An Invitation to Mathematics
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Unit
Functions
Lecture - 23
Function on the plane, Rigid motions

Welcome back, what we talked about last time was trying to understand functions in terms of various ways of picturing them. Now, what we do today is to talk about function R 2 to R 2.

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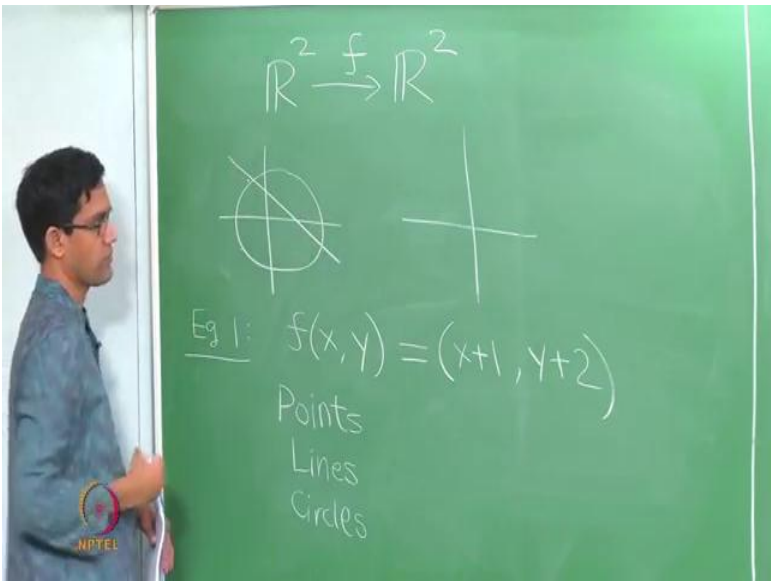

So, R 2 is a plane to the plane and let us do this by examples, so, I am going to write out a sequence of examples. So, let us take first example to be the function f of x y equals x plus 1 comma y plus 2. Now, let us try and picture this function, try and understand what it does. So, typically how does one want to understand functions from R 2 to R 2, So, here is the, here are the various you know ways of thinking about R 2. You know, what are the various subset of R 2. So, R 2 has points, so each of this is a point in R 2. The other subset of R 2 of interest might be say lines are more general curves.

So, for instance I could think of lines or maybe I could think of circles or maybe when more general curves like parabola or ellipse, just more even more arbitrary curves. So, I could look at lines, I could look at circles. So, these are points are sort of what we often called zero dimensional objects, lines are just one dimensional, circles are also just one dimensional. The plane itself is two dimensional and we could also look at regions in the plane.

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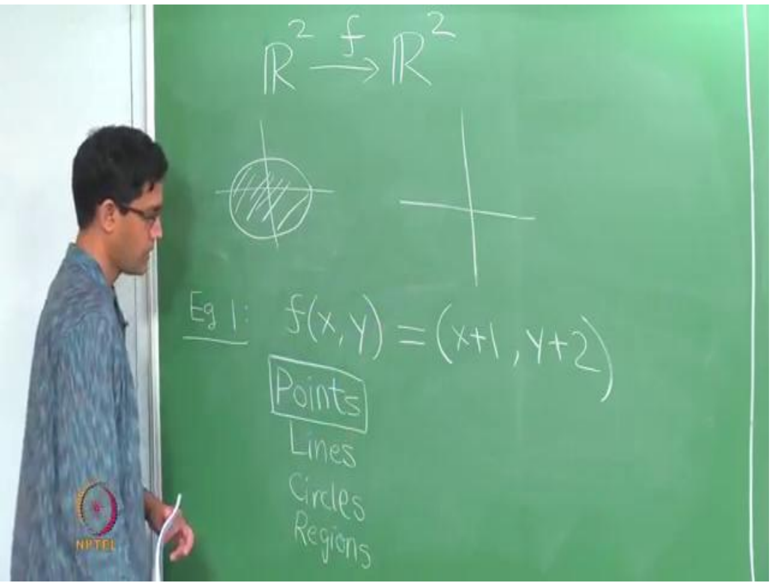

So, by a region we mean sort of a two dimensional thing, so let say you know, you could take the region that is inside a circle. So, this disk is often what you would call a region in R 2. Of course, a region need not have such a nice shape, it could be a bit more arbitrary shape region. So, here are various subsets of R 2 and one way of trying to understand the function f is to see, what it does to each of these kinds of thing.

