An Invitation to Mathematics
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Unit
Functions
Lecture - 21
The Intermediate value property

Welcome back, so last time we talked about the Intermediate value property of continues function.

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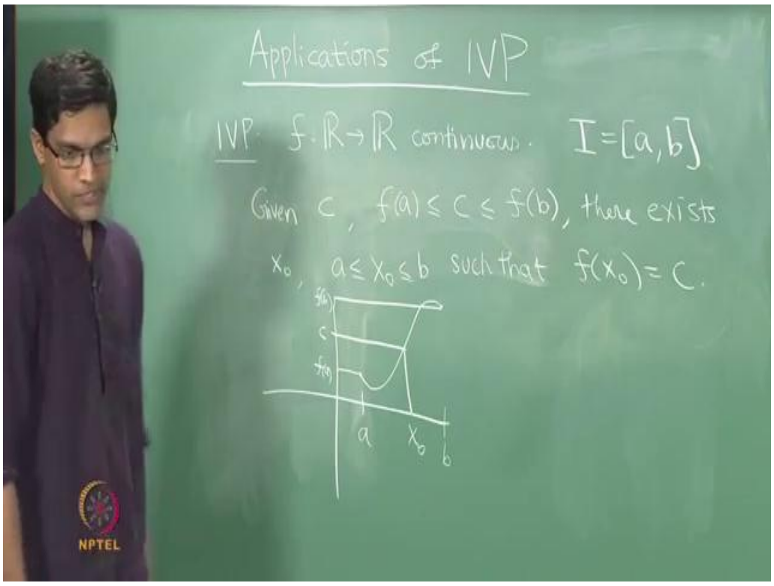

So, I am just shorting it into IVP. So, here is the intermediate value property itself, if you given a continuous function and you have an interval I. And given any y value, so here is a b and if you have given a function which goes between the boundary value f of a and f of b. Given any y value between the two, there is an x value at which that y value is attained, that is basically what the intermediate value property has. So, let us look at some very interesting applications of the intermediate value property.

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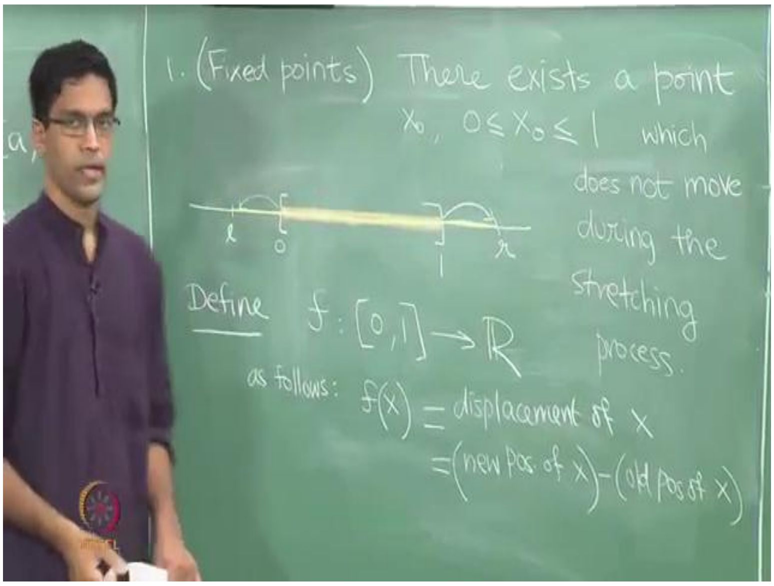

So, the first application is sometimes denoted or sometimes goes by the name of fixed point theorems, trying to find fixed points in various situations. So, here is the general set up for a fixed point theorem, so here is an example of fixed point theorems. So, let us take imagine the interval between 0 and 1 and let us for the moment written that, this is some sort of a rubber band. So, in the sense that it can now be stretched, so I have the interval between 0 and 1 and what we now do is the following.

We will imagine stretching this rubber band, such that the end point 0 moves to something to the left of it, so you know you sort of take it and stretch it. So, let say 0 moves to some point to the left of 0. So, let give it a name, so let say 0 moves to a point l, the left end point and 1, the right end point moves to some real number r. So, let say l and r, now denote the new. So, this l and r now denote the new positions of the end points, so that is what the rubber band now looks like.

So, the orange denotes, what it look like initially and after the stretch, you got the yellow rubber band. Now of course, the stretch itself is assumed to get a continuous process, so you just sort of stretching it the whole way. And now here is the interesting statement, it says, there exists point on the original rubber band which does not move during this stretch. So, there is so called fixed point which does not suffer any displacement during this transformation.

So, let us write it out, there exists, so here is the statement which we will try and prove there exists a point. Let us call it x naught on the original rubber band, which means x naught does a point between 0 and 1, which does not move from it is place during this stretch process. So, here is one way of stretching it, here is there exists a point on the rubber band, which does not move during the stretching process.

So, let us try and prove this using the intermediate value again. So, this depend on how you think of it, it might sounds somewhat counter imputed that and also the key thing here is no matter of how this stretching is done. So, I do not assume anything about l and r, you can imagine stretching this to any length for instants l and r can be, l could be arbitrarily small, r could be arbitrarily large and no matter, how this stretch is performed, how long the final rubber band is and so on.

