An Invitation to Mathematics
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Unit
Functions
Lecture - 20
Functions on the real line, continuity

Welcome back, today we will start talking about Functions.

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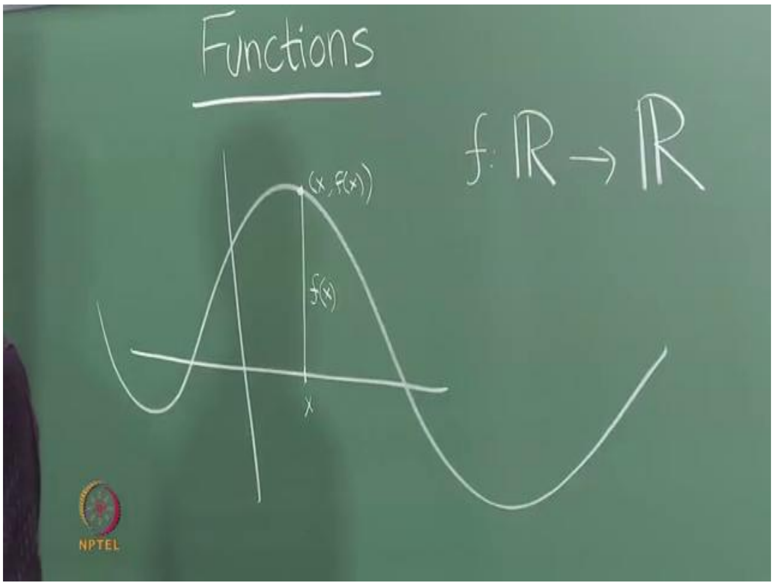

So, the most familiar examples of functions are, what we will call functions from set of real numbers to the set of real numbers and the most familiar pictorial representation of course, is as a graph. So, this is how we mostly think of functions and what this really means of course is for each value of x, there is the function value f of x is what is denoted along the y axis.

So, the point here is just x comma f of x and we just join all these points x comma f of x as x takes all possible real values. So, recall that the formal definition of the function requires two sets, there is a domain and a co domain.

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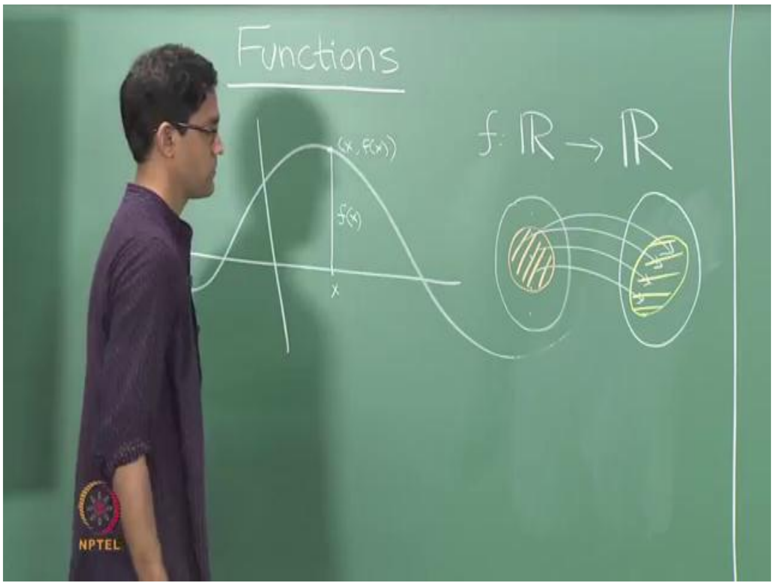

So, in this example both domain and co domain are just set of real numbers and what we really want to say is, well here are the various points of the domain, the function is thought of as mapping each point of the domain to some point of the co domain. So, now, of course, the natural question we might ask is, well what is the function do two subsets of the domain?

So, for instance, you pick out some chosen subset of the domain and ask, what do all the points in this subset map to under the function. So, you just collect together the images of points which come from the chosen subset and put them together and ask, what set is this. So, this is often a very natural and an instructive way of looking at functions, beyond just what it does to points. You also sort of want to think of it in terms of what it does to regions or what it does to various subsets of the domain. So, let us try and do this for the familiar function from the real numbers to the real numbers.