So, at the very basic level what you really given is, how a function x on a point x comma y to produce another point. So, let us first think of what this function does to a typical point in R 2.

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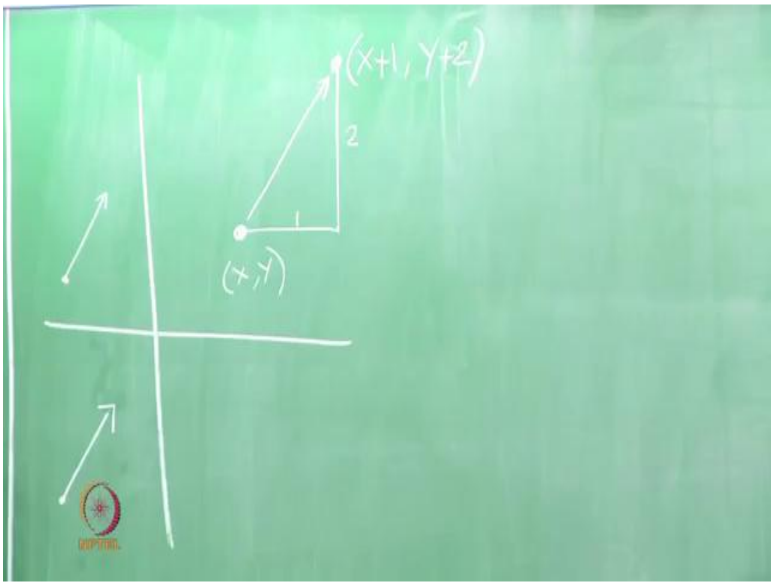

So, I have the point x comma y, here is the point, so it is one unit to the right and two units above point of x y. So, what this function does is really the following, it takes this points x comma y and moves it or maps it to this point here and it sort of does this uniformly. It does this to every single point on the plane it takes it and moves it by the same distance and the same direction. So, it take this point, it would similarly move it parallely, this guy would also move parallely.

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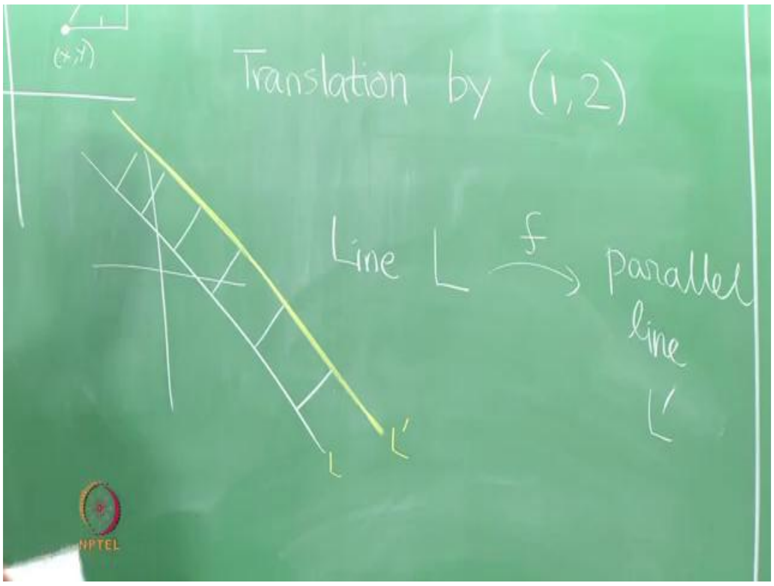

So, sometimes this map here is called a translation and it is a translation by the vector 1 comma 2, if you wish. So, it translates everything by that amount, one unit to the right and two units along. So, now, let us ask the, so this is actually a very, very simple sort of map, what it does to everything is pretty much has a same description. What does maps do to a line for example, so suppose I have a line on the plane asking, what it does to the line is like asking you know, what is it do to all the points of this line.

So, every point of this line moves by the same amount, so all points of this line slide up to those points and so, this line here let us call it L maps to a parallel line, so this is L dash. So, what it does is let sort of also translates the line L by the same amount 1 comma 2. So, lines map to parallel lines, so line L what is it go to under the function f, it goes to a parallel line. L dash which is shifted by the amount 1 comma 2 and well, you can think of the same thing, if you have the circle the same thing will happen.

