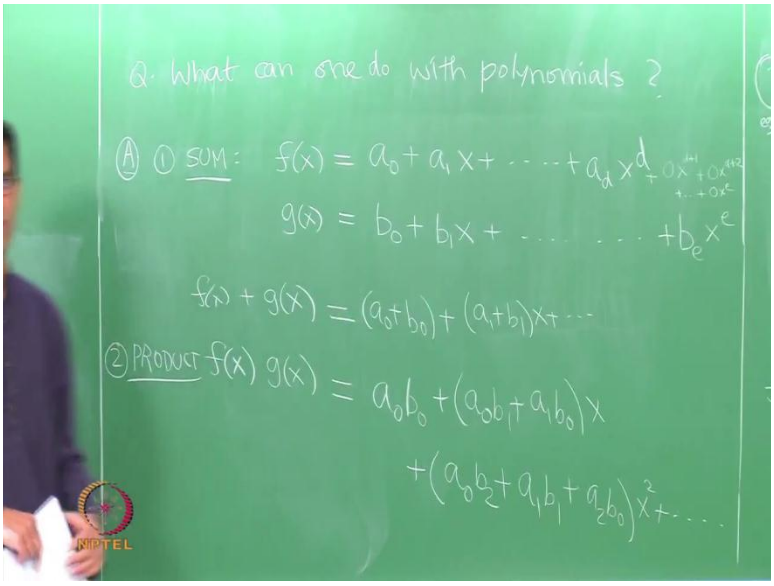
An Invitation to Mathematics
Prof. Sankaran Vishwanath
Institute of Mathematical Science, Chennai

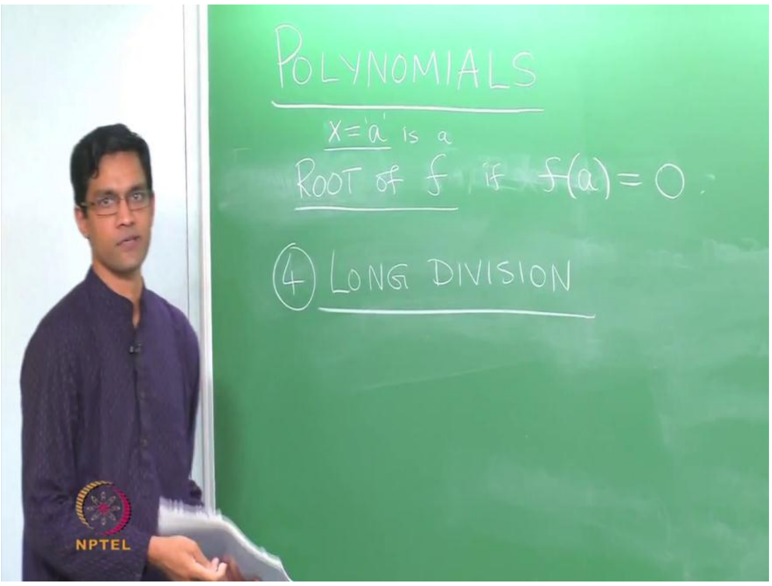
Unit - I
Polynomials
Lecture – 1B
Long Division

(Refer Slide Time: 00:19)



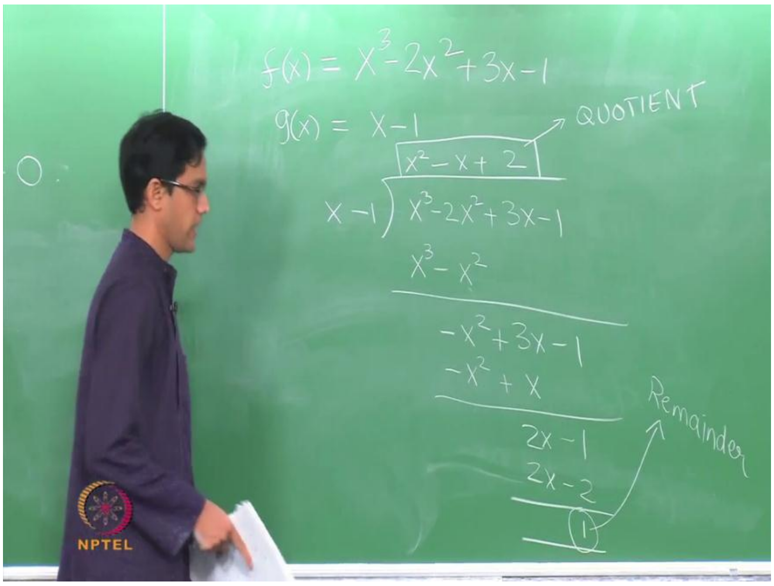
We have looked at three things you can take the sum of two polynomials, you can take the product, you can substitute values for x.

