An Invitation to Mathematics
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Unit
Functions
Lecture - 25
Composition of functions

Last time we talked about following map from R 2 to R 2, which is given by f of x y equals x minus y comma x plus y.

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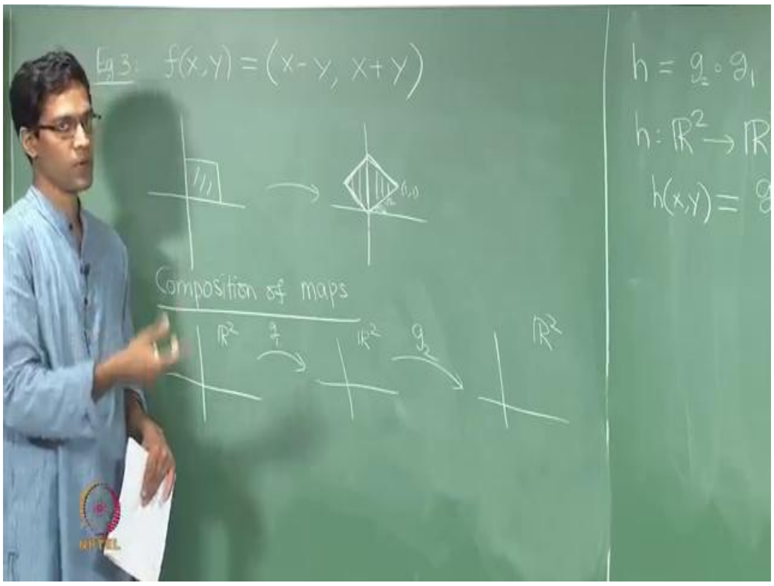

And we notice the following fact that, it does the following to square of side 1, it maps it to again a square, but of a larger side length square root 2 and also rotated by an angle of 45 degree. So, this is the image of this unit squared under this map, so this point is 1 comma 1. So, this is in fact, a side square of side root 2 and which in fact makes an angle of 45 degree with the x axis 5 by 4. So, we therefore said that, it seems like there are two operations happening here at once, so this is natural notion of composition of maps.

So, recall what it composition mean of functions, suppose I have functions, let us call it g 1 is a function from R 2 to R 2 and I have another function g 2, R 2 to R 2. So, all these are R 2, then their composition, so it is call it g 2 composition g 1.

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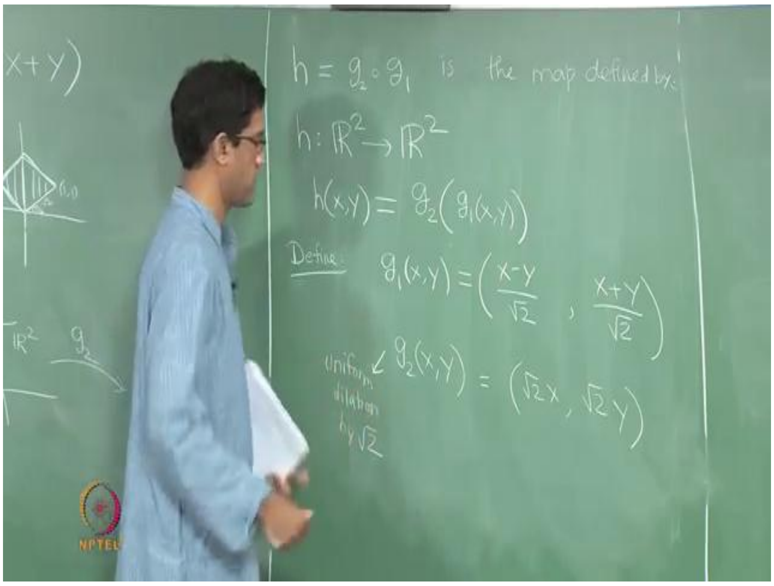

So, let us define h to be g 2 composition of g 1, so what is this, this is the following map is a map is the map, which is defined by so firstly, where is it map from it is again map from R 2 to R 2 and what is it do, it takes a point x comma y on the plane and sense it to, so g 2 composition g 1. So, it is g 2 evaluated at whatever g 1 does to x coma y, this is the usual notion of composition, you follow one map up by the second one. So, this is the other way of saying you applying g 1 first, and then you apply g 2.