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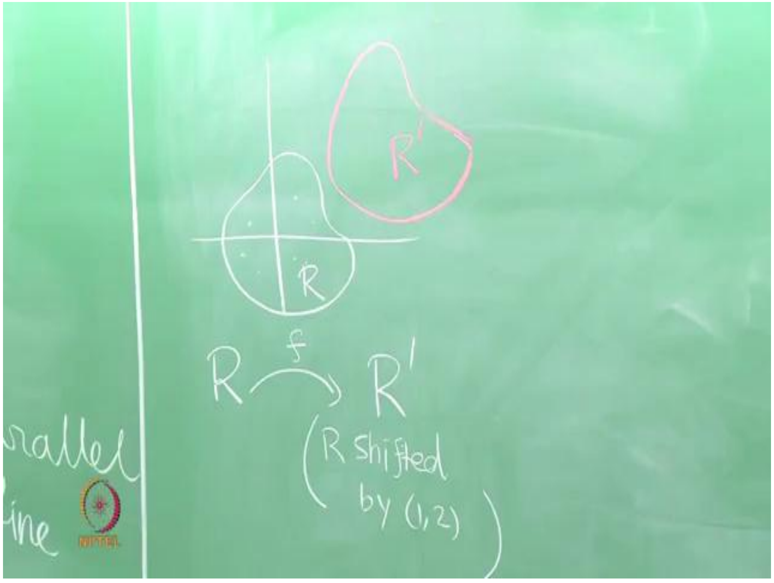

It will go to a shifted circle, something shifted by the amount 1 comma 2 or more generally if I favorably generate the same thing. So, if I have some sort of the reason on the plane, then what is this function do to this region, so you look at what it does to all points of this region, all points move by that amount. So, of course, what this function n's are doing is really the same thing. So, it is moves to region up by the same amount, so this is the region R dash, the original region R.

So, R maps to R dash under this function f and what is R dash, it is just a shift. It just, R shifted by 1 2, that specifically R dash, so our translation is very, very easy to describe. It just does sort of the same thing everywhere, it just move everything by some fixed amount.

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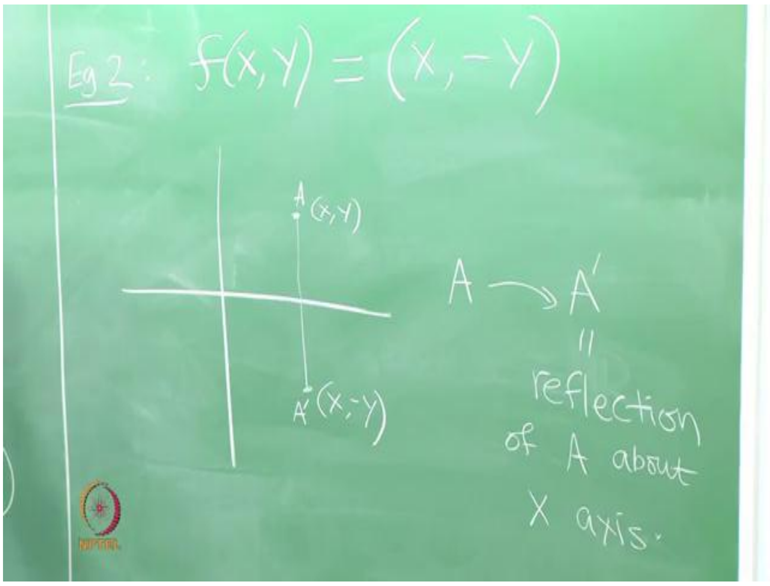

So, let us look at the second example, if I have a function defined as follows x comma y equals x comma minus y. So, again first let us try and understand what this function does to points. So, if I have a point x comma y on the plane, what this function does is well it keeps the x coordinate the same, that maps it to the point x comma minus y. So, the point let us call it A maps to the point A dash under this function. So, A goes to A dash and what is A dash, well one way of describing A dash is to say it is just the reflection of this point A about the x axis.

So, observe A dash is just the reflection of A about the x axis. So, this map f here, it is again very easy to describe, it just does a simple reflection about the x axis.