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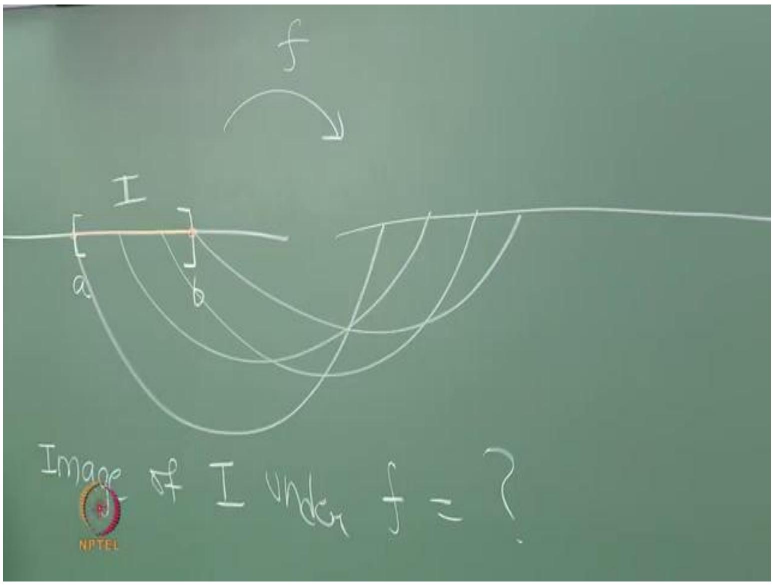

So, the question here we will ask is, what does function typically do to a set of the following form to an interval. So, what is this mean? What I want to do is the following, I want to take an interval, closed interval a, b. This just means the set of all real numbers which lie between a and b, including the two end points both a and b. So, here we do include a and b and now the question is, what are the images of the points in this interval under the function.

So, now, we will think of the function as being a map from the real numbers to real numbers and ask, well you know collect together, what it does to the various points of this interval and ask, what sort of set does one get on the right hand side. So, if we call this interval as I. So, the question here is really is the following, what is the image of I under f. So, let us try and answer this, more or less from the graph of the function.

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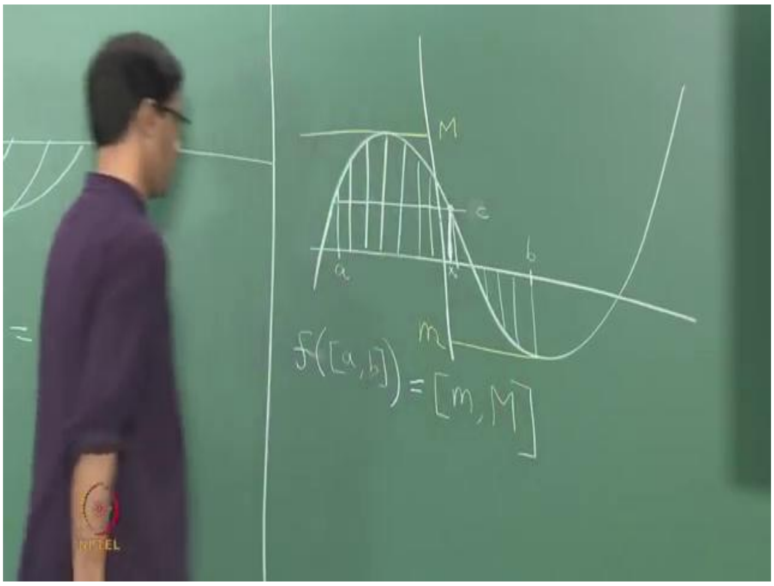

So, what we will do is, so imagine there is some graph function f and we just pick some interval, pretty much any interval will do. So, let us take for instance the interval between a and b, this is a and that is b. So, what value is does the function take when x ranges between a and b, so we will just write down all the y values. So, pictorially the lengths of these vertical lines indicate the values of f as x various between a and b.

So, we ask well, what are the various values that the function takes? So, here is what it seems to be from this graph, there it seems to be a maximum value. So, there is this largest vertical line. So, this maximum value let us call this maximum y value as M, there is a minimum y value, let us call it small m and observe that between capital M and small m, any choice of y value is always taken by f.

So, you pick any y value between capital M and small m, let us say this number, let us call it y or maybe you call it c and we ask does the function take the value c and we notice, yes it does. So, here is our choice of x that which it takes the value c and in fact, there is another choice of x, let see. So, here could be a choice of x, but this value of x here does not lie within this interval. So, we do not worry about this one.

So, the only there seems to be just a single value of x, this guy here, let us call it x naught at which the function takes the value c. So, in general there could be more than one value of x. But, the key point here is the following, observe that from this graph at least it seems that, if I take the function f and I take the image of the interval I, so here is how one writes this. So, you take the interval between a and b and when we write f of the interval a, b is just means the image of the interval under the map f.

