An Invitation to Mathematics
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Unit
Functions
Lecture - 22
Visualizing functions

Welcome back, today what we are going to talk about is a notion of Visualizing functions.

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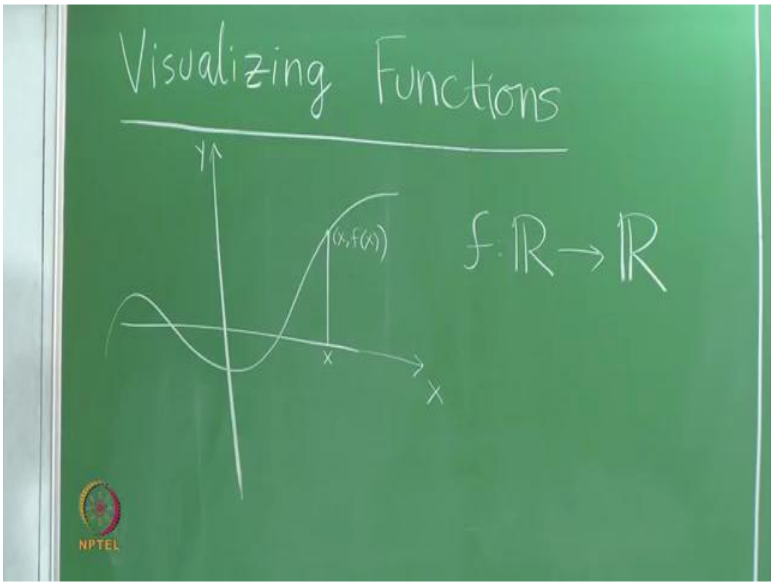

So, recall last time we have talked about the usual definition of functions and the usual context in which function appear as functions from the set of real numbers to the set of real numbers. So, here is a usual mental picture of a function, it has domain to be the real numbers and co domains also to be the real numbers and such a function is best represented as a graph. So, we talked about what a graph meant and also what continuity meant last time.

So, for each value of x, you look at the corresponding point x, f of x on the the graph and you sort of let x vary and what you get, what it traces out is what is what we called the graph. And what really is meant by visualizing functions, what I would like to mean by it is how do you do something similar for functions, which are not defined on R to R for instance. But rather say functions which are defined on the plane, which is what we should denote as R 2.

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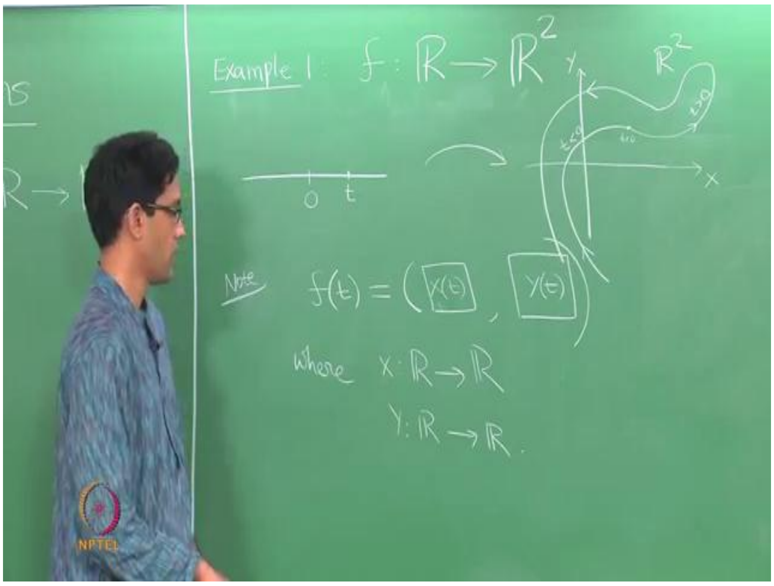

So, let us do various examples, so what I really want to do this lecture is to consider few examples of functions. But not from R to R, so let us do the first one function from R to R 2, so R of course, is just the real line and so this is function maps, each point in this real line to point on the plane, so the plane is what usually denote R. Now, the question is, how do you really try and you know form a mental picture of the function or how do you draw the picture, which gives you some idea of how the function behaves and so on.