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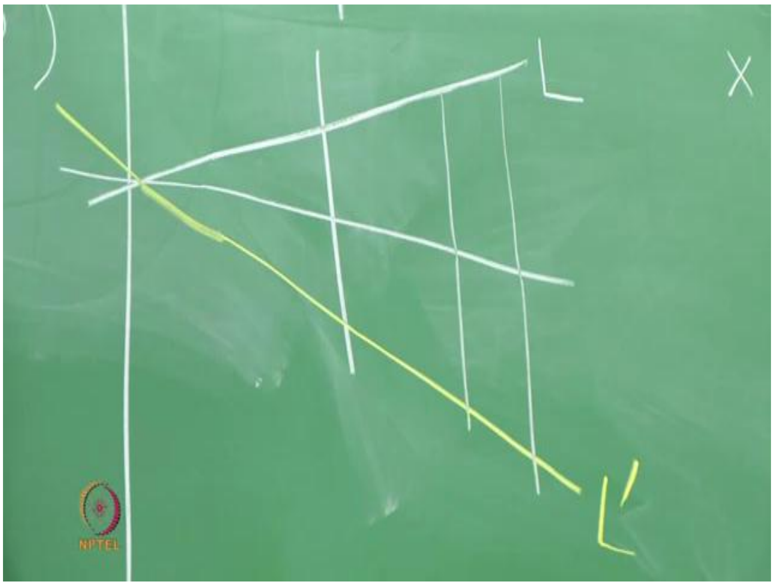

So, again similarly if I have, say a one dimensional subset, say I have a line the L R 2 and I want to know, how does this function transform this line, what it do to this line. So, I just see what it does to each point of this line, well similarly it just maps, it reflex each point of this line about the x axis. So, it just gives me a line, it give me the reflected line, so this is what a function could do, it just the mirror image.

So, this is a line L maps to the line L dash and observe of course, these lines will have, will intersect somewhere on the x axis. So, that is the picture for what it does to lines. Now, and similarly if you want to do regions and so on, it is not much harder.

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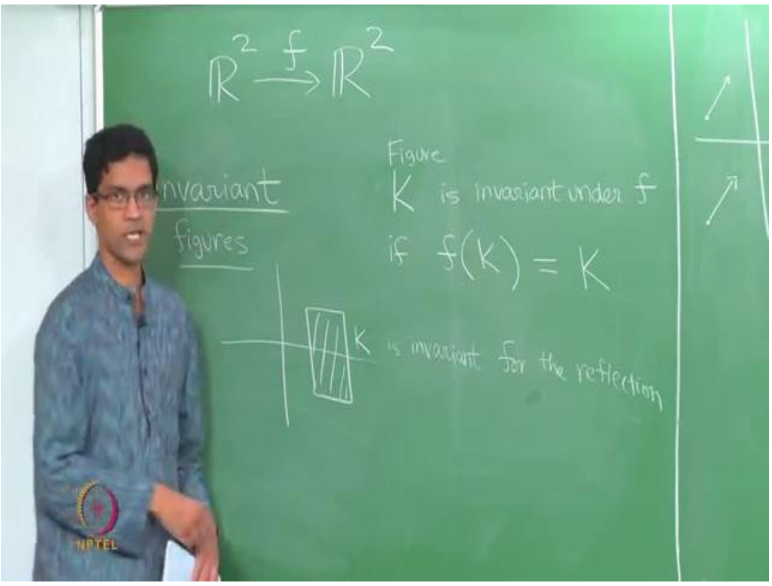

But, here is often an important and eliminating aspect that helps us understand how this functions, that is a notion of an invariant figure. So, what is an invariant figure? We will call a figure; by figure I mean it in the most general sense could be points, lines, curves, regions and so on. An invariant figure is one which is, which does not move under the function in some sense, which is map to itself under the function. So, what is an invariant figure, I will call a figure f, let us call it as f, so let call it…

An invariant figure is the following, you say that a figure k is invariant under f, if when you apply f to that figure, which means if you apply f to each point of that figure, that you get back is that same figure that you started with. In other words, it somehow maps that figure onto itself. So, let us do some examples to get a better sense of what this does, let us do it for the case of the reflection about the x axis.

So, I will come back to the translation, so if I take the reflection about the x axis, observe that any figure that symmetrical about the x axis. So, for instants if I take say a rectangular figure like this which is symmetrical about x axis, so if I take this region for instants to be k. Then, when I reflect it about the x axis, what I get back is a same region, because the top half will map to the bottom half and the bottom half maps back to the top half.

