An Invitation to Mathematics
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Unit
Combinatorics
Lecture - 19
Solution to the Integer valued
Polynomials problem

Welcome back, so last time we looked at Permutations and Combinations.

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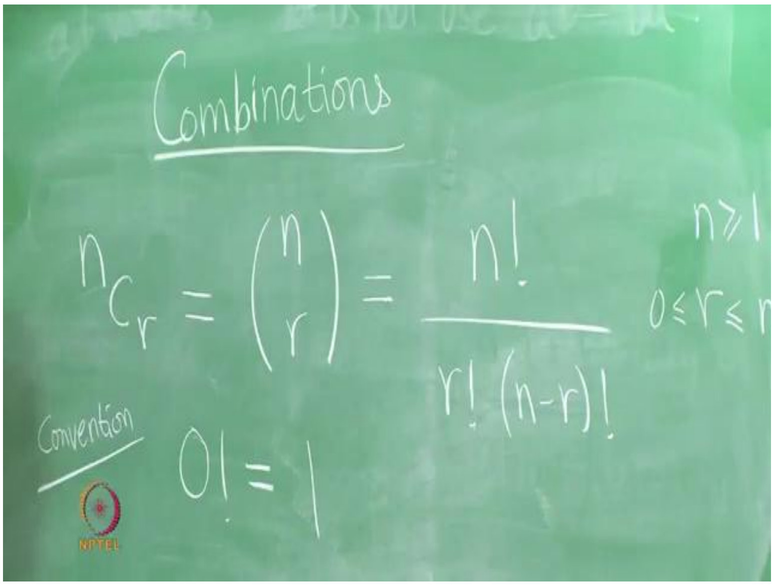

So, I just want to recall the formula here. So, n c r, so let me just mention one other things sometimes more commonly this is denoted n r with in bracket like this and pronouns n choose r, so this again notation, which is fairly common. So, this is n factorial divided by the product of these two factorials and this formula holds for any n. Well, what is it hold for? So, here really this is let say n is at least 1, r is a number between 0 and n.

So, that is the usual, those are the usual ranges of n and r, which appear when we try and use this for counting problems. Now, so of course, if you choose r equal to 0; that is 0 factorial will appear. The 0 factorial is, the convention for 0 factorial is that it is 1, so this is the convention. So, observe this add in fact, occurred earlier when we talk about Taylor polynomial and so on.

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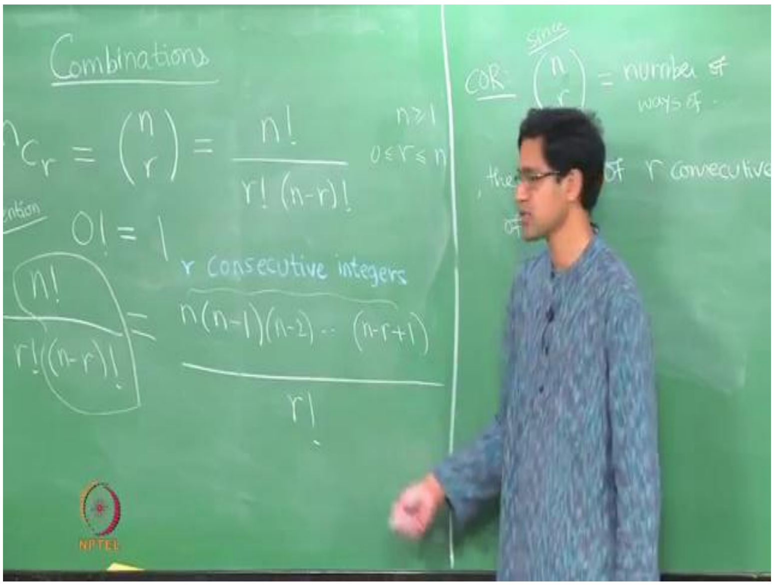

So, 0 factorial is 1 and so of course here, let sort of try and rewrite this formula as follows. So, if you sort of cancel of the n minus r factorial with the n factorial. So, this is the product of all numbers from 1 to n minus r and this is the product of all numbers from 1 to n. So, what you will get will be the terms which do not cancel out, when you try and cancel these of is n, n minus 1, n minus 2, all the way till the number just after n minus r. So, that is this one, n minus r plus 1 and the r factorial remains the same.