So, the obvious thing to do is to sort of try copying something like what we did for graphs, we just to trace out all the points that you can. Now, but this slightly different, so here is what we would try and do maybe to get an idea of what this function does, think of this domain as being like time. So, instead of calling it as x as this conventional, let us call it t for now, to remind ourselves that it is like time and think of this function as doing the following for each value of time, it gives you a certain position on the plane.

So, this is the x axis in the y axis and you should really think of this function as describing the motion of the particle on the plane as time varies from say minus infinity to infinity or maybe only all some intervals, say time going from say 0 to 1 and so on. So, the motion of the particle, if you want to really describe it here, here is what you want to say, take time 0 and sort of see, where are the particle is, it say at some point on the plane.

And as you increase the value of the t, you would look at what f of t is, so f of t would trace out some curves as time increases, the position of particle varies and it becomes you know may be something like this. It could trace out some strange curve, this is what happens as time increases. But of course here, we also have t negative; t can also be all the way to minus infinity, so maybe it traces out some curve like that. When, time comes from minus infinity to 0, so this is the position at time t equal to 0. So, these are the positions that it traces out, when the time is positive and these are the positions where the particle was when time t is negative.

So, you should really think of this function, a good visualization or a good mental picture to form of this function is in terms of the curve that it traces out on the plane. So, the curve of course, also comes with the direction, it tells you, you know which is the direction, which you move as t increases. So, let us do an example, so let us take the function f defined as follows f of t is, so if you want to write formulas for functions from R to R 2.

What you would want to do, for each value of t, you must specify a point on R 2 and a point on R 2 is actually two coordinates x and y. So, here is what you would say, let f of t be some function of t, so what I should write here should be a function of t, varies as t varies and here, I should write a other function of t. So, this is sort of the most general way of writing a function which goes from R to R 2, which is we should say it essentially f of t equals the pair x of t comma y of t, where x of t and y of t are just functions from R to R.

So, where, so I am not able to make example, so let me just say, note the function x from R to R 2 is really always of this form x of t y of t, where what are x of t and y of t, you should think of x as being the function from R to R. For each value of time t, it gives you a single real number, the x coordinate and y is now a function from R to R again. For each value of time t, it gives you some y coordinate.

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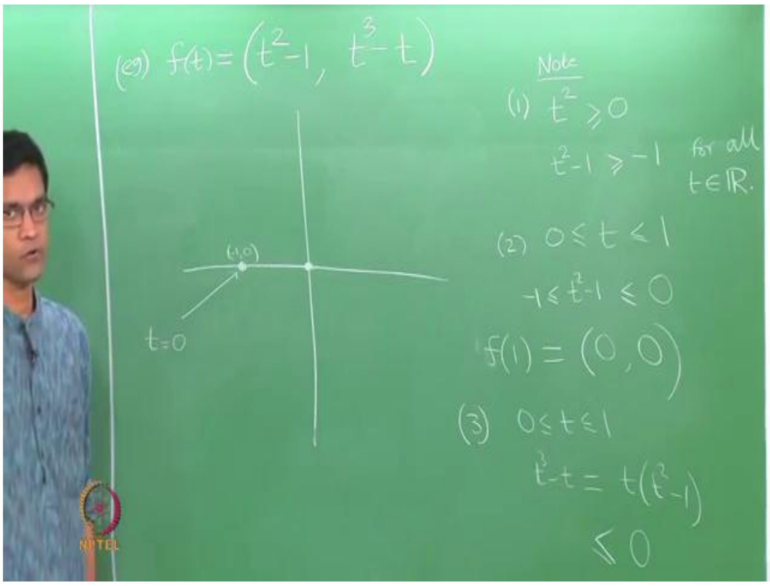

Now, let us take an example with actual functions, so suppose I take f of t equals t square minus 1; that is the x coordinate, y coordinate is t cube minus t. So, here is your function f and so what we want to do is really figure out, what is the curve that is traced out by this function as time t changes. So, let us do the following, let us look at where this particle is when t equals to 0. So, if you plug in t equals to 0, you get minus 1 comma 0.