So, observe this k is invariant, for the reflection, for under the reflection about the x axis. Similarly, pretty much any figure that symmetrical about the x axis will be invariant, now let us think about translation. So, here is the first question is there anything at all that is invariant under the translation, because it seems as of that translation moves everything, it seems to everything in one uniform fashion. So, no matter which region you try to think of anywhere, it seems as of the translation will move it, but observe a line for instants.

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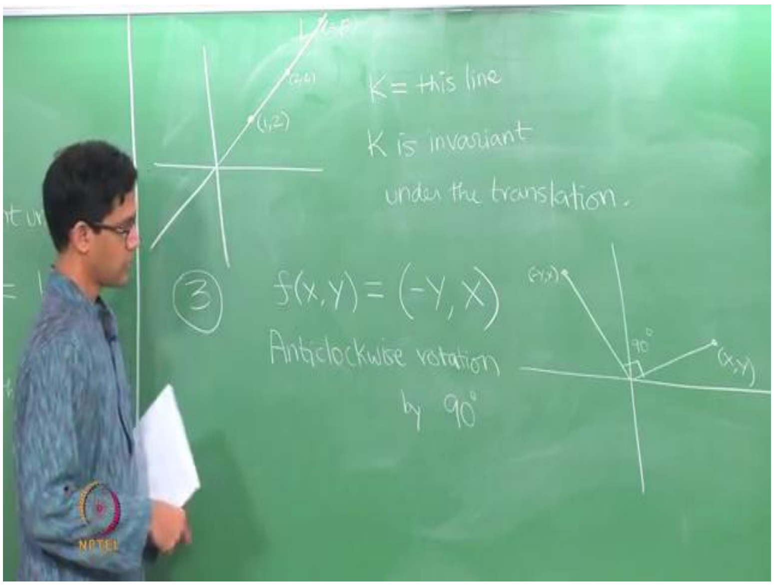

So, if you took, so imagine the following, you take a line L which passes through the point 1 comma 2 and through the origin, now imagine what the translation will to do this line. Well, what will do is to move every point on this line by 1 comma 2. So, the point 1 2 moves to the point 2 4, 2 4 will move to the point will be 3 5 or 3 6 and so on. But, pretty much no matter which point you take on the line, what you will get is another point on the same line, so the line sort of slides up by 1 comma 2.

So, here is an example of a figure. So, if you take k here to be this line or in fact any line parallel to this one, then you apply the translation will find that k in fact does not change, k is invariant under the translation. So, good we have talked about translations and we have talked about reflections, so here is another one f of x comma y equals minus y comma x. So, let us try and see what this does first lead to points.

So, if you imagine a point, so the here let call it x comma y minus y comma x. So, minus y is that is the new x coordinate and then x, so let us this and if you sort of try and compute angles and so on, you will find exactly this in 90 degree angle. So, what is this map to, well it takes every point and rotates it counter clockwise by a 90 degree angle. So, this is nothing but the anticlockwise rotation by a 90 degree angle. So, that is what it does to each point on the plane.

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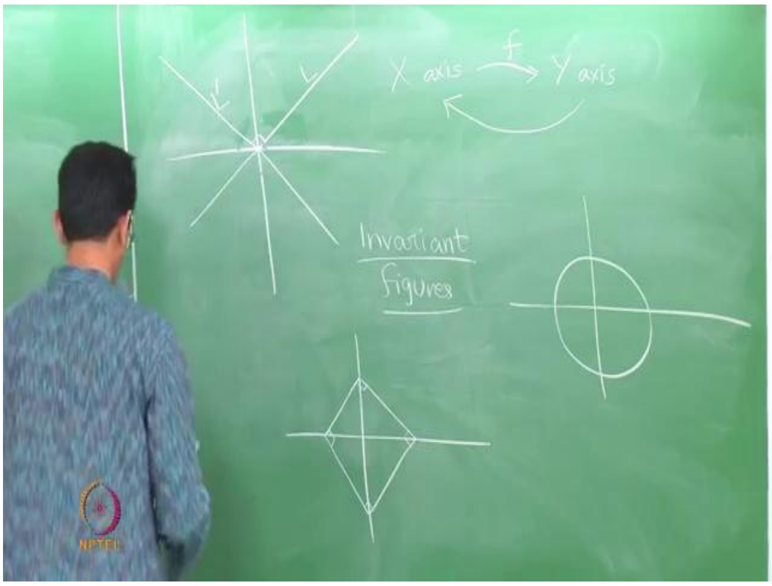