So, if you do first g 1 and then g 2, so this is sort of what we mean by last time by saying that, map f that we looked at seem to be two things at once, what it really was doing was doing you know two things one after the other. So, what I want to do next is to try and express this map f as a composition of two simpler maps or two more familiar maps that we already looked at. So, for that purpose let us actually define these two maps, so I want to takes specific examples of g 1 and g 2, let me define g 1 to be the following map, it is x minus y divided by square root 2 comma x plus y divided by square root 2.

So, that is g 1 and g 2; that is g 1 and g 2 of x y, I define it to be just square root 2 x times square root 2 y. So, I am taking two particular examples of g 1 and g 2, one of them given by a first formula and the second given by the second formula. So, observe that g 2 something that we already looked at, it is just the uniform dilation by a factor of square root of 2.

So, g 2 of course is familiar, this is just a dilation or a uniform dilation by square root 2. So, everything gets expanded by square root 2. So, we will return to g 1, but first observe, what is the composition of these two maps.

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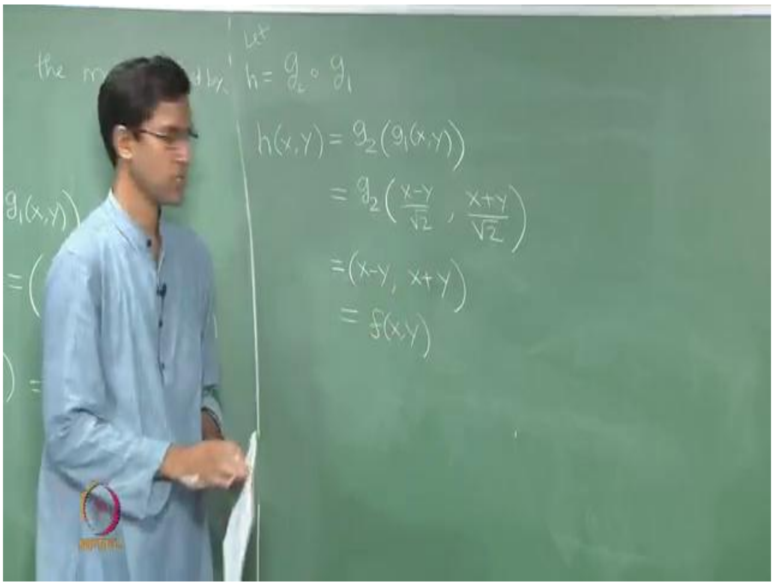

So, suppose we did g 2 composition g 1, so let us called a just g 2 composition g 1 and see what just map h is. By definition it is g 2 of g 1 of x y, so let us evaluate g 1 first, g 1 is x minus y by root 2, x plus y by root 2; that is what g 1 does and g 2, simply multiply each factor by root 2, so this then gives you x minus y comma x plus y. So, in other words, what is this; well this is the original function f that we started at, so this is exactly this function f. So, what this means of course is that the original function f is rather it really is a composition of two simpler functions, it is g 2 composition g 1.

Now, let us try and understand g 1 better, g 2 of course, we have now understood, so it is dilation. Now, what is g 1 really do, that is the only remaining part of right now.

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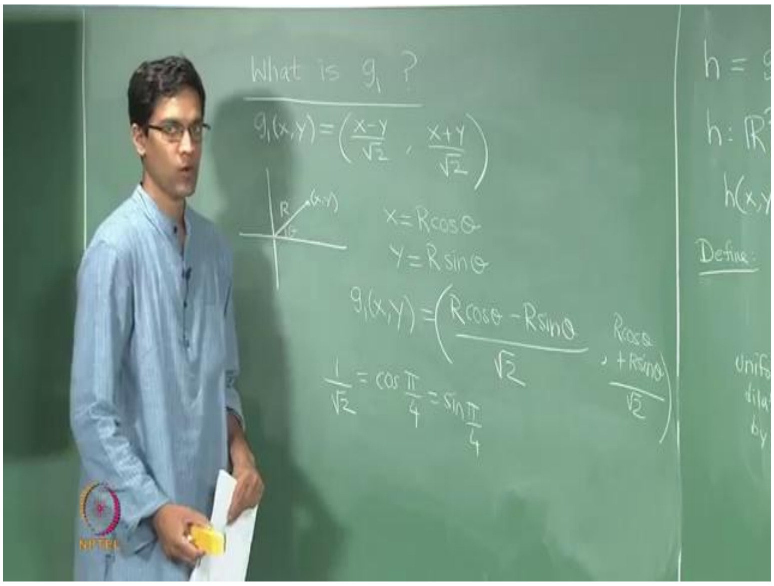