This statement is independent of all that, the moment what you do, there exists a point which will never move during the stretching process. So, let us try and prove this, it is a nice application, it gives us an idea, what the intermediate value theorem really does. So, of course, in order to connect this up with the intermediate value theorem, you need a function.

So, let us do the following, let us define a function f from the interval 0 1 to the real numbers and what is my definition, so 0 1 remember is like your rubber band. So, I will define the function as follows. For each point x on the rubber band, let f of x denote the displacement of x. So, what is displacement of the point x, which by definition means the new position of x minus the old position of x.

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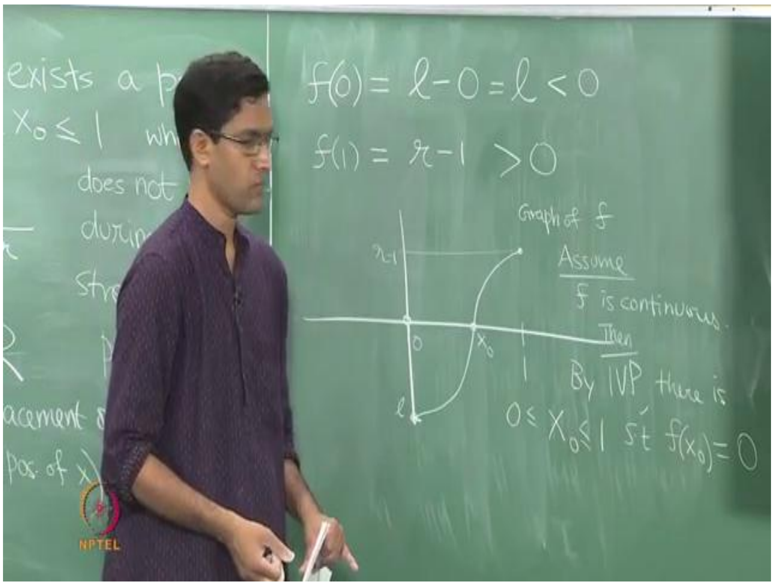

So, my function f is just the displacement function and let us just figure out, what this does to the two end points. So, the left end point 0 is now displaced by what l, the new position is l, the old position of those was 0. So, this is l, the right end point 1 is now displaced to the, well it is now at the point r, it was originally at 1, so this is r minus 1. So, here are the two values of displacement, the left end point is displaced by l, the right end point is displaced by r minus 1.

And observe that l is a negative number, because l of course is to the left of 0, r minus 1, since r is to the right of 1, r minus 1 is a strictly positive number. I do not care, what they are, all I need to know is that l is negative and r minus 1 is positive. So, let us imagine drawing the graph of this function f, so now, I have 0 1. So, you can see an important thing, the function f here is not really defined on all real numbers, but only on the interval 0 and 1.

But, the intermediate value property continuous to whole, like just while I stated it, I sort of set between a function from R to R, but it is enough for it to be a function from the interval 0, you know a b to R. So, let just see, what the graph should look like, at 0 so I am going to draw the graph of the function. So, at 0 it is the value l which is negative, at 1, it takes the value r minus 1, whatever that be, it is just some positive value.

So, here are the two end points from l to r minus 1 and what I know is that, there is some graph, I do not quite know what the graph look like also. Because, a graph would just tell me what the displacement is of individual points on the rubber band and the actual shape of the graph will depend on how the stretching was done. You might stretch the left half of the rubber band, a little more than the right half and so on and so forth.

So, depending on what exactly was done to the rubber band, the shape of this graph would change, I could you know instead of going up like this, it might have gone a lot more of initially and down later and so on. So, and again I do not really care as I said, it is independent of how the stretch was performed. The key property now is the following, the only thing we need is that the stretch is continuous. So, we are going to assume and there is reasonable assumption to make that f is continuous, just says that you deformate in a continuous function.

Now, the fact that at 0 it took a negative value and at 1 it took an positive value means that by the intermediate value property, there exists some inter meaning point at which it takes the value to 0, like this graph must cut the x axis; that is basically all it means. So, there exists a point x naught at which so then since f is continuous by the intermediate value property, there is a value of x naught between 0 and 1. Satisfy such that the value of f at x naught is 0. So, 0 is the value sees that we are now chosen, it is the value between that two end paths. So, there is a point at which f takes value 0.

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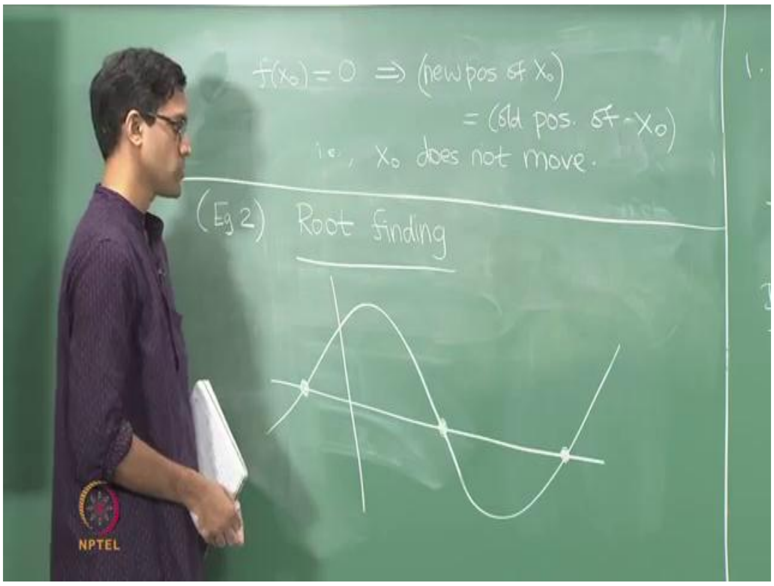