So, of course, it is easy to see what it does to other things, for instants say the x axis the line which is the x axis will of course, map to the y axis, because it is a 90 degree of rotation, the y axis similarly will map back to the x axis. Or in general, so let just write this, so the x axis under this map f will map to the y axis. In fact, the y axis also maps to the x axis and in this map and in general, any line will just map to the line that is perpendicular to it.

If you take a line through the origin, you rotate it 90 degrees, so it maps to a perpendicular line, L will map to L point. So, let us what happen to lines, region similarly you imagine a region anywhere on the plane and just, what you get is the rotated reason. But, again what interesting here is to look for invariant figures, so thus rotation had invariant figures at all. So, here is one obvious figure is just the circle about the origin.

If I take circle about the origin and imagine what happens when you rotated it by a 90 degree angle, well you just get back the same circle. So, circle through the origin is in fact invariant under this transformation, but of course, one can even find other such invariant figures, as all you are doing is just a 90 degree rotation. So, imagine placing a square in this fashion, sort of a very good figure.

So, imagine this is square and it sort of look at what happens when you rotate this by 90 degree angle, all that will happen is that you know this vertex will now coincide with the next one that will coincide with the next one and so on. So, here is an example of figure which is invariant under 90 degree rotation. So, a circle is special in the sense, it is in fact invariant under rotation by any angle, not just by 90 degrees. So, this figure is very special, it is invariant if you rotated it by 90 degree, is 180 degree or 270 degrees, but not by you know in other angles.

If you rotate it by, say of 45 degree angle this would not give you back the same figure, whereas a circle is sort of perfectly invariant under every possible notation. So, we looked at three functions, the translation and they have the following interesting properties.

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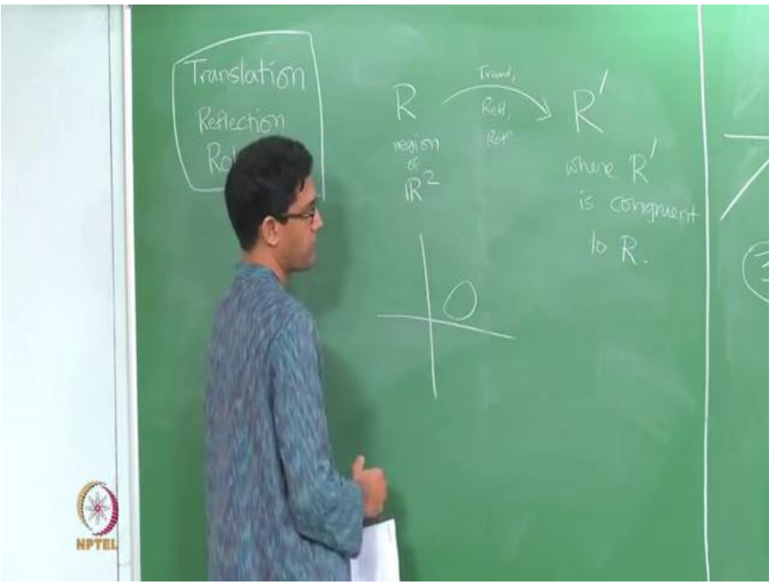

So, for now what I have done I given you examples of three functions, the translation function, the reflection about the x axis, rotation through 90 degrees. So, you looked at three such functions and one interesting thing aspect of all of them is the following that they all map a region R to a congruent region R dash. So, if R is any region in R 2 and you take any one of these three fellows translation, reflection or rotation, what does it do to R, it maps it to R prime.

So, this can be a translation, reflection, rotation any one of these three acting on the region R will you a congruent region R prime. What is congruent mean? So, where R prime is congruent to R, congruent remember means the following that you can super impose R on R prime in such a way, that they coincide exactly, so that is what congruent means. So, observe if I had, so imagine you have a region, so of course, if you are translated by some amount, clearly it is a same region.

So, the translated region R prime is it is clear it is congruent to R, it is just to copy of R that is just physically moved by that amount. Reflection, well in order to make these two things coincide, so if I have let say.