So, the only question that remains here is to understand, what g 1 is. Now, let us start out with the formula g 1 of x comma y is just x minus y by square root 2 comma x plus y divided by root 2. But, to understand it more geometrically, let us do the following; that is take the point x y on the plane, let us say imagination certain distance R from the origin and which makes an angle theta with the x axis.

So, we have the following trigonometric relations that x is just R cosine theta, y is R sin theta and now, we apply g 1 to the point x y. So, by the formula this is just x minus y by x square root 2 comma x plus y, so x is now R cos theta plus y by square root 2. Now, let us do the following, let us analysis each of these. Remember 1 by square root 2 that appears in this formula can actually thought of as a following. It is cosine of 45 degree, it is cosine of pi by 4 are in fact, the same as sin of pi by 4. So, cos and sin of 45 degrees are both just 1 by square root 2.

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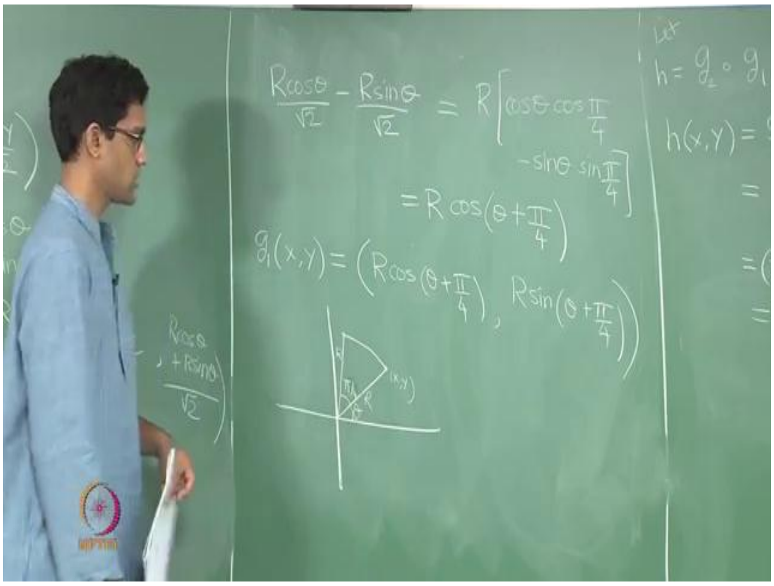

So, what we can now do is to rewrite the x coordinate. So, what did we have, we have R cos theta by root 2 minus R sin theta by root 2 as the following, it is really R times cosine of theta cosine of pi by 4 minus sin of theta sign of pi by 4. And this is just R, we now use the trigonometric identity for the cosine of the sum. So, cosine of theta plus pi by 4 would exactly be cosine theta cosine pi by 4 minus sin theta sin pi by 4.

So, in fact, what this tells us is that, if you take g 1 of x y, if you look at g 1 of the point x comma y, what it mappings 2 is a point x dash comma y dash, which is given by the following. The x coordinate of the new point is just R cos theta plus pi by 4 and if you sort of work out the y coordinate similarly, you will find that it is just R sin theta plus pi by 4. Again, you need to use the addition formula for sin.

So, what is this mean, geometrically it is say that the original point, which was an angle theta from the origin and at distance R has now been map two the new point is still at a distance R, but the angle theta plus pi by 4, so this is the original point x y has now been map to well a point here. So, it is just moved like this by an angle of 45 in this, the length remains unchanged.