(Refer Slide Time: 00:25)



And there is another important operation called long division, so let me recall what long division of polynomials is, again the procedure which must be very familiar. So, we let say do it by example, so let me take some more space.

(Refer Slide Time: 00:49)



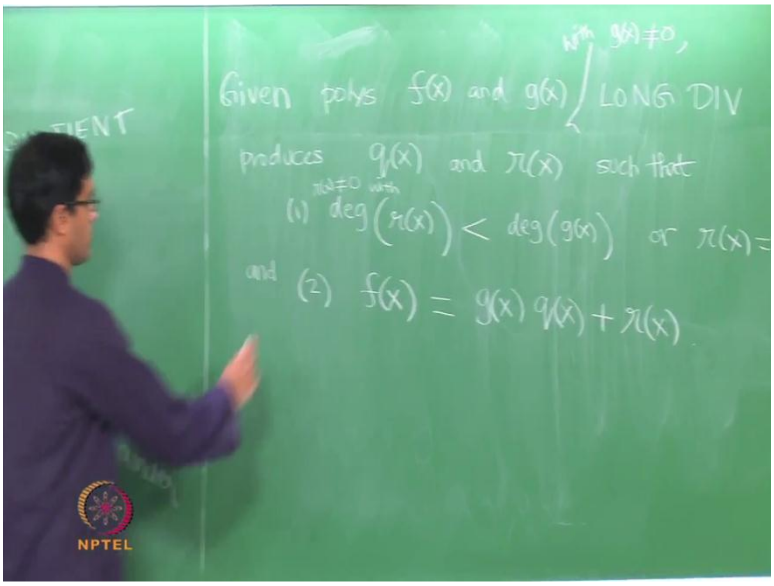
So, let us perform long division of the polynomial f of x which is, I am going to divide f by g perform the long division procedure. So, what do we do we will write f of x first x cube minus 2 x square plus 3 x minus 1, you want to try and divide it by the polynomial x minus 1. So, the algorithm the procedure says, first divide the highest power here by the highest power here. So, x cube divided by x, you see a quotient of x square you write that on top and we multiply this x square by the x minus 1.

So, x square times x minus 1 is of course, x cube minus x square, then you subtract. So, when you subtract you will end up canceling the highest power of x that is the way we it is design is essentially. So, we now subtract will get minus x square plus 3 x minus 1 and then we continue the same procedure, we divide minus x square the highest power here or the highest degree term with the corresponding thing there. So, the quotient is minus x square divided by x, so minus x.

And now, you multiply x minus 1 by the minus x, so that gives you minus x square plus x, then again you subtract, you see a 2 x minus 1 and we do the same thing. You divide 2 x by x you get a 2 and you multiply the 2 by the x minus 1 and gives you 2 x minus 2 and again you subtract. So, when you subtract get minus 1 minus of minus 2 is a plus 1, so we have just done this procedure all the way to the end and what this procedure produces is, well it is two things. One is the quotient, so this guy here is the quotient and the last thing here.

So, why do we stop there, because we cannot quit continue the process, you cannot divide 1 by x minus 1, the highest power there is a 1 in it is x power 0 which is smaller than the highest power here. So, you cannot continue the division, so what is obtained that the very end is what you called the reminder. So, long division produces a quotient and reminder that is the key thing to remember.

(Refer Slide Time: 03:36)



So, let me just say what long division does, so given polynomials just often abbreviated to polys, polynomials f of x and g of x. What is long division do, long division produces of f by g produces two more polynomials. So, I started with two or then it produces with two more polynomial, let us call them q of x, q for quotient and r of x, r for remainder such that the following properties are true, such that well what. Firstly, r is a reminder which means it is something which is obtained at the very last step of the division, which means you cannot continue the division anymore.