So, in this example at least it seems to be the following, it is all values between small m and capital M. So, at least in this example this seems to be true. So, you just figure out the maximum, you figure out what the minimum is and every value between them is taken by f and that is exactly the image of this interval. Now, what is the key property of the function f, which makes this true? So, why is this really true? So, here is an example of a function for which this may not be true, for the graph that I drew, it happens to be correct.

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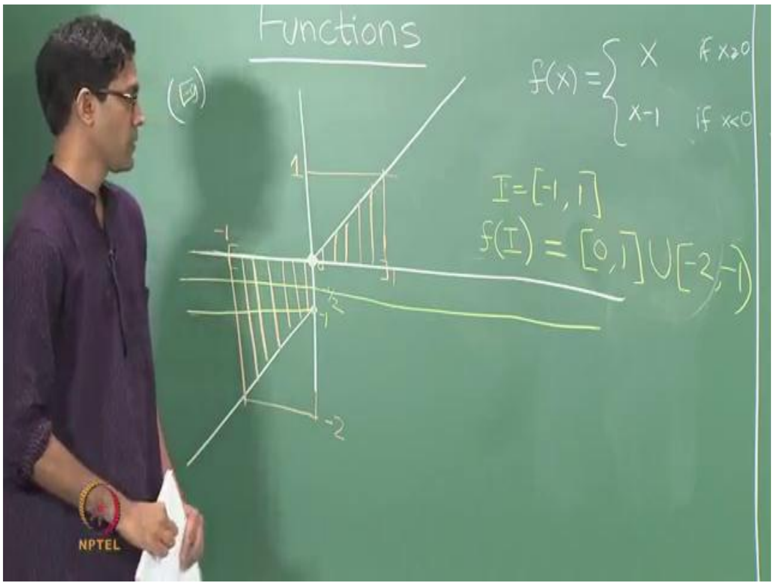

Now, let us do the following, let us take another example. So, write out the formula for the function f of x equals, where it is define in two pieces, it is the value x, if x is positive and it is the value x minus 1, if x is negative. So, it is a piece y define function and if you draw the graph of this function, here is what from fines, it is just the graph of the line y equals x. So, this is the graph for the function, when x is positive and when x is negative, it is more or less the same line just shifted down by 1.

Here, is the graph and the key point observe is that this point here, where the bottom line meets the y axis is not on the graph. So, we often denoted by a circle like that and this is sometimes shaded in, to indicate that this point is on this graph. So, what we now have here is the graph this function f. And now, we could ask the same question, if I take an interval, so let say take the interval between minus 1 and plus 1.

So, just consider this closed interval here and ask, well what are all the y values, what are all the values in the function, when x ranges over this interval. So, we do the same thing, we draw all the, so here are the various y values that you get and when x sort of ranges in the positive side at x equal to 0, you get this point here and then, you get these vertical lines. So, these are the various y values that you obtained and now, again observe that the maximum value is a 1; the minimum value is in fact a minus 2.

So, those are the maximum and minimum values, but here is a key difference as compared to the previous case, the values of y between 0 and minus 1. So, these y values, so you take any y value between 0 and minus 1, those y values are never obtained by this function in this interval. So, for instance if I take y equals minus half and ask, well what value of x would possibly give me a value of minus half.

So, I just draw this horizontal line and ask, does this horizontal line meet the graph of this function at all and I find that of course, it does not. So, here is a function for which the, what is the… So, if I take the interval I in this case to be interval between minus 1 and 1 and ask, what is the image of f, what is the image of I under f, the answer seems to be the following. I get, well of course the y values here are, everything between 0 and 1 certainly occurs, so it is the set of all x.

So, here is one way of writing this, it is all values of x between 0 and 1, those are certainly there and if you look for what happens when x is negative, you get all values between minus 2 and minus 1. So, we say that, this interval union all values between minus 2 and minus 1, but notice minus 1 is written as an open interval. So, the value minus 1 itself is not obtained. So, here is the image of this interval under this function f, so this turns out not to be an interval, this is sort of a union two different pieces.

In the earlier example, the image of an interval just turned out to be an interval itself, it was just the interval of all numbers between the minimum and the maximum. So, in other words, there are no breaks in between. So, somehow that is the key intuition to keep in mind.