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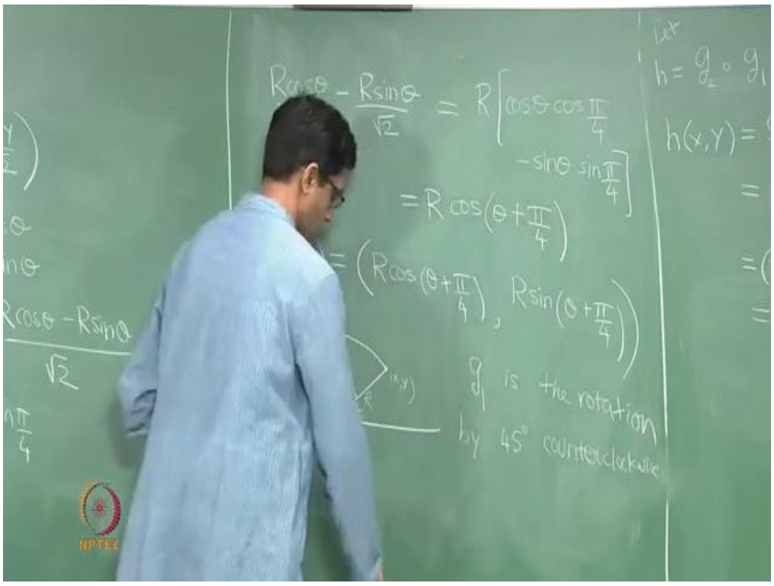

In other words, this is just a rotation by 45 degree. So, therefore, what is we have managed answer the original question g 1 in fact, is nothing but is the rotation by, g 1 is just the rotation by 45 degrees anticlockwise. So, finally, coming back to our analysis of f, remember f was just the composition of these two maps.

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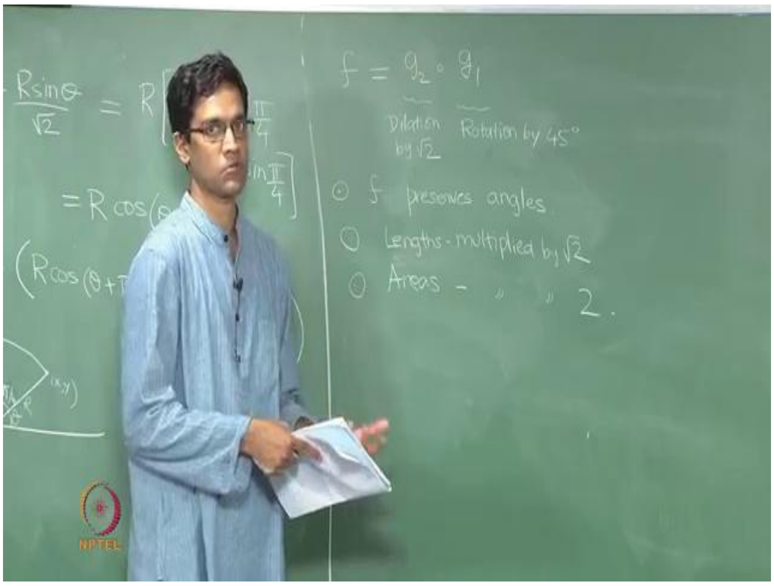

So, recall that this function that we started out with is just the composition g 2 composition g 1. So, in other words, the function f that we looked at is just the following you first perform a rotation by 45 degrees, and then follow it up with a dilation by a factor of square root 2. So, that simple looking formula that we wrote out for the function f, it just x minus y comma x plus y, but if you really analyze with little deeper, you find that if the function f can we broken down into two simpler consequence, one being rotation by 45 degrees and the other being just a simple dilation.

So, what is this tells us? Well, it tells us lots of things, because we know many properties of rotation and dilations. So, here is a consequence observe this means that f preserves angles, if I have say two lines which makes some angle theta and I applying f to it, what I will get will be two more lines which still the make same angle theta. And why is that, because the first operation rotation does not change angles, the second operation dilation as we just saw also does not change angle.

So, when you do the first followed by the second, the angle cannot change neither of the two continuance pieces can change the angle. So, f thus preserve angle, what is f do to lengths, f dilates lengths by a factor of root 2. So, lengths are multiplied by a factor of root 2, again why is that, because well a rotation does nothing to length, it preserves lengths, keeps lengths same. Whereas, then when you follow it up with the dilation, the length will become multiplied by square root 2.

And of course, areas so when lengths get multiplied by square root 2, you sort of except the areas 2 get multiplied by factor of 2; that is what happens in this cases, the areas 2 get multiplied by a factor of 2. So, what this tells us is often you know the simpler sorts of transformations, we talked about dilation inhomogeneous dilation, rotation, translation is things like that are often very good building blocks with which you can try and understand somewhat more complicated transformation of again.