So, what is that mean that of course, means that the displacement of x naught is 0, so this means that, so what is f of x naught 0 means that, it suffers 0 displacements. In other words, it is new position is the same as old position, new position of x naught is the same as old position, in other words, x naught does not move that is exactly what we wanted to prove, so at e x naught does not move.

So, this is often called fixed point theorem, it says that there is a point which is fixed under some transformation. So, use the first example, it terms out to just be an application of the intermediate value property. Now, here is example 2 and this is sometimes, what is called a process of finding roots, so root finding. Now, of course, it is a special case of root finding, so what is a root mean, so suppose I have a function f, a root of course is a value of x at which the function takes the value 0, it is a point at which it cuts the x axis.

So, here for instants is the root of this function and so the way I drew it here, there are three roots three places, where it cuts the x axis. And often given a function, somewhat complicated function for instants, it may not actually be possible to explicitly solve for the roots, it solving for roots for instants is easy if you are doing quadratic equations and so on, but more generally finding a root exactly may not really be feasible.

So, what one wants at least is some sort of approximate root, we want some sorts of approximations to the roots and here is an algorithm for finding roots, which more as uses the intermediate value property.

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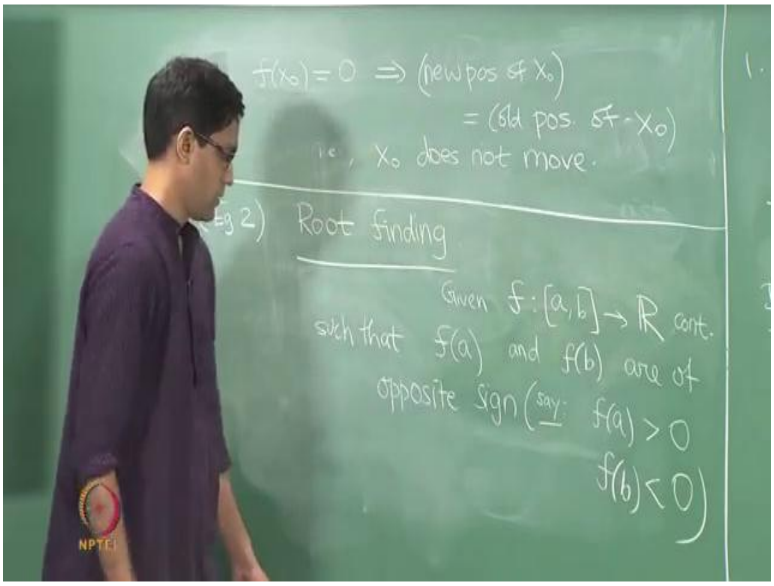

So, here is an assumption, suppose I given as a function f, a continuous function f, so let see what our assumptions are. Given f, let say it is not even defined on all of the l numbers, suppose I just knew that, it was define the interval f from a b to R, f continuous and with an additional properties such that at the end points f of a and f of b are of opposite sign. So, such that f of a and f of b are of opposite signs.

In other words, let say it is positive on the left hand point and negative at the right hand point, I can always assume, say let us just make a following assumption, let say that f of a is positive, f of b is negative. If this is the case, then observe firstly by the intermediate value property, we know that there must be a value of x between a and b at which the function takes the value 0.

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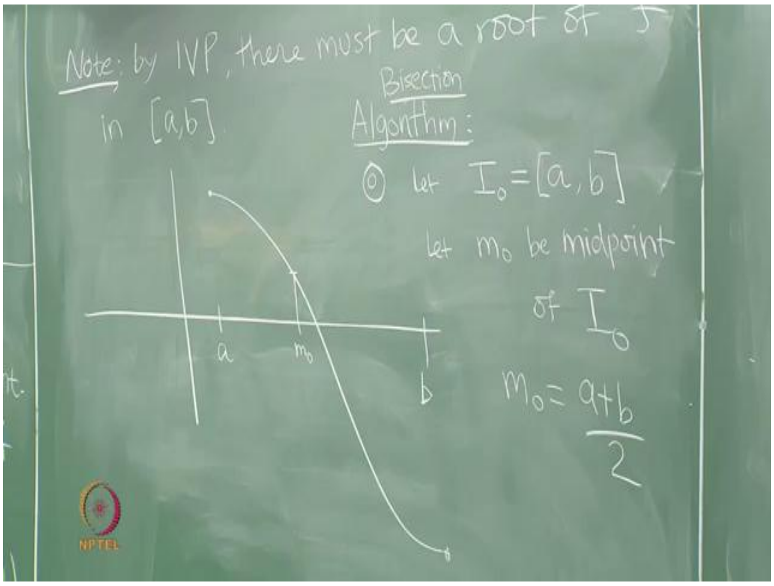

And observe this model as like what we en counted in a first example, note by intermediate value property, there must be a root of f between a and b, there must be a root of the function f in the interval a b. Now, the question is, how do you find this root to some given degree of accuracy, we just want to figure out a way of determining this root. So, here is the idea, there must be a root between a and b and what we assume f of a is positive and let say f of b is negative.