So, that is of course, just an equivalent way of writing this formula here. But, now we observe that the numerator here is a product of well it is n, n minus 1, n minus 2 and so on are consecutive numbers. So, what is in the numerator here? So, these are all consecutive numbers, there all consecutive integers starting with n, I am going down wards. Now and how many of them are? Well, there are r consecutive integers. So, this is a list of r consecutive integers in the numerator and what is on the bottom is r factorial.

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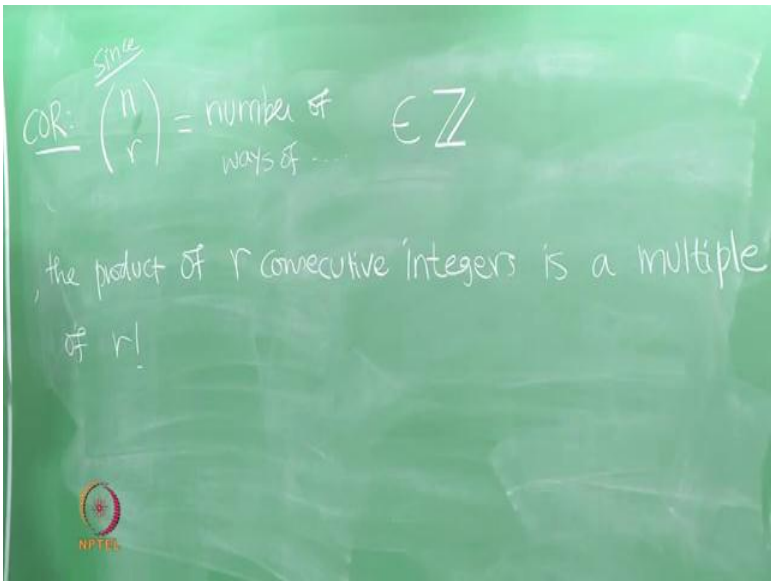

So, since n choose r, so here sort of a nice corollary to this fact here. So, recall n choose r is of course the number of ways of picking r numbers out of n numbers. So, since this is the number of ways of something out of the other, number of ways of picking r numbers out of n numbers. So, it does not really matter what it is, whatever this is, this is surly an integer. In fact, it is a positive integer or a non negative integer at least.

So, n choose r is certainly an integer that much is clear, because it of the interpretation, that it is a number of ways of doing something. Whereas, on the other side, what you have is a product of r consecutive integers. So, observe that a product of r consecutive integers starting with any number n, can always be thought of us really being n choose r. So, here is sort of corollary to this facts, since n choose r is an integer, we conclude that the product of r consecutive integers is always divisible, is always multiple of r factorial, it must be an exact multiple.

You take any r integers, the product is always going to be the r consecutive integers, say product will always going divisible integer by r factorial. And of course, it is you know, it is not strictly something that you can see in just the instantly from here, because n is assume to be a positive integer that at least r and so on. But, you should be able to convince yourself that it is in fact, true as stated, that even if n is smaller than r, the product of r consecutive integers staring with n downwards will still be divisible by r factorial.

In that case will actually be the product will be a 0. Similarly, if n is a negative number, it is still true that the product of r consecutive numbers starting from n downwards will still be divisible by r factorial.