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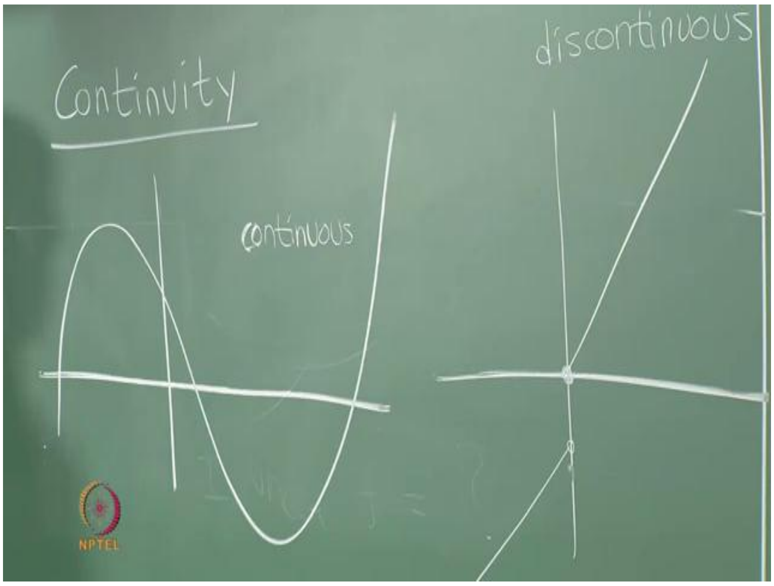

So, why was this property true for the first function and false for the second one? The key thing which made it work is, what is called continuity, the fact that the graph of the function did not have any breaks. So, in formally continuity just means that when you draw the graph of the function f, so like the very first graph we drew something like this, the graph was just drawn continuously; there is no break in the graph itself. So, this is an example of a continuous function.

As suppose to the second graph which look like this and this is an example of a discontinuous function. And in fact, we often say that the discontinuity is at the value of x equal 0 that sort of where the break occurs, here there are no breaks at any value of x. So, the key property of continuous functions, so continuous functions are sort of very, very important, they have a lot of very nice properties.

And one of them, we have just seen which is that, if I take an interval of values of x, the image of that interval under this function f will again be an interval, which will be all y values lying between the minimum and the maximum. So, here is in fact the statement that we have for a continuous function, so let us write this out again.

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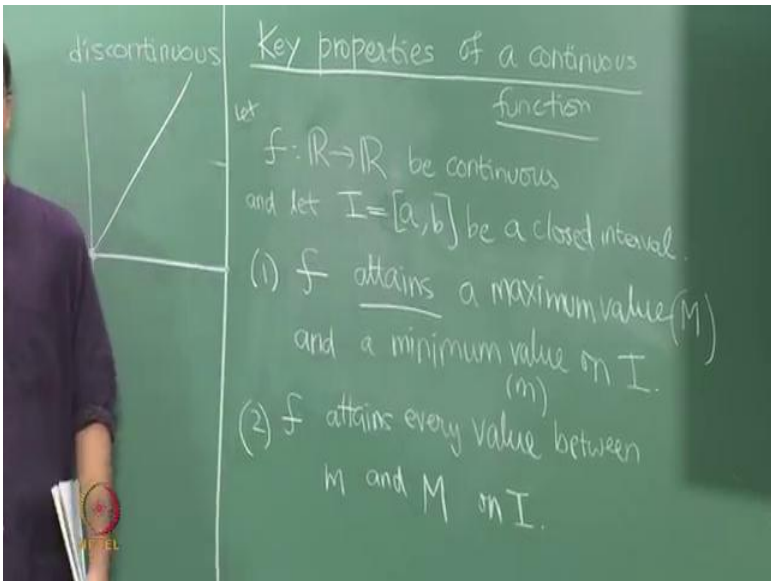

So, here are key properties of the continuous function. So, let f be a continuous function. So, let f from R to R be continuous and let I be a closed interval. So, let us say it is all. Now, there are two important properties, number 1; that f attains the first property with somehow looks rather obvious, but there are example, easy examples, where we can see it may fail.

So, the first thing says in fact, there is value of x at which f attains it is maximum and a value of x at which f attains it is minimum value on I. So, we do have a maximum value and a minimum value; let us call these M and small m. So, that is fact number 1 and fact number 2 is the think that we just said that, it attains every value in between. So, let us also say f attains every value between m and M on the interval I. So, there is a maximum value which it attains, there is a minimum value which it attains and in fact, it attains every a single possible value between smaller m and capital M.