So, this point here on the x axis minus 1 comma 0, this is the position of the particle at time t equals 0. So, let just denote that, this is position at t equal to 0, and then we see what happen as time increases, t becomes say 0 or well greater than 0 or in fact, thus time decreases. Whatever you do to time, you are either increasing or decreasing; observe that t square will always be positive.

So, here are various points to note about, so let us see what all we can introduce, t square is always greater than equal to 0, no matter, what t is positive or negative. In other words, t square minus 1 is at least minus 1, for all values of t, for all t in real numbers. So, this of course means that this particle behaves in the following manner; the x coordinate is always minus 1 or higher. So, it is always to the right of this point minus 1 0, if you wish, so that something to keep in mind.

Now, what happens as you increase t for instance, so let us figure out, if t goes from 0 to 1, so as long as t is smaller than 1, the x coordinate is still negative. So, let us analyze, what happens when t is between 0 and 1, t square minus 1 is greater than equal to minus 1, less than equal to 0. So, the x coordinate varies between minus 1 and 0 and it in fact, it reaches the value 0 correctly at t equals to 0.

So, let us also figure out, what happens at t equals 1, so let us call it f of t equals 1 is, if you plug in t equals 1 into the formula, well t square minus 1 and t cube minus t are both 0. So, f of 1 in fact 0 0, which means at t equals to 1, it is at the origin, so this happens for and how does that happens as t goes from 0 to 1, the x coordinate increases from minus 1 all the way to 0. And the y coordinate is in fact, what we know about the y coordinate in this stage, so if I take t from 0 to 1, let us look at the y coordinate; it is t cube minus t, which is valid t times t square minus 1.

So, in fact, this is negative, because t square minus 1 as we just said it is the x coordinate is negative and you multiplying it by a positive quantity which is t. So, t cube minus t is negative, when t goes from 0 to 1.

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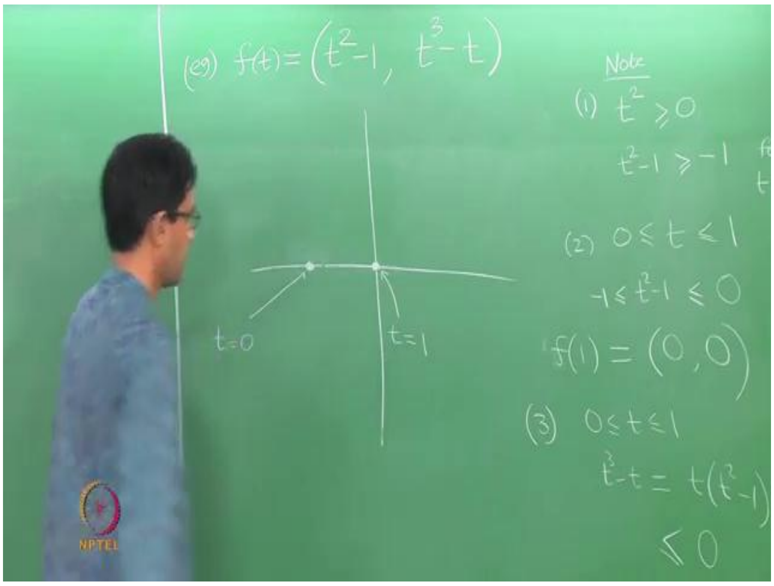

So, here is really the question, so let us just remember this, this is where the particle is at t equals 0, this is where it is at t equals 1; that is the origin. Now, the question is, how did we go from here to there, we are trying to get an idea of what the shape of the curve is. So, when t changes from 0 to 1, the x coordinate is of course, negative it continues to be negative, whereas the y coordinate is positive, so the particle sort of stays in this quadrant, it somewhere here the entire time. So, now, all that remains is to sort of to try and get an idea as to what the shape of the curve might be.