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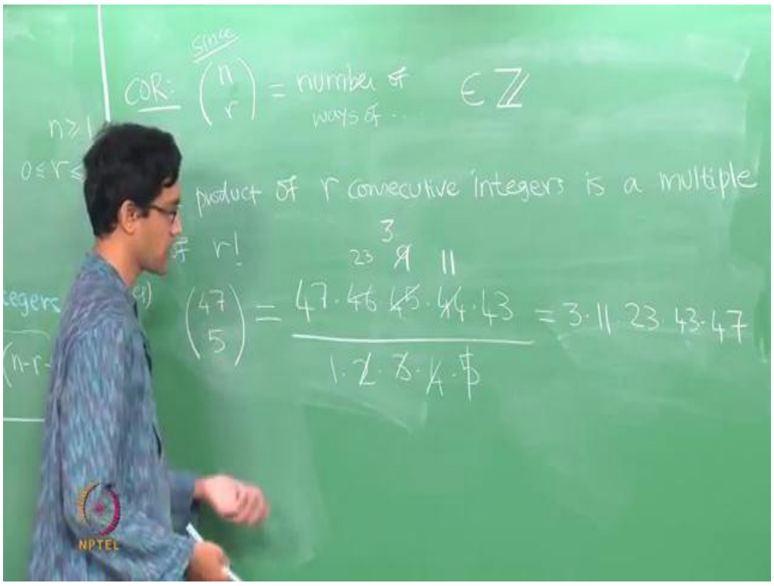

So, this is sort of the nice fact it is somewhat surprising that something like this is true, if you know unless one is sort of looking at this context, it is somewhat surprising I would say. So, for an example 47 choose 5 is the product of 5 numbers 47, 46, 45, 44, 43 and the claim is that this product is always divisible by 1, 2, 3, 4, 5. The product of 5 factorial, I mean the number 5 factorial and see, why it is divisible.

So, let see where the calculation will occur, for instance, the 5 cancels the 45 leaving as with 9. Let see what else will go, the 4 cancels 44 giving you a 11, there is a 3 which will cancel of the 9 gives a 3 and I guess the 2 can cancel of well the 46 give you a 23. So, observe the final answer, after all this cancellation is 47 into 23 into 3 into 11 into 43. So, this is 3, 11 into 23 into 43 into 47, which is certainly an integer, but it is kind of a priory little surprising that there were always be such exact cancelation.

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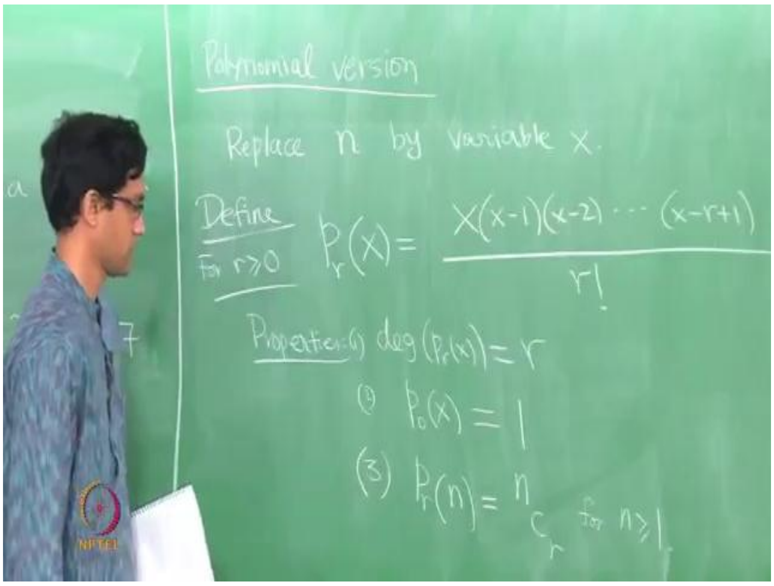

So, now, let us do the following, since we sort of talked about polynomials, just before this. So, here is the polynomial version of the number of combinations. So, what do I mean by that? Let us do the following, let us replace the integer n by the variable x in the formula for n choose r. So, what do I mean by that? Let us give this a name, let us define the quantity, so I want to start from this definition of n choose r.