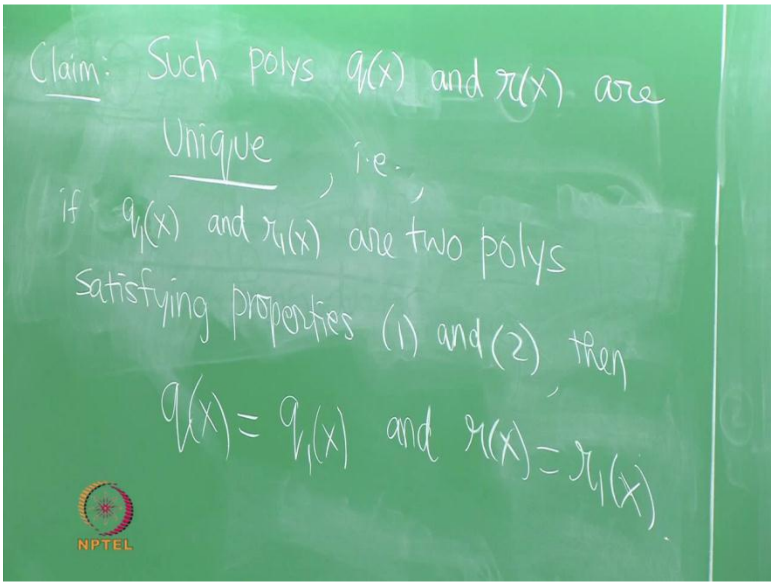
So, that is the reminder is something whose degree is smaller strictly smaller than the degree of g of x and so I should also say given non zero polynomials. So, I do not want to be dividing by 0, yes at least I want that g is a non zero polynomial. So, may be let me just be more to precise here. So, given polynomials f and g with g non zero, so I cannot divide by 0 in the world of polynomials.

So, what it does it, it produces two polynomials q and r, so that the degree of remainders strictly smaller than the degree of what you dividing by and what is property 2, property 2 is the original polynomial f. So, what is it mean to divide it, it is just is the quotient that you get on doing the division, then r of x is the x again. So, that is the key outcome of doing the long division process, you produce two other polynomials with these properties, that f can be return as g times, the quotient plus reminder and with the remainder having degree strictly smaller than the degree of g.

So, notice so I should again qualifies it little bit more, so the degree of the remainders strictly smaller than the degree of g or the remainder could be 0. So, in our case the remainder was something non zero, but it could very well happen that in when you divided it an exact division that the remainder just 0 and so observe that I said degree is not defined if, you know the polynomial is 0. So, I am talking about either the remainder is 0 or it is non zero and has degree strictly smaller than degree of g of x.

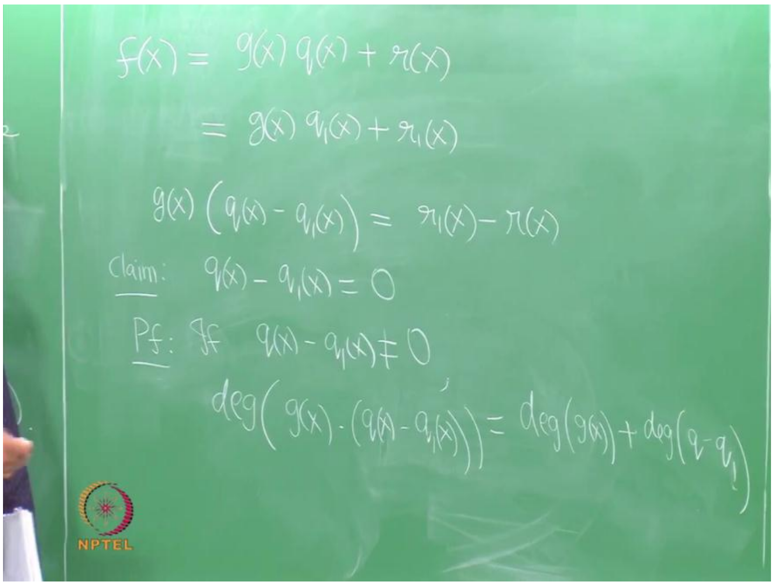
So, here I should say r of x non zero with degree strictly smaller or the remainder just 0, so that is what long division does. So, we are ready to sort of do our first proof, let us prove the following fact that the quotient and remainder are really unique.

(Refer Slide Time: 07:17)



So, what is it mean, so here is what I want to prove, so here is the claim such q and r are unique such polynomials. So, what is the mean to say that it is unique, if you find another pair of polynomials which has these two properties, then those polynomials will actually co inside with the polynomials q and r that is, if there exists say if let us call it if some other name q and x and r 1 x are two polynomials satisfying the same two properties. Satisfying properties 1 and 2, then q is the same as q 1 and r is the same as r 1. So, let us try and prove this, it is just a very elementary argument.