So, this is say how the graph may took and we are trying to find that point. So, here is what we do, let us make a sequence of approximation, let us make sequence of guesses. So, let us define, let us take the midpoint of the original interval, so here is an algorithm, so sometimes called the bisection algorithm to find the root. So, what it says is the following. So, let us take the original interval, let us call the original interval as I 0 as we now construct a sequence of new intervals.

So, let I 0 denote the original interval, now let us do the following, let us take the midpoint of I 0. So, let it is call it m 0 be the midpoint of I 0, what is that mean m 0, therefore, is that just a plus b by 2, so as the average of the end points. So, you have the original interval and you take the midpoint, let say this is what the midpoint is m 0. Now, you compute the value of the function of the midpoint, the function value in this example at the midpoint is positive.

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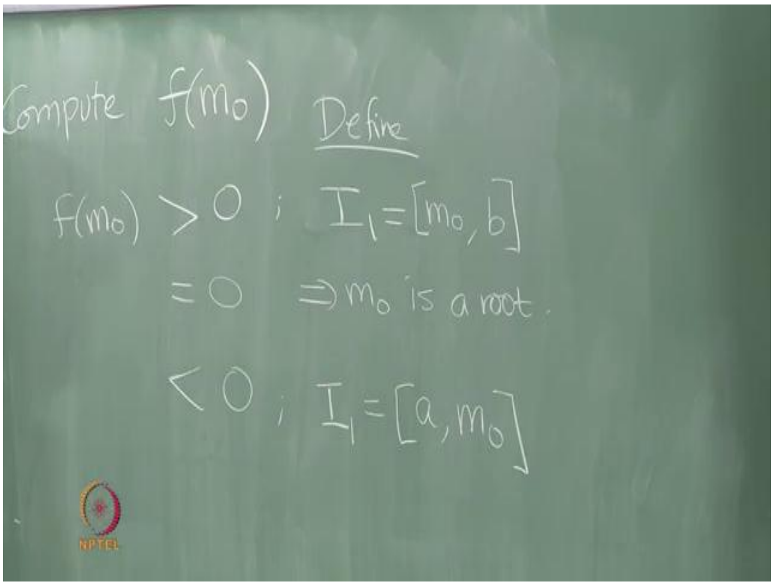

So, we compute, so here is it is call it step 0, if you wish of the algorithm. Now, what we do, we do we compute the value of f at a midpoint. So, the function f itself is presumably given in some nice way, there is some formula value. So, we compute f of x 0, now what are the possibilities f of m naught could be either positive negative or 0, f of m naught could be positive, it could be 0 or it could be negative, so there are three possibilities in general. So, of course in this example that drew, I assume, I drew it in such a way that in point value is possible.

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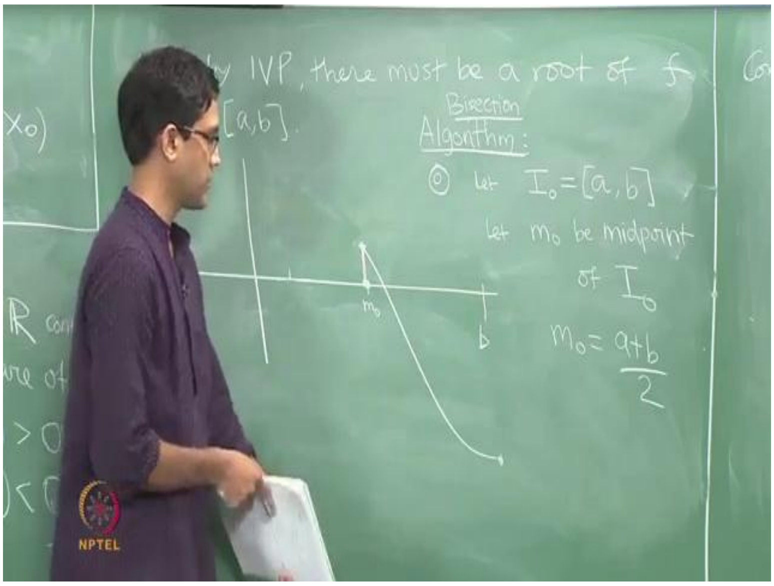

Now, what do we do in each of these cases, if f of m is positive, so let us look at this one here, the value of f and m naught is positive, what is that mean, it means the following, let us get rid of the function in the first half. So, now, just focus attention on the portion of the function between m naught and b. Now, here again, we have f of m naught, the left hand point value positive, the right hand value is negative.

So, it short of get the original function by intermediate value property that must be a root of the function between m naught and b, so we could conclude the same thing again. So, here is the thing to do, if m of m naught is positive, let us do the following, you define a new interval, so you get rid of half of the interval, you define the new interval. So, it is called the, new interval is called I 1 now to be the interval between m naught and b.