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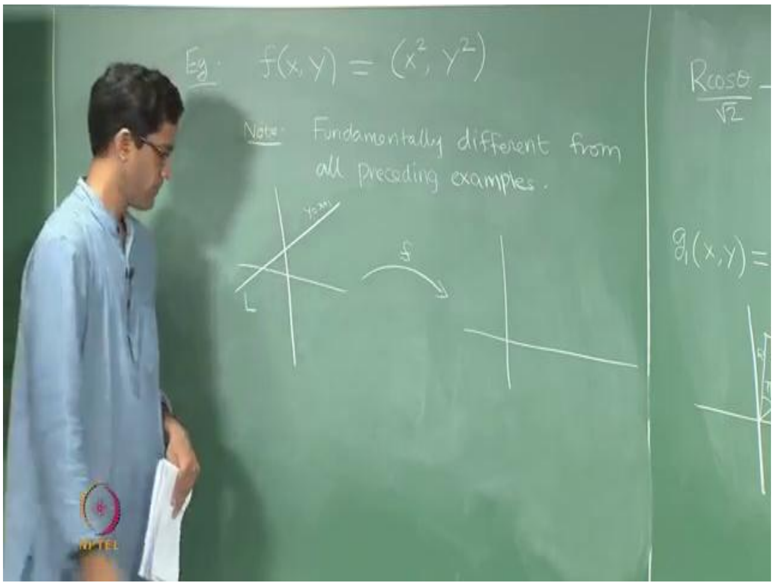

So, now, one final example straight the function f of x y define to the x squared coma y squared. So, this formula is now again simple, the sense, but fundamentally different from all the example we are looked at until now. So, observe everything we have done until now has never had squared of the variable occurring in these two places. So, always had something which is look like some multiple of x plus some multiple of y for instants or some multiple x plus are constant.

So, what we had so far have always being linear polynomial if you wish in the variables x and y in the first component and in the second component. Whereas, here the first time we see square appearing, so point to note here is that this is fundamentally different from all the examples so far. So, this is certainly very different, so I should say, this is fundamentally different from all preceding examples of transformation and we will soon see this difference being expressed.

So, for instance here is the first thing, let us try and figure out, what this map does two lines; that is often being our first step in all these prier examples. So, let us take a line. So, I am going to pick the line y equals x plus 1 has slope 1 and y intersect set of 1. So, here is my line L and let see, what this function f does to this line L, what sort of thing you get on the other side. So, what is the approach, let us write just as we did earlier will try and write equation for the image.

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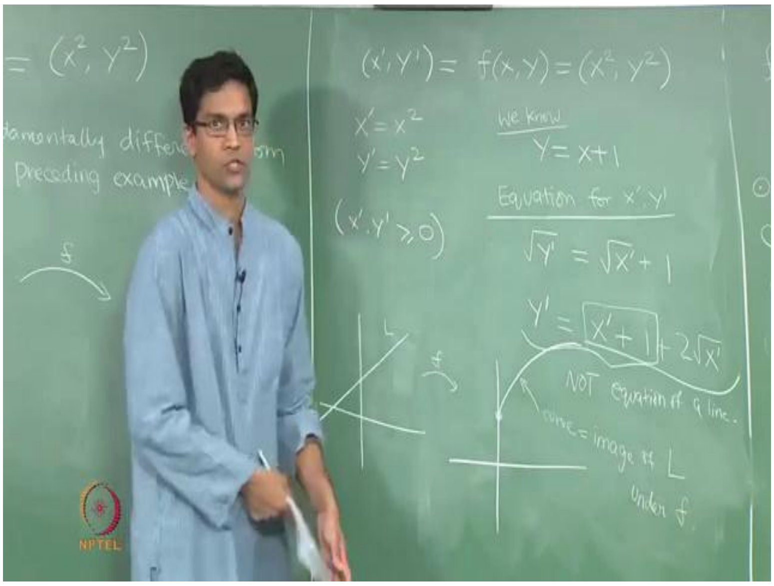

So, let us call x prime y prime to be the image, it is call it f of x y, in other words, this is x squared y squared, so x prime is x squared. So, observe this just means of course, x prime x squared y prime is y squared. So, now, what we know is that is variables x and y are connected by in equation, then they vary, when x and y vary they vary in such a way that y is always equal to x plus 1, this is the equation that we know.