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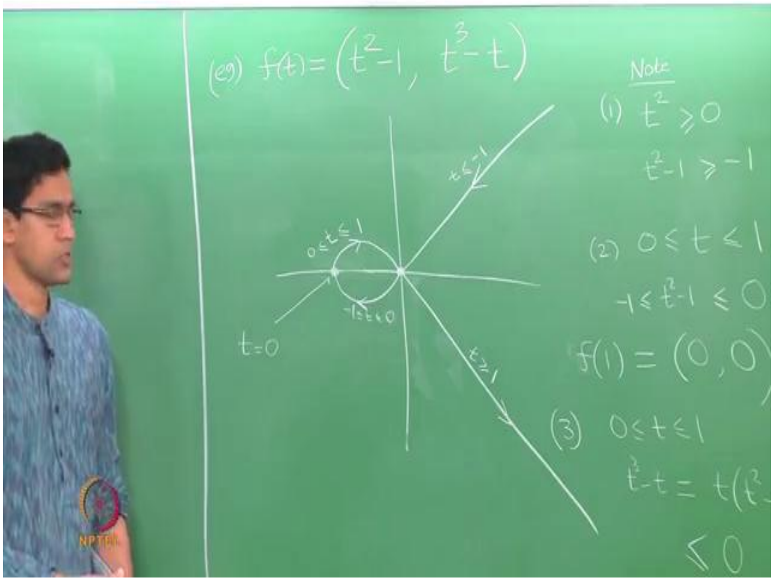

So, let us see, but the way before we do that, here is another thing to note as well that when t goes in the other direction, if t say negative. If t is minus 1, then again we will let us do the following, let just see what you can get when t is minus 1, you still get 0 0, was t square minus 1 is a 0.

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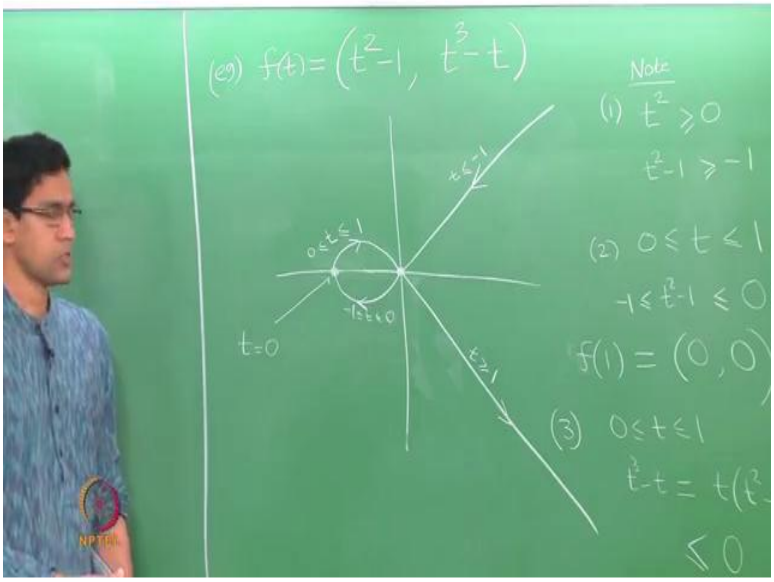

So, here is an additional point to keep in mind that the particle is actually at the origin, when t equals 1 also when t equals minus 1. So, for the moment, let say we want to analyze, what the particle does between minus 1 and 1. So, maybe I will leave this part as an exercise just try and figure out, what the shape of the curve must be, may be just by plotting few points and so on.

So, when t goes from 0 to 1, so let me just still you what you expect to get. So, if t goes from 0 to 1, here is what happens, t is it starts at 0 and goes to 1. Then, as we just say the y coordinate is positive, whereas the y coordinates is between minus 1 and 0. Whereas, if t is the other part that it is between minus 1 and 0, then here is what you get, here is the shape of the graph or the shape of the curve, when t is between, this is minus 1 and 0. It traces out these two things.

And now, here is again something that will require some more analysis of this kind, what happens, if t increases further, say I go from 0 to 1 and what happens beyond that, erase this now. If you increase t, if beyond one here is what will happen, so sort of to do something like that and what if t is say, even smaller than minus 1, then it is curve like that, it is not quite a straight line, it is some curve, but it lies in that quarter. So, this is basically when t comes from minus infinity down to minus 1, so this when t is less than equal to minus 1 and this what happen if t at least 1.