Similarly, the value of f at m naught is negative, then you take the other half of the interval, you take the interval between a and m naught, where again you have the same feature that the value at the left hand point is positive, the value of right hand point is negative. And of course, if the value of f at m naught is 0, you have already done, you already find a root here in fact you have find it exactly. So, this of course, means that m naught is a root.

So, you have found the root and you can just stop the algorithm right there, but of course, in practice you almost always end of the either the first of the last cases. And in those cases, you define I 1 to the half of the interval here, you take the right half and here you take the left half.

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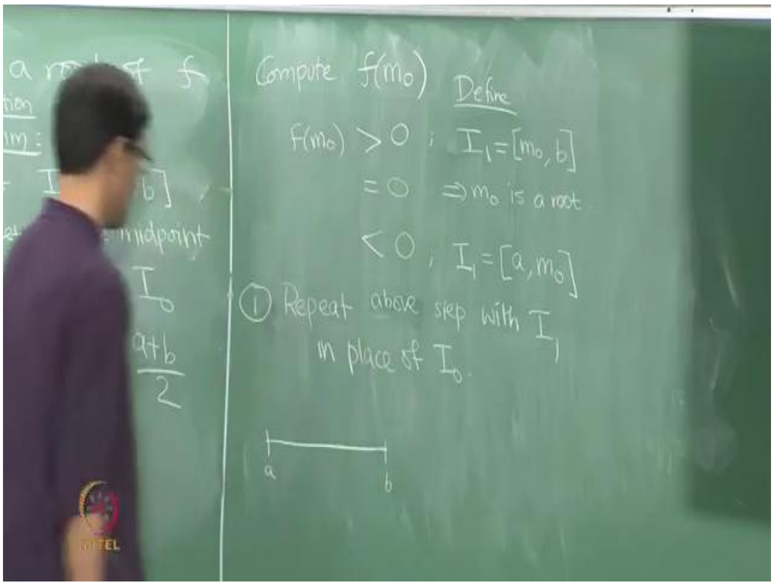

And now, you just repeat the process, so step 0 is basically the prototype. Now, you repeat the same thing with I 1 is instead, repeat same step, repeat about step with interval

I 1 in place of I naught. What is that mean, let just think of it once, so I had I naught between a I have I 1, so maybe I just do it in this figure right here.

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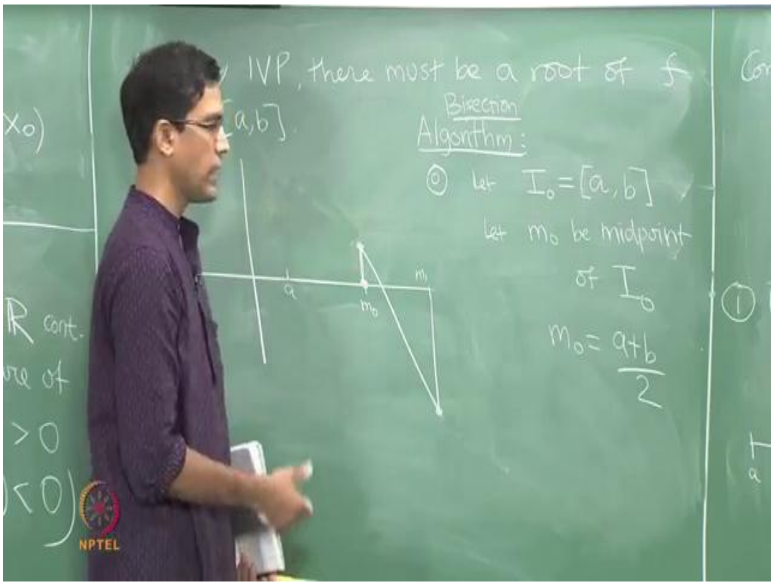

So, I had I naught to start with and I compute at the value f at the midpoint and took only the right half of the interval as I 1. So, is my interval I 1 and now, I do a same thing, what is it mean, I take the midpoint of this interval. So, the interval I 1, what is it midpoint, it something called m 1 this case, what is m 1, it is m naught plus b divided by 2. So, I have some midpoint, I compute the value of function at the midpoint and I see, whether the function value is negative or positive.

So, in this case, the function value turns out to be negative, the way I through the graph. So, again that means, that I should do the following, I get rid of the half of this interval and only focus attention on the left half, the part between m naught and m 1. Call this I 2 and now again by the intermediate value property, there must be a root between you know the left hand point m naught and right hand point m o and so on and so forth. Now, when you keep doing this algorithm for large number of steps, what you end of with this following that the interval that you are left with at each step is only half the size of the preceding interval.