(Refer Slide Time: 08:48)



So, let us start with the second property there, it says f of x can be written as g of x, q of x plus remainder r of x that is the original polynomial q and r that you obtained by the long division procedure. And now, what we are claiming is, there is another pair of polynomials q 1 and r 1 which have the same property. So, I am only using the property 2 now. So, the same thing has two different expressions, given this what we will do is subtract these two equations.

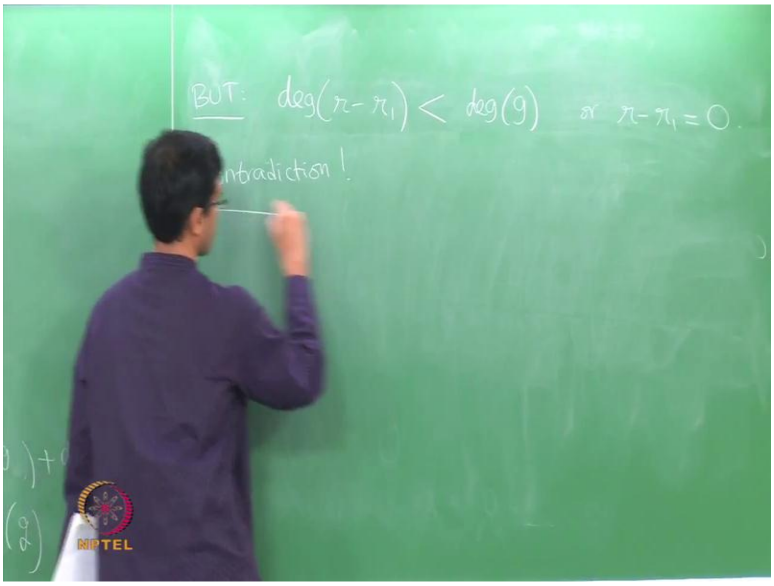
So, from this we conclude the following by subtracting that g of x multiplied by... So, I am going to subtract the top thing minus the bottom thing q minus... So, g of x into q of x minus q 1 x equals, now I will push the r of x over to the other side. So, I get the following equation that g of x multiplied by q minus q 1 is r 1 x minus r of x. So, the claim from here is that, this term here q minus q 1 had better be a 0. So, we now claim this implies that q minus q 1 is 0 and let us prove it.

So, this is the standard methodologies of proof will prove it by contradiction, so we will assume that this is not true and then obtain some; obviously, observed statement from that. So, let us assume this is false suppose not, if q minus q 1 let say is non zero polynomial, then look at the left hand side we have got g of x which is a non zero polynomial, you have got q minus q 1 which is an non zero polynomial and you are multiplying these two polynomials out.

So, I am taking the product of two non zero polynomials, remember we had a property of degree. So, what can we say, if both these are non zero polynomials, the degree of their product, the property of degrees says degree of g times q minus q 1 must be the sum of the degrees, it is degree of g of x plus degree of q minus q 1. I am just suppressing the x there, often we will just want to use the symbols rather than of x each time.

Now, so let me also do that for g, so it is degree of g plus the degree of q minus q 1. Now, what about the right hand side, the right hand side is r 1 minus r and each of those is what we know about r 1 and r. Now, we will use property 1, each of them is either the 0 polynomial r and r of x could both be 0 or they could have degree or they have, they are both non zero or in 1 or the other is non zero and the degree is strictly smaller than the degree of g of x, so that is property 1.

(Refer Slide Time: 12:02)



Now, what is that imply, the difference of r and r 1, but by property 1 we conclude that the degree of r minus r 1 is strictly smaller than the degree of g, because both r and r 1 have degree which is strictly smaller than the degree of g. So, the difference would also have the same property or of course, it could be 0 that is also fine. So, r could happen that r minus r 1 is 0, but notice that we already have a contradiction, because on the one hand the left hand side has degree equal to the degree of g plus the degree of something.

So, the left hand side has degree at least the degree of g, so this guy is at least degree of g whereas, the right hand side has degree strictly smaller than the degree of g. So, these two things could in possibly be equal, so that is the contradiction. The degrees on the left and right hand side down note to be different and why do we obtain this contradiction, because we made an assumption which must therefore be false. So, we assume q minus q 1 is non zero, but in fact that is the false assumption, because that leads to an observe statement. So, that is done, so we have done with the proof of this claim of uniqueness.