So, I have quite done the entire analysis, it something that you should certain try and do yourself. So, the point of this; however is following, when you want to try and understand a map from R to the plane, what you should really try and view it as something is describes the motion of the particle. And therefore, what gives you may be very good idea of what the function does, it actually drawing the part that traced out by the particle as it most.

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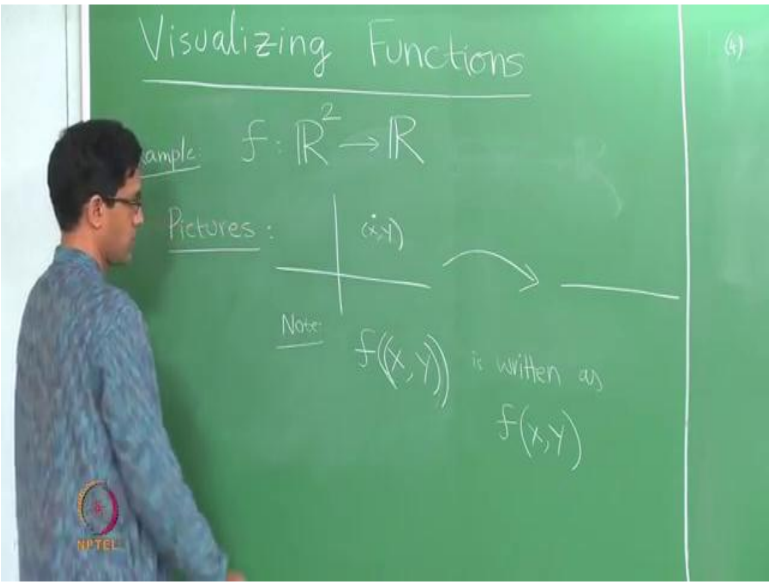

Now, let us do, can another example of a function f, sort of the opposite thing that we going from R 2 to R. And now, we want to again see, what sort of picture can be draw of this function, what is the graphical representation. So, how do we picture such a function, so let say there are actually two different ways of understanding functions which goes from R 2 to R.

So, firstly R 2 remember is just the plane, R is just the real line and the function does the whole following, it associates to each point x y on the plane, it associate some real number. So, you would often want to say, if I say this function f acts on a point on the plane and gives me some real number. So, the real number I get, when I act the function on the point, you should know, this is probably how we should denoted, if you wanted to be absolutely proper.

But, you know there are too many brackets here and we often just write this as, so note this probably what would constitute you know absolutely correct notation, but this is often written as is simply written as function acting on the pair x comma y. So, this is what you would really call it as a function of two variables. So, the two variables x and y you want to think of being the x coordinate and the y coordinate. And so as this going to say, what sort of pictures does wonder of such pictures, well the first picture is of course, just the graph itself.

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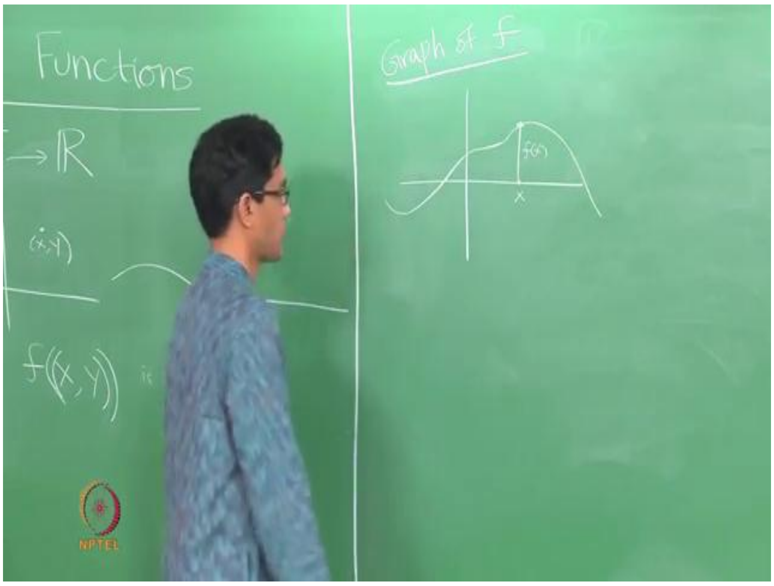

So, what is the graph, well recall what the graph meant in the case of the functions from R to R, one axis, you drew the domain and on the other axis, you just drew the co domain. For each point of the domain x, you mark of on the vertical side, the value of the function at that point, and then you join all of these. So, this was the procedure for obtaining a graph, if the function came was a function from R to R.