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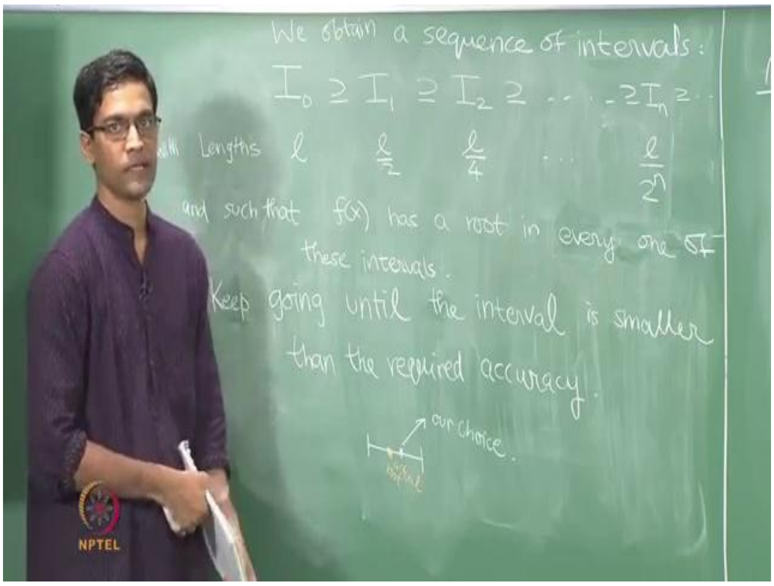

So, observe the following facts about, so what we really have is a following the, we obtain a sequence of intervals, so we obtain a following sequence of intervals. So, I naught, so this contains the next interval I 1, which contains the next interval I 2 and so on and the length at each step becomes half. So, if the original length, so let say here are the length of this intervals. Suppose, I naught had length l, which is b minus a, I 1 has l by 2 the next guy has length l by 4 and so on.

So, if I take the do this for n steps, I get an interval whose size is just l divided by 2 power n. So, the intervals are getting smaller and smaller and smaller and what you conclude is the following at every step. So, what you get, we get an obtain sequence of intervals with these lengths and what is the key property and such that, f of x has a root in I n for every n as a root in I n for all values of n that you have you obtain.

So, it is you keep doing this, so has a root in every interval, so it is write it like them every one of these interval. So, this is of course, assuming that you did not get find a root. So, observe if you evaluated f on some midpoint and you actually got the value 0, then that is already you have done, you found the root exactly and you going to root exactly keep doing this algorithm any more.

So, assuming you do not succeeding in finding a root at any given step, you can keep going on, on, on, on. For let say, you keep going for 100 steps, then here is what you know at the 100 steps, the interval that you obtain the I n or I 100 contains the root of the function f. Because, that is exactly the property using which you define the next interval.

So, if say you are goal was define a root of the given function f to within an accuracy of let say in a point 0 0 1 or point 0 0 0 0 1 or whatever be the given accuracy. All you must do you just keep doing this process until the interval sizes becomes smaller than the required accuracy. So, the general algorithm now says is keep going, until the size of the interval, until the interval obtain becomes smaller is smaller than the required accuracy, whatever be the given accuracy.

And once you reach that step, you can just take you are approximate root to be any point inside the interval higher, I mean you could take the midpoint for instance or any point in interval I n will still do. Because, if the interval I n has you know length which is smaller than you know whatever be the accuracy point 0 0 0 1 and so on, their surly exist root somewhere in this interval that much we know. We do not know where the root is, let say this is where the exact root is.

So, this is the actual root, but let say we pick the midpoint as our choice of approximate root. So, suppose this is what we pick, so this is our choice of approximate root and here is the actual root, the difference between these two is at most this size of this interval. So, because they both lying in this interval the difference between the actual and our approximation is at most the length of the interval. But, since the interval this already smaller than the required accuracy, we have done, we do not need to worry any more. We are obtained the root within the required accuracy. So, this algorithm is often called the bisection method and it can be a rather powerful way of finding an approximate root of a function f.

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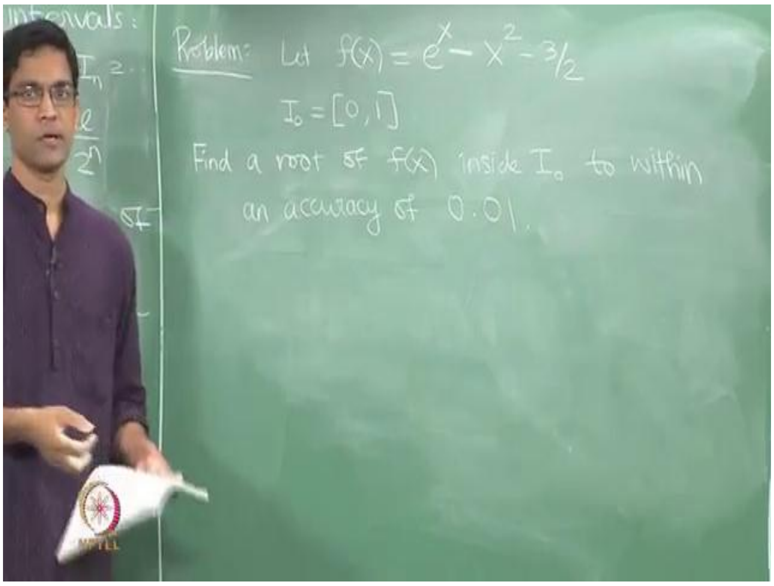

Sort of try out get a sense of how this works, so here is the problem like you try out on your own to get a sense of how this works. You define a function, let say I take the function f of x is, so I am going to use a function e power x minus x squared minus 3 halves are I am going to take by interval I not to be just the interval 0 1. So, here is the problem, find the root of this function, find the root of f of x inside this interval to within an accuracy of 1 over 100 to within and use the bisection method to try and do this. So, I leave this an exercise. So, here is this was the second application of the intermediate value property.