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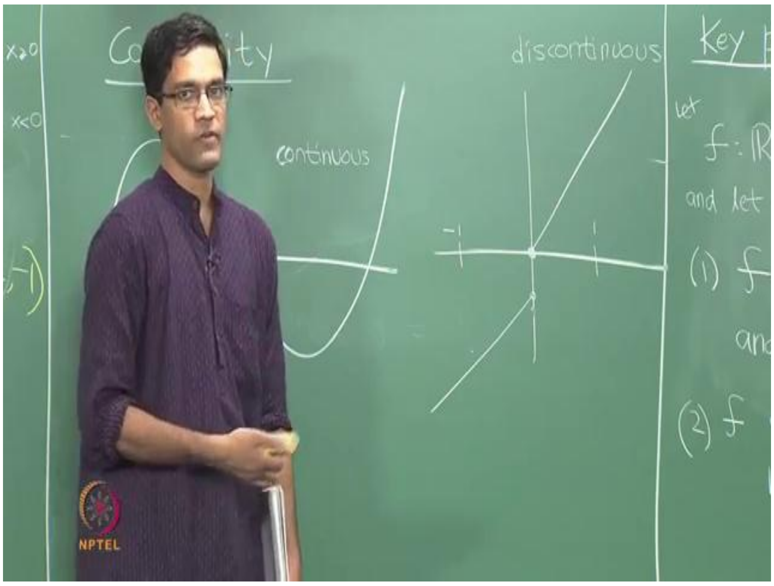

And both these statements are very easily false in examples of discontinuous functions. So, we have already seen an example, where the maximum and the minimum value are attain, so here is the function m, small m and capital M, where both attained. When, you take the interval to be the interval between minus 1 and 1, but values between, every value between the minimum and maximum was not attain. There were values for instance between 0 and minus 1, which were not attain by the function.

So, that is an example, where f is discontinuous and the second property fails. Now, let us also look at an example, where the maximum and minimum are in fact not attain. So, in other words, they are really is no maximum or minimum.

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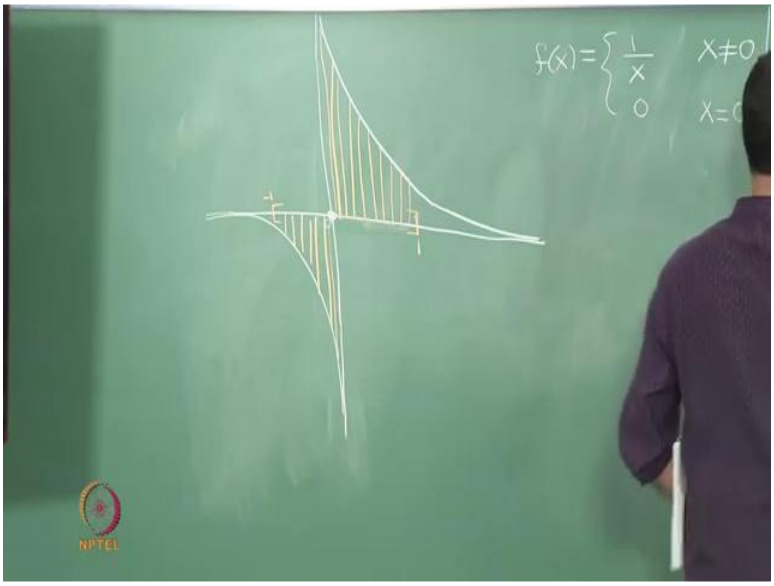

So, here is one of the standard examples. So, we could take the interval between 0 and 1, let us define the function first. So, let us take the function f of x to be just the function 1 over x, but of course, in order for this to make to sense, I cannot allow x to be 0 and so, x equal to 0, I redefine it as 0. So, here again is a piece wise defined function, if x is positive, what you get is the graph of 1 over x, which looks like that. If x is negative, again it is this graph; the same thing 1 over x and 0 alone is something like that.

Now, let us do the following. So, at 0 the value is 0. So, again let us try and take an interval. So, let us take an interval, let say between minus 1 and 1. So, I will just take the same thing is before. So, let us look at the interval between minus 1 and 1 and now, we ask, what are the values of f in this interval and we observe the following, if you look at the y values, then the keep getting larger and larger and larger and larger. So, they go off to infinity is how we often say this.

Similarly, the y values for negative values of x go down to minus infinity. So, now, if you ask, well what is maximum value of f on this interval between minus 1 and 1. So, at 0 itself, it is a 0, so there is no problem really at 0. But, if you ask what is the maximum value, well we observe the really is no maximum, in some sense, it is like infinity the value of f is larger than any number, you can write down.