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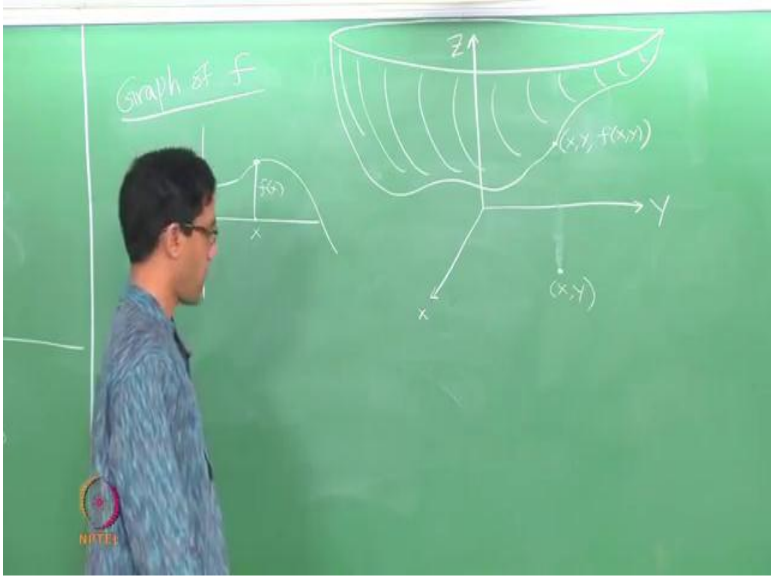

Now, if you want to do something form R 2 to R, well it is more or less as same idea instead of this, the domain now a plane. So, let us think of that the domain as being the x y plain and for each point on the x y plane, what you have is some function value, which you know plot along on the z axis. So, you got z axis here and so for each point, you will have something like this, so this is vertical along the z axis.

So, this point here as the coordinate, those x and y coordinate and this z coordinate is f of x y and you do this for every single point x y on the plane and doing that, we will give you several such points. And what you is really join all the map and what you will typically get some sort of surface, two dimensional sort of this surface and that is what we call the graph of this function. So, let just get the one dimension graph. So, this is how you plot the graph of a function of a two variables.

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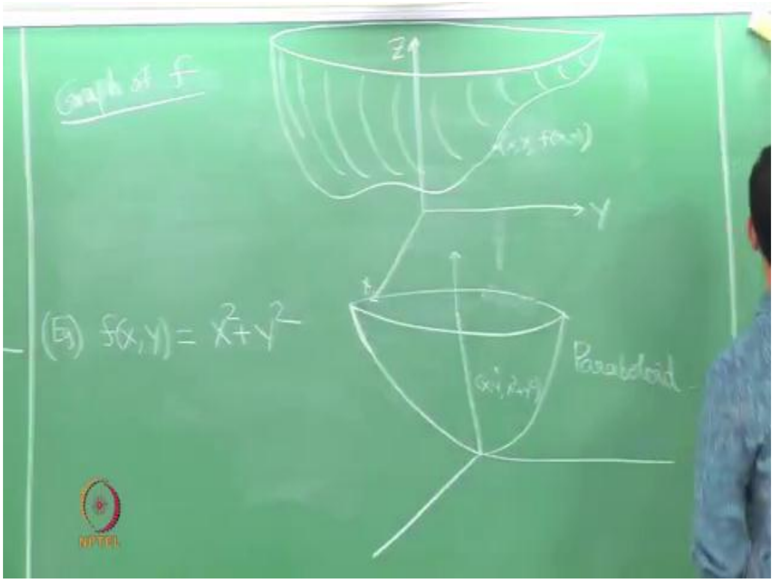

So, let us see, what is an example actual function? If I take a function of f of x comma y, just be the function x square plus y square. Then, if you try and draw of the function, what does is that for each value of x and y it gives you sort of the distance square from the origin. So, the further point is from the origin, the larger the value of the larger the z values actually. So, if you sort join all of these up, so let me tell you, what you get, it get a sort of get a surface like that, sometimes called a paraboloid.