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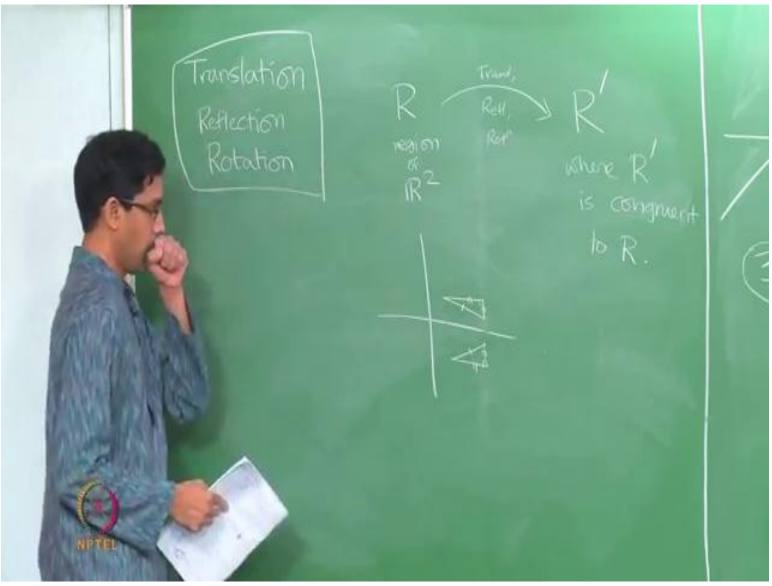

So, I have a figure like this, say that is my region R and if I look at what happen to it under the reflection, well this will map here. The, you know this point and this point will map to point here and a point there, so you just draw it a little better. So, that is the reflection about the x axis of this original figure R, but again it is clear they are congruent in order to make them super impose, we just need to align the correct sides together.

So, for instants this side here as to be aligns with this hypotenuse here and similarly, so I will imagine is a right angle. Similarly, this horizontal side aligns with this and likewise for the third and similarly for a rotation, when you rotate of course, the entire figures move by some amount, but you need to be oriented, you need to sort of rotate it back in order to make them align with each other.

So, it is again a sort of visually clear that the rotations also give raise to congruent regions, if you take a region R it maps to a congruent region R prime. So, in particular these are the property that they do not change areas of regions, so you know, what are the property of translation, reflection and rotation.

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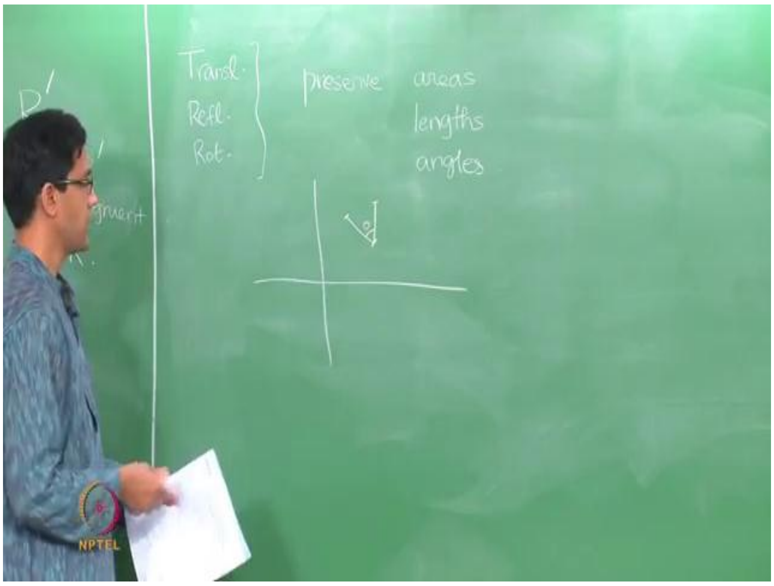

Well, translation, reflection, rotations they preserve areas, so R it give you back another region of the same area, in fact it gives you back region which is congruent. In fact, it preserve lengths, it preserves angles, so it preserves all of these. So, what is it mean to preserve length and angles, well if I have a line segment somewhere on the plane and say, let say this is segment and if I see what happens to it under translation, I get back another line segment on the same length.