Similarly, there is no minimum, because the value of f becomes more and more negative, it is sort of tends towards minus infinity and so, well the really is no maximum or minimum in this example. So, and notice again with this is an example for discontinuous graph, because clearly there is a break at 0. So, we draw the graph really in three pieces, there is the negative 1 by x piece, the positive 1 by x piece and the piece is at 0.

So, this again is an example of a discontinuous graph and one in which, there is an either or maximum nor a minimum on the interval between 0 and 1. So, having a nice continuous function will give you both these properties that you will actually get a nice finite maximum value or minimum value and every value in between will always be attained.

Now, this second property here the every value in between is attain is sort of like the intuitive notion of continuity, it is a formulation of something very intuitive about continuity. That when you draw graph in a continuous fashion, you really must pass through all intermediate values.

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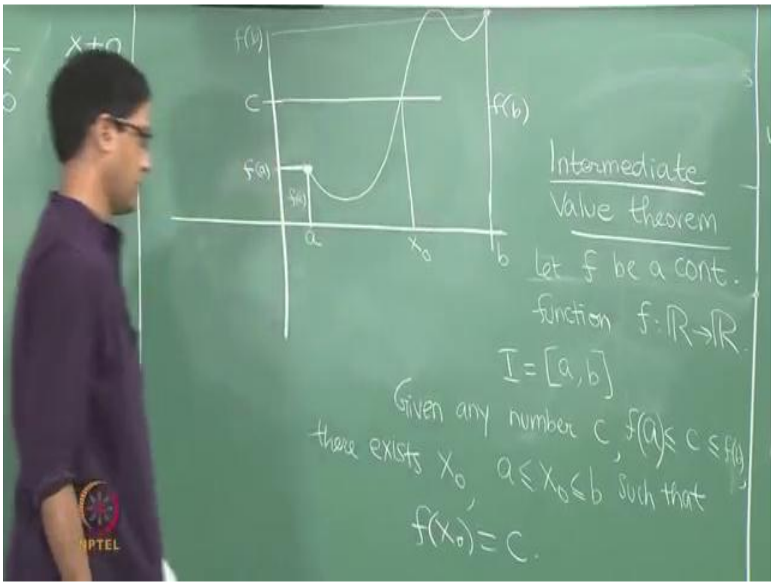

So, here is another way of stating this. So, if I have the interval between and b. So, there is a way of stating the same thing without bringing in the maximum and minimum values. So, here is the statement, suppose I have a graph of a continuous function and I want to see, what it looks like between a and b for instance. So, at a, I have some value. So, let say this is the value f of a, set the y value at a and at b, I have again y value. So, may be just to make it a little clearer, let us just think of both values as been positive.

So, I have let say f of a, let us y value at a, f of b is the y value at b, it is to intuitive notation of continuality, you try and draw the graph between the point a comma f of a and the point b comma f of b. Now, when you do this, you pretty much have to pass through every inter meaning y value between f of a and f of b. So, these are the two boundary y values and it is sort of clear intuitively, there no matter how you draw the graph, you are going to have to pass through every value of y in between these two boundary values.

So, this is sort of another reformulation of the property that we just said that f in fact attains every values between f of a and f of b on this interval a b. So, this formulation is sometimes called the intermediate value property or the intermediate value theorem. So, let me just formulate this. So, it is call the intermediate value theorem says that, let f be a continuous function, let say from the real numbers to real numbers, you in some sense you really only need it on the interval a b, but for now let us not very too much out of this.

So, let f b a continuous function from the real numbers to the real numbers and let us just restrict attention, it is call I to be the interval a, b. Then, given any value between a and given any number c, any number real number c, c lie, it is a number between a and b. What can one do? So, let us take a number c between c is between f of a and f of b, I should be a f of b, given a real number c between f of a and f of b.

So, it is a y value between the two boundary y values, what I can conclude is there surely exists a point on the graph. So, let us call it x naught, there is an x value at which the function takes the value c. So, given any number c between f a and f b, there exists an x value solve, solve it real number x naught, x naught between a and b, such that f at x not is exactly equal to c. This theorem here is call the intermediate value theorem, the proof uses some very important properties of real number and so on.

So, we want get in to the formal proof itself, but this much should be clear that, it is a rather intuitive property of continuous functions, somehow, capturing the feeling that you sort of do not really lift your pen of the paper. When, you draw the graph from two end points, if well going from one end point to other. And this somewhat simple looking property has lots a very interesting application and we will look at least a few of these applications next time.