So, this is the paraboloid and what is the typical points on the paraboloid, what is going look like, it has x y has x coordinates y coordinates on the z coordinates is x square plus y square. So, typical point on the paraboloid and observe that the paraboloid it has the following feature that if you take cross sections, if you cut it by a plane that is parallel to the x y plane, then what you get exactly a circle.

So, no matter where you get it as long as you cute it parallel to the x y plane you always get cross section which is a circle. So, this is sort of the second picture that one tends to draw of functions from R 2 to R.

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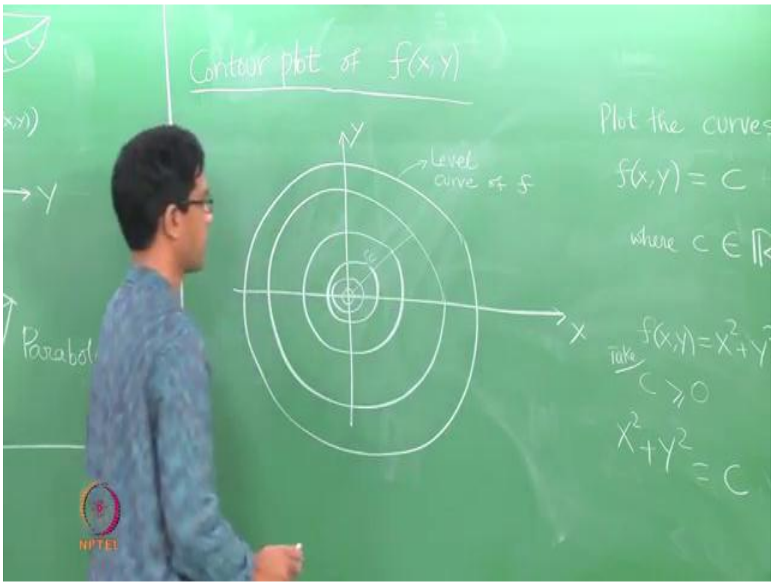

So, this is sometimes called the contour diagram or the level curves, so these contours plot of function of two variables. So, what you do is, well you do not read a three dimensional diagram at all, you all the plane, just take all values x y, well you take the x y plane and on this plane, you plot the curves which are given by the equation f of x, f of x y equals C. So, plot the curves C, when C is every possible real constant, so where C varies or the real number.

So, let us look at what this would have meant in the example t that is we just wrote out, if I take the function f of x y, x y square plus y squared. Notice that being as the sum of two squares, the value of the function is always positive 0 or higher and if you take pretty much any constant C, the curve f of x y equals C is just the curve. So, let us take C, it is enough take to C to way a positive constant and the curve x square plus y squared equals C is just a circle of radius square root of C.

So, here is the circle, center the origin of radius square root of C and if I change the value of c, so this just the radius of the square root. So, of I increase the value of C, I will get larger and larger circles, whereas, if I change whether decrease the value of C, what I get is circles of smaller and smaller radius. So, a plot like this in which you sort of also keep track of what see each thing corresponds to.

So, for instance this you know outer most thing, might have been C equals 100, C equals 50, C equals to 50, C equals to 20 and so on, it gives you some sort of mental picture of what the function is doing. So, these curves that we are drawn or sometimes called the contours level curves, this is one level curve of f, why is it called level curve, it is curve on f takes a constant value.

So, for instance this outer circle here could have been the curve on which f takes the value 100 and the inner circle is say the points on which f takes the value 50 or 25 or 10 and so on. So, it gives you an idea as to you know where f takes certain values and how that set changes as you change the value. So, this is again extremely useful piece of information drawing the contour plot in general, there some most of that one can actually reduce form the contour plot, but I get it in the plot right now.