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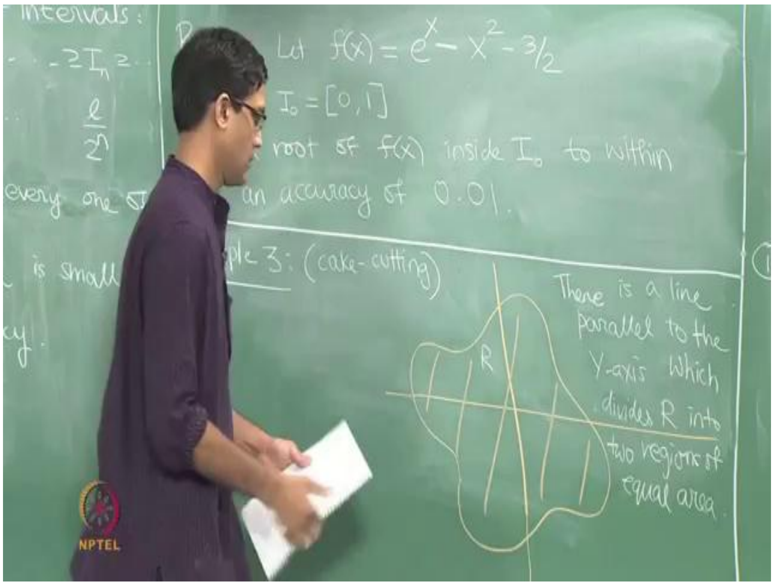

Now, let us look at more geometrical application, so something to do with how regions in the plane can be divided and so on. So, here is an example which is sometimes goes by the name of cake cutting problems. So, set up is a following, you imagine you have shape region on the plane of some arbitrary shapes. So, I am naught really imposing any condition, except that, let say the boundary some nice close curve, it is a bounded region.

So, imagine you have this regional on the plane. So, this is an arbitrary shape k if you wish and the goal is following, what you want to do is do, cut this in to 2 equal halves by using a line that is let say by a vertical line parallel to the y axis. So, here is the statement itself it says, there exist line parallel to the y axis, which cuts this cake in to two equal halves.

So, cake cutting problem says, well here is the assertion that will try and prove, there is a line parallel to the y axis, which cuts this region. Let us call this regions R which cuts which divides region in to two equals halves or which bisects which divides R in to two equal regions a two regions of equal area. Let us try and proof this statement, see why it is a really an instance of the intermediate value property.

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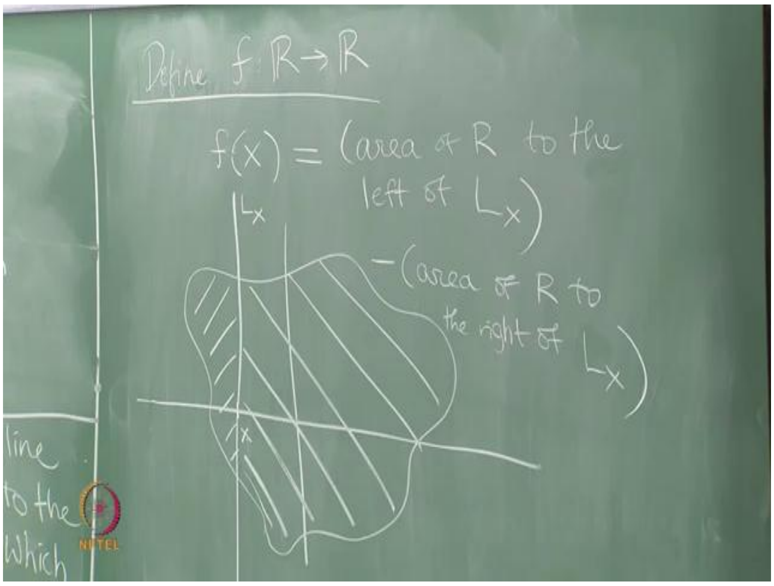

So, in order to use intermediate value property, what one real needs do is to define a function as usual like. So, let us do the following, let us define a function, so let me define a function f from let say R to R. So, what should you do well, so imagine we have, so what should f do, for each real number x in R, let us do the following. So, let us again draw this figure. So, I have this now for each choice of x, I will do the following, I will draw vertical line through passing through the point x comma 0. So, let us say this is a value of x, now what you do is, if you do this following, you draw the vertical line here.

So, let us call this vertical line as l, l through x comma 0; let us call it l x. So, imagine you draw line like that and you define the function as follows that line well presumably divide this given reason are in some arbitrary proposal. For instance, line could be even all the way to left of this region, which case, it really does not do any division at all. But, let us do the following, let say f of x is the following, it is the area to the left of this line, that is the area minus the area to the right of this line.

Let us keep track of what proposal, it divide a given region are into or the difference in the two pieces. So, f of x is just the area of R as a region R that lies to the left of this line minus the area that raise to the right of this line, so area of the region R to the right of this line l x. So, having defined the function, what we now need to do is to just check that, so first we observe this function is continuous, because as you m o move the value of x, as you change the value of x the line of course moves.