And what we know want to derive from here is an equation which tells you how x prime and y prime are connected, how to they vary is the question. So, let us try and figure out what this implies in terms of want an equation in terms of x prime and y prime and so of course, that is equation for x prime y prime. So, it is again easy just like we did earlier. So, we will just rewrite y as the square root of y prime, y prime is y squared. So, square root of y prime equals square root of x prime plus 1.

So, imagine, so this is relay the equation that you have which connects the variables y prime and x prime, observe here y prime and x primes, since there squares these variables are definitely pass. Observe that x prime and y prime are defiantly greater the equal to 0, so here is the equation that tells you how y prime and x prime vary. So, best way to do this of course, the both sides we will see what that uses tells you y prime equals x prime is square root of x prime plus 1 square.

So, that is this plus 1 plus 2 root x prime; that is the equation which tells you, what you know, how y prime x prime vary. And observe, this is clearly not the equation of the line, because of the this factor here, you have the two root x prime, this is clearly not the equation of a line. So, what this really imply is the original line that we had, y equals x plus 1, a that is the line L, when you map it under this function f, what it transform to, what it deforms to is something very different. And it some curve y prime and x prime are both positive and they are connected by this equation y prime equals x prime plus 1.

So, observe if you only had the first two terms of course, that this same line you know y equals x plus 1, but there is an additional term there. So, if you sort of draw it something like that is the curve. So, this is the curve, which is the image of the line L and this curve is the image of the line L under this f. So, the very first point here is that, it does not map lines to line. So, that is already a very fundamental difference, which means, it actually makes lies somewhat harder for us.

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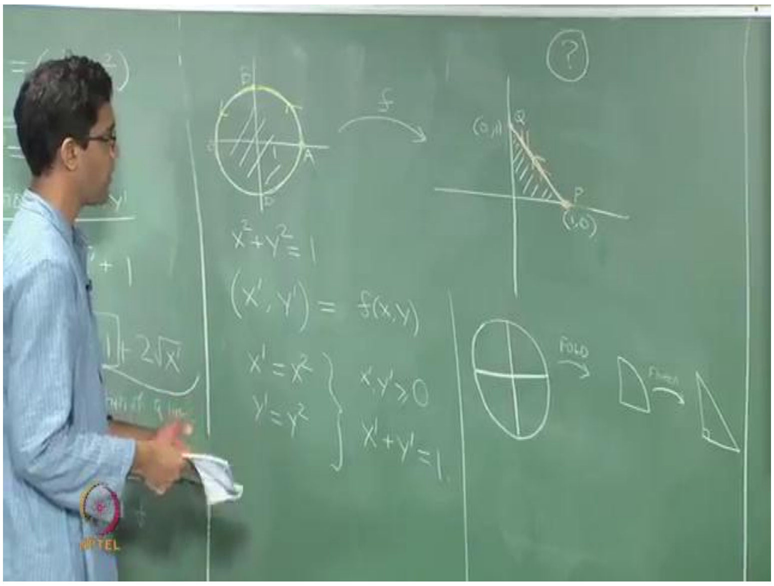

Because, you no longer can figure out, what happens to polygonal regions just by figuring out what happens to just the end point, just the vertices alone, because the line connecting those two points could have deform to some funny curve connecting in the vertices, but never mind. So, let us look at one further region, so let us look at this circle of radius 1. So, I want to take circle of radius 1 and figure out what happens to that under this map f.

So, my question now is, what is the image of circle under this particular map, so let us try and do same thing, now circle again is a curve that is given by the following equation. So, it is all point x comma y on the plane, it satisfies x squared plus y squared equals 1. So, that is the circle and now again I do the same thing, I will look at the image of this point x y, I will call it x prime y prime. And now, so what is the mean x prime x squared y prime is y squared.

Now, the question is can you know find the equation that tells you how x prime and y prime vary as x and y satisfy the given equation. So, observe that x prime is x squared y prime is y squared of course, means the following as before x prime and y prime are positive, at further since the original point lie on the circle x squared plus y squared is 1. So, in other words, x prime plus y prime, it is actually here is the conclusion about the point x prime y prime as x comma y runs around the circle, the point x print y prime, it is image lies on well it first it lies on the first quadrant, because x and y are positive, but further, it lies on particular line lies on the line x plus y equals 1.