And what I want to do is sort of something like define x choose r, where x is some playing the role of n. So, I am going to define a quantity x into x minus 1 into x minus 2 till x minus r plus 1. So, let us define x times x minus 1, x minus 2, x minus r plus 1 divided by r factorial. So, what is this? Well, it is a polynomial in x. So, let me call it p of x and it is a polynomial of what degree, well there are exactly r terms in the numerator, recall there were in fact, r consecutive integers if you wish.

So, this is actually a polynomial of degree r and what are the choices of r, let us define for r greater than equal to 0. So, take any r greater than equal to 0 and define the following polynomial p r of x to be just x into x minus 1, x minus 2 till x minus r plus 1 divided by r factorial. So, what are the various properties? So, let just see what good this is. So, the first property of course, which I already said is that the degree of p r of x is exactly r, so that is more or less clear.

Further, so what is p 0 of x for instance, so it is just the border case, if you put r equal to 0, then you want to think of this as really being 1. So, it is an empty product on the numerator, that r into any terms and the denominator it is of course, the 0 factorial. So, this is sort of like n choose 0, if you wish. So, p 0 of x is just the constant polynomial 1, which of course has degree 0; that is as we wanted to be.

Now, of course, what is the whole point of defining this new polynomial here? If you substitute an integer or natural number for x, what this gives you is exactly n choose r, for n natural number, let say n greater than 1. So, it is just a polynomial which when you plug in x to be an integer, it is gives you back you know the usual function n choose r.

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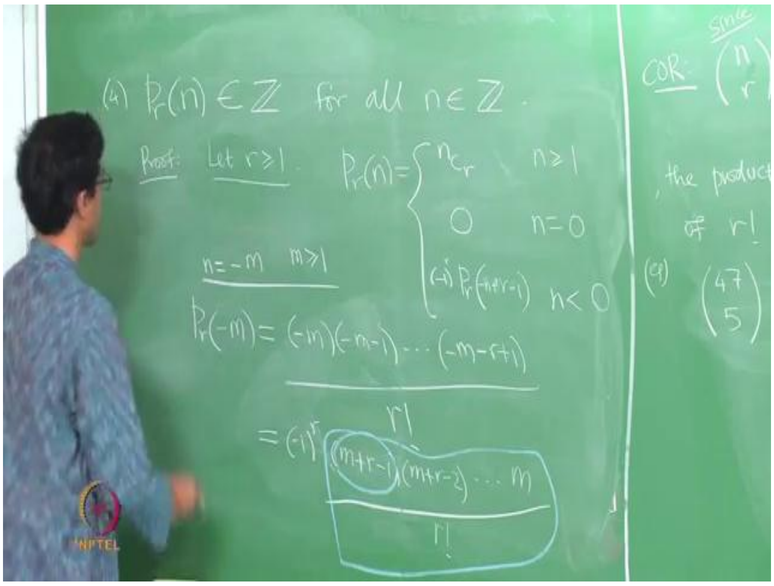

Well, so in particular what does it mean, it says that this polynomial has the very nice property that when you plug in, so here is the fourth property. If you take the polynomial p r and you plug in an integer, the answer is an integer and well, proof we just sort of saw, what happens when n is greater than equal to 0 or n at least 1. So, observe that p r, if you plug in, so let us just take r at least 1.

So, let us just do this. So, let r be at least 1, because if r is 0, then p r is just the constant polynomial 1 which satisfies this property. So, if r is at least 1, then let us look at, so p r of n is the following. So, various possibilities, if n is at least 1, then this is just what we would calling n choose r, so that surly an integer. If n is 0, then by definition, if we plug in 0 into this polynomial, so observe p r of x is just x into x minus 1 into dot, dot, dot.

So, there is an x in front, if you put n equal to 0, of course, I mean, if you put x equal to 0 this is just going to be come 0, provided r is at least 1. And so in order to finish this proof of this assertion, you need to worry about what happens if n is negative. So, let just see, so this is, if you plug in p r of n, so let us write n as minus m, where m is no positive. And let see, what happens, when you plug in a negative number into x in place of x, this is x into x minus 1, x minus 2 till x minus r plus 1 divided by r factor.