Likewise, if I reflected or I rotated the length of the line segment that I get does not change, I always get it maybe it is oriented slightly differently, but I get the same thing that is what length preservation means. Angle preservation means imagine I have two line segments and now, I take this figure here and I say translated or I reflected or I rotated, now doing any of these three will never change the angle between the resulting lines segments.

It cannot deform the angle between them; it can only move this entire figure somewhere else. So, this property of translations, reflection and rotations is sometimes encapsulated by same that these are all rigid motions.

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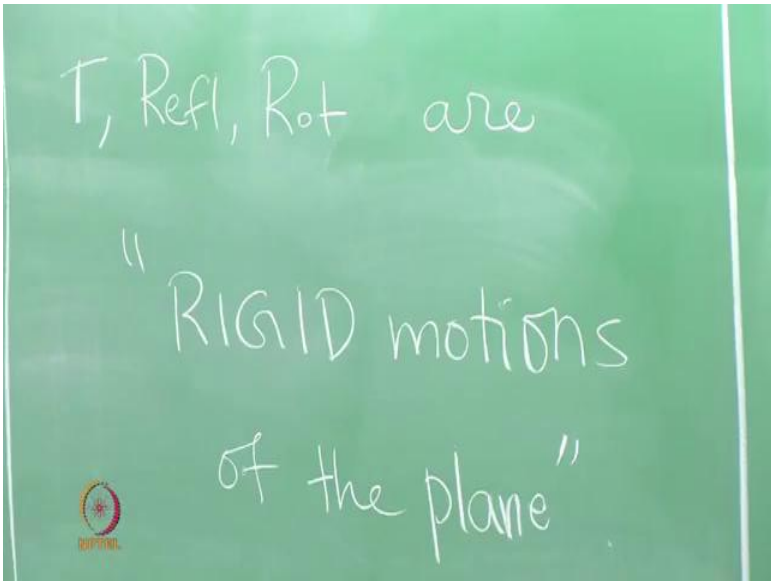

So, because of these properties that they do not really change either lengths or angles, we often say that translations, reflections and rotations are often called rigid motions of the plane. So, by rigid motions we mean you just move to think of moving the entire plane in some way as sort of like a rigid body, imagine the whole thing is made out of you know metal or something which cannot be deform, then you sort of move the whole thing or maybe you reflect, you rotate these operations do not really deform, they do not changed length for instants or they do not shrink, they do not sort of make angle smaller.

So, that is what rigid really is supposed to bring to mind. Now, what we will do next during the next lecture is to also look at other example. So, of course, these are not the only interesting examples, what about maps which deform.

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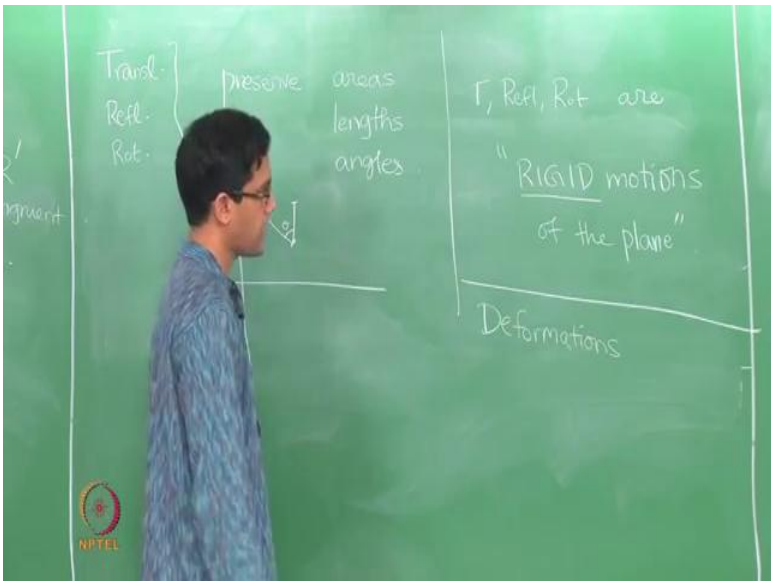

So, this is to be contrasted with maps which are sometimes called deformations. They deform lengths or angle or areas, so some times also called there are many different kinds of these. So, I am just using this as a lose term to mean something that does not act as a rigid fashion, so we look at examples of these. So, there are things called dilations and then there are more general linear transformations, but then we for next time.