So, there are in essence two different ways of understanding, what the function looks like in a graphical fashion. One you either draw the actual three dimension graph, let us somewhat hard to draw in general and also hard to visualize, you know in this case, it is easy, but an alternative would be two sort of draw the sections cross sections. So, the cross sections notice are exactly level curves, they give you the curves that you get, when you demand that of x y has to equals some value you C.

So, these cross sections parallel to the x y plane, they give you, so if you plot all of them, that is what called the contour plot or the set of all level curves and this again has advantage of just being the two dimensional plot. So, it that much easier to handle and also contains quite lot of information about the function. So, these are two different graphical ways.

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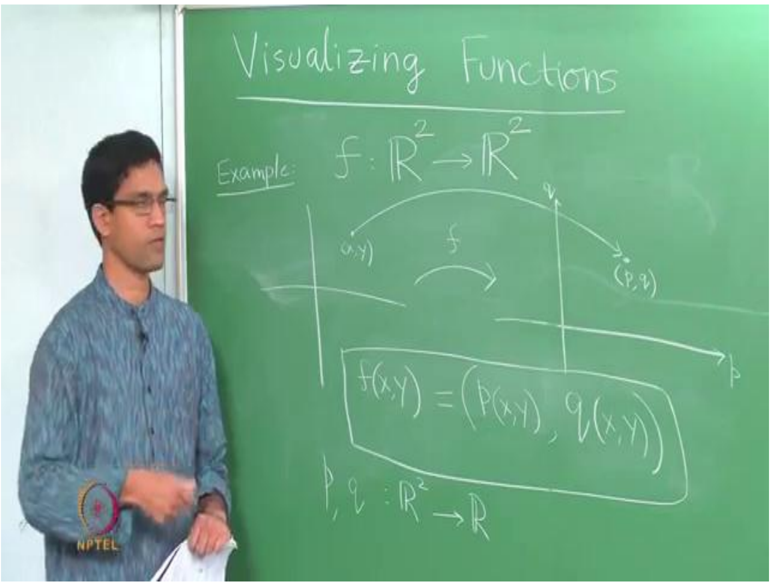

Now, sort of the final thing, since I have been playing with R and R 2, the thing you want to do is a finally this. So, I have looked at R to R functions from R 2 to R from R to R 2 and then finally, I want to sort of slowly get into this case of looking at functions from on the plane to the plane. So, what does it do for the each point of x comma y, it associates again point on the plane, in other words it associates two real numbers. So, that may be call it t comma q.

So, I will think of it as sort of being like a b axis and the q axis. So, at each x y it associates p q. So, how do you think of or how to visualize functions, so one of course, you know observe, what is f doing to the point x y, it is giving you an x coordinate p or you know the first coordinate their, which depends on the input x y. And also it giving you q value, which again it depends you depends on x y. So, this is probably the most general way of writing out a function from R 2 to R 2, each x y, it gives you pair of functions.

So, you should probably think of map from R 2 to R 2 as being a package of two functions, there is the p function and the q function and notice that the p and the q, well what are p and q, they are just functions which to the following. For each point in R 2 they only give you the real number here, so it just a function from R 2 to R. So, each of these is just a function of kind wave already studied a map R 2 to R and f is the really the package of these two functions p and q.

So, one way maybe of trying to understand this function of f might be to understand these two functions p and q separately. And since, they are functions what R 2 to R, you could either draw their graph as a three dimensional curve, as a dimensional surface or two dimensional surface in a three dimensional functions or you could do the contour plot of each of them.

And then somehow try and understand f as being made up of these two pieces, but those all these out to be rather than compression and not so eliminating ways of understanding, what this function does. And what I want to talk about next time is to think more in terms of you know think little bit more geometrically about what these functions do in terms of not just what does the points on the plane.

But, may be to what it does to say certain curves like line curves on the plane, what it just do the regions, for instance say a circular region center of the region or let say rectangle or square and so on. How this function transforms certain regions of the plane into other regions of the plane on the other side, so this geometrical point of view is a often gives you sort of better feel for what the function really does. And it is sort of also leads naturally to notions of linear transformations and to matrices and various other connections to other things that we have talked about, but more on that next time.