are those four constants which must appear and so there must the matrix of the transformation. So, the question is what is the matrix of the composition? So, it is a natural thing to worry about and so what one must do really not do this is write out the composition in full. So, you must compute f of g of x comma y and then extract the four constants from it and here is the answer.

So, again I am going to leave this out leaves this an exercise, this is just the product of two matrices m of f time m of g. So, exercise check by computation that this is exactly what map it is and recall. So, what is product of matrices mean in this case, so if you have sort of seen before you will recall that if I have two matrices a, b, c, d with say a p, q, r, s.

So, here are two matrices multiplying them works by this somewhat strange rule at least strange at first sight which is you sort of have to go across rows and columns. So, you need to multiply a with p b with r, so this is a p plus b r and add the two answers. Similarly, you take the first row with second column, so that is a q plus b s, this is c p plus d r and this with this is c q plus d s. So, here is the definition of matrix multiplication, multiplying these two matrices would give you that somewhat funny looking matrix and it is not to clear or intuitive at first site what you know might lie behind this strange looking definition.

But, maybe this is this fact here is one of the important reasons behind the definition, you should really think of matrix multiplication has being some kind of shadow of composition, what it attempts to express or encode is the operation of composition. So, single linear transformation is just a single matrix. So, of course, two linear transformations become two different matrices and composing those two linear transformation produces as the third linear transformation and that third linear transformation, you know at the level of matrices what is that taking these two matrices and multiplying them together produces a new matrix and that new matrix is exactly the matrix of this third linear transformation that you produce.

So, you should keep the following thing in mind that matrix multiplication is in some sense the same thing as the operation of composition or encodes if you wish the operation of composition. So, check this property out we have almost done the calculation required when we did the composition, we computed the composition of to affine transformations, so morel as a same computation. So, try this out your own and check that in fact what you end up with is the matrix of the composition being the product of the matrices. So, we will talk a little bit more about matrices and so on next time.