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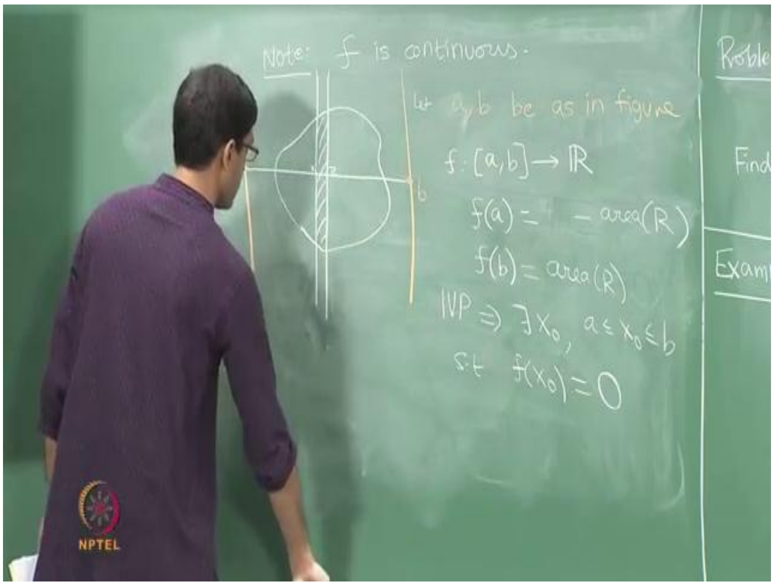

So, observe firstly, so here is the key point to keep a mind this function that we have define is in fact continuous, why is that, because what happens as you have change the values of x, so I have some value of x, so here is the line l x. And let say, I change x slightly, so I have another line, so let us call this line here l x and the line to the right of x. So, let us call this value of x and let us call this may be as a z. So, I have a line L z, x and z are very near each other.

Now, what happens is the area that is to the left of x minus to the area to the right of the x minus the area to the right of x; that is the function we have defining and similarly, the area to the left of z minus the area to the right of z. Now, observe that the area that really sand which in between these two lines; that is really the difference that you going to get between the function f of x and the functional value of x and z. And this area is really going to become very, very small as you let these two lines come very close each other.

So, I am just sort of giving you some kind of heuristic for why this function is continuous, but what we really want to do now is to figure out what the intermediate value property says. So, imagine if x is some number to the left of this region, so let us do the following take l, so let l and r be the following, l is a value of x or may be just called a and b. So, let a, the some x value that much to the left of this region R, let b denote x value which is much to the right of this region R.

So, let a b, b any numbers as in figure to be as in figure and we will just a focus on the interval between a and b; that is all we really need to worry about. So, let us look at the function f as define before, the only on the interval a b. So, observe that, if I take the value of f at a, what will give me is area to the left of a minus the area to the right of a, there is no area to the left of a, the entire area lies to a right of a.

So, this is just going to give me 0, which is to the left minus the area to the right, which is the full area of the region R, so that is for a. Now, for the right n point b, here the area to the left is the full area and area to the right is the 0. So, f of a and f of b are in fact of opposite signs, one of them is area of R and other is minus area of R, one of them strictly negative, the other is strictly positive. So, what is this mean by the intermediate value property, there must be some value between a and b, there exists the value x naught mean a and b.

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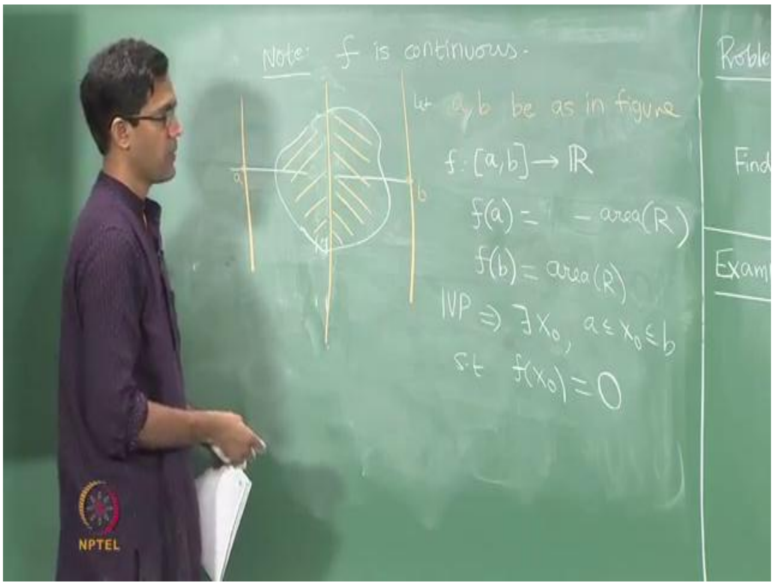

Such that, f at x naught is 0 and observe that, the value at x naught been 0 precisely means that, so let us draw the line through x naught. So, f of x naught is 0, just means that the area to the left of x naught is equal to area to the right of the x naught; that is exactly how the function was define. So, this is the exactly the bisector that we are looking for something which gives you two pieces of equal area.

So, there are many, many more such geometrical applications and maybe I will indicate some of these and some of the feature like just. But, for now these already give as a good sense of how to the intermediate value property can be use to prove some rather interesting and may be at first non trivial facts about geometrical configuration is transformation and so on.