So, here is the line x plus y equals 1 and this point x prime y prime must lie on this line, it can only vary on this line; that is what you depends, but further remember it lies only in first quadrant. So, it do not even have the parts which are it is not even line anymore, it is only this is the particular line segment that it must be as. So, let us look at the two end points, here is the point 1 comma 0 and here is the point 0 comma 1 and you have the line segment join in this two points.

What this transformation f must be doing is, it must somehow map this entire circle just to that line segment alone, so it somewhat surprising at first glance. So, let us figure out what happens you know how can circle really just map to a just small line segment like this. So, imagine what happens here as x. So, let us look at the four parts this circle the four quadrants. So, here is the yellow portion as x and y both live in the first quadrant and you know let say, I move from this point here towards the point alone. So, I move like this along the circle, let see what happens to those points.

So, this point here A just the point 1 comma 0, under this function f, maps to well 1 comma 0, I square both components, I get this point let me call it p and q. So, when I move from A to B along my circle, A maps to p B maps to q under this map and so what I get is, I will move along this line or line segment from p towards q. So, this is how the point moves, when you move from A to B on the circle.

Now, it is keep doing this, so you move between let say B and C. So, observe C is now the point minus 1 comma 0 and under this function f, C again maps to p, because it is going to square both components. So, as you move from B to C, you will observe that all you doing is just moving back down along this. So, you first move from p to q, and then you move back down p and well it is similar, when you move from C to D, you move back up and when you move from D to A, you move back down.

So, you sort of do four iteration you go up and you go down you go up again, and then you go down again. So, traversing the four quadrate of the circle, amounts to under this map f, what it does is it traverse the line segment up and down four times and to really understand, what this is doing. So, if you also look at the region inside the circle, what it is going to do is, it is just map it do this triangular region here.

So, again something for you will check, so exercise take point inside the circle for instance the origin maps to the origin and so on. Now, what is a geometrical way of really understanding, what this map does after all of these various example, we are doing are with following thing in mind, we want to get some geometrical intuition for how this function deforms the plane in to other points of the plane. And to do that really looking at certain sub regions of the plane and figuring out what deformation they suffer under this function f.

So, here is what happens, so let me just give you the description and leave it for you to convince yourself that this is in fact, what it is. So, it is really the following step, you first take the circle. So, imagine I have a circular disk, paper say which is in the shape of the circle and imagine, you have these to diameters now you fold this circle, so you really perform the fold say along one diameter then along the other, such that, it just quadrant.

So, there are two folds that you perform, the first along the first diameter and make it a semi circle, and then along the other diameter to then make it in to the just a single quadrant. And now, what you do is this quadrant under this function f, you sort of flat and out this circular arch in to a triangular region. So, of course, the flattening is not, it is mean that a just geometrical term, we just tell you that you sort of doing something to that circular portion and make it a square.

So, the exact thing is of course, given by formula x squared y squared, but what you doing here is your flattening this function f is now flattening this quadrant into a triangle. Finally, that is the right angle triangle that you get, which you joining the point 0 1 and 1 0. So, it is really a two step process here again, one the circle is folded along the diameters I to single quadrant, and then this quadrant is flatten in to a triangle and each other.

And so this combine deformation is really what the function f achieves it does both in one show. So, the key thing here is with this example at least the key point is to realize that this thing of lines mapping to lines that the keep you know talking about or we kept talking about all the earlier examples is not something to be taken for granted. It is not always true that any given transformation may not map lines to lines, it can do some rather strange thinks to lines make them other kinds of curve and so on as we saw in the example.

And so every time, you have a transformation which are trying to understand, it is important to check for instance that it does in fact, transform lines to lines, if at all you want to see what is going to happens to triangle. So, R squared is other polygonal region checking that lines map to lines is somehow the very first step that something that one must always do before once starts you know manipulating playing with plane these function little bit more.

So, we will return to this theme of lines mapping to lines n, specifically next time we will talk about transformations which have the property that they map lines on the plane to line on the plane. These are called a finely transformation and you know sort of special case of those are what are called linear transformation. So, these form very important class of transformation, which can be understood of many different point of view and using other tools from other part of mathematics, specifically things like matrices and so on.

So, the think for next time this to try and understand the so called affine transformation and how matrices naturally make an appearance only on to study of affine transformation.