So, the numbers on top are all negative numbers from minus m downwards, but we can just pull out the negative sign from each of these and write it as minus 1 power r and then, these now become m into m plus 1 into m plus 2 till m plus r minus 1. So, let me write the highest term first, which is this guy m plus r minus 1, m plus r minus 2 and so on, a decreasing sequence which finally m z n, divided by r factorial.

So, in other words, p r of minus m is nothing but, minus 1 power r times an expression more or less of the same form, except this is now the same polynomial p r evaluated at m plus r minus 1. So, this is just, so let me write out the answer, it is minus 1 m power r times the polynomial p r, but evaluated it different point, which is let see m plus r minus 1 and what is m, I should worry about that, m was actually minus n. So, let me just substitute in place of m, let me also put a minus n.

So, this is minus n plus r minus 1, but of course, the precise form does not matter, all we worried about right know which is improving the p r of n is an integer, provided n is an integer; that is now true. Because, p r of negative number is just plus or minus times, well again, this is now positive number times p r of positive number, which we know already to be an integer. We have done with the proof of this that p r has this property.

So, observe this is exactly the question; we post earlier when we talked about polynomials. So, recall we talked about real valued and integer valued polynomial and real valued polynomial, we obtain criterion said that polynomial which takes real values, when you plug in real numbers for x, such a polynomial must have real coefficients, there is no way out. Whereas, if you had a polynomial which takes integer values, when you put x an integer, need not have the property, it could happen that, it does not have integer coefficients and we looked at one example back that, which was x into x minus 1 by 2 but.

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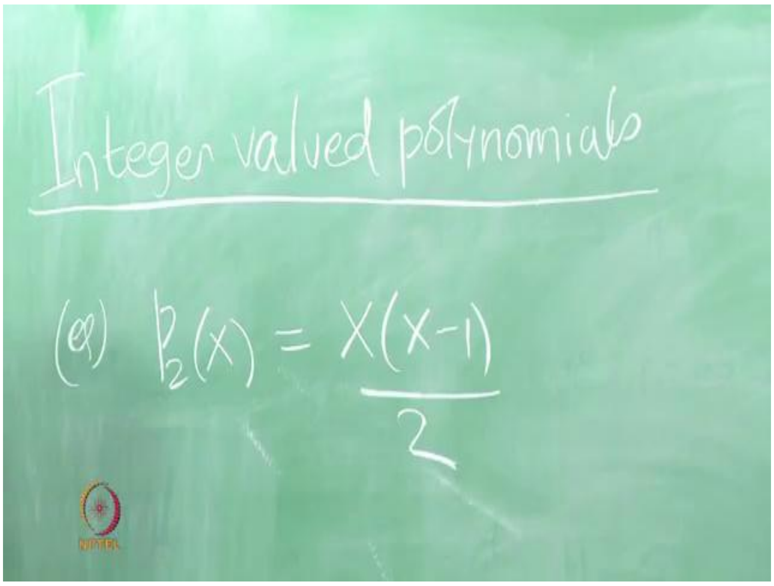

So, observe that the example, we had back when we are talking about integer valued polynomials. So, the back them, I am give you an example of the polynomial x times x minus 1 by 2, but observe that in this new notation that is what we are calling p 2 of x, it is just sort of like x choose 2. And this polynomial as we are no seeing has sort of generalization, you can look at p r of x for every r. So, you can look at x into x minus 1 into x minus 2 divided by 3 factorial and so on.

And each of these polynomial would have same property that you put x an integer the answer will be an integer, but the coefficients are not integers. For example, here the coefficient is 2, I mean it is a half, similarly if you talk r of x in general, the coefficients will involve 1 over r factorial, so clearly there not integers. So, now, nevertheless there is something very nice that one can say about these, which is the following.

So, observe the key property, one of the key property if I mention is that, the sequence p r of x has 1 polynomial of each degree and this is a very, very important, this occurred back when talk about polynomial, when we looked at the Chebyshev and the Legendre polynomial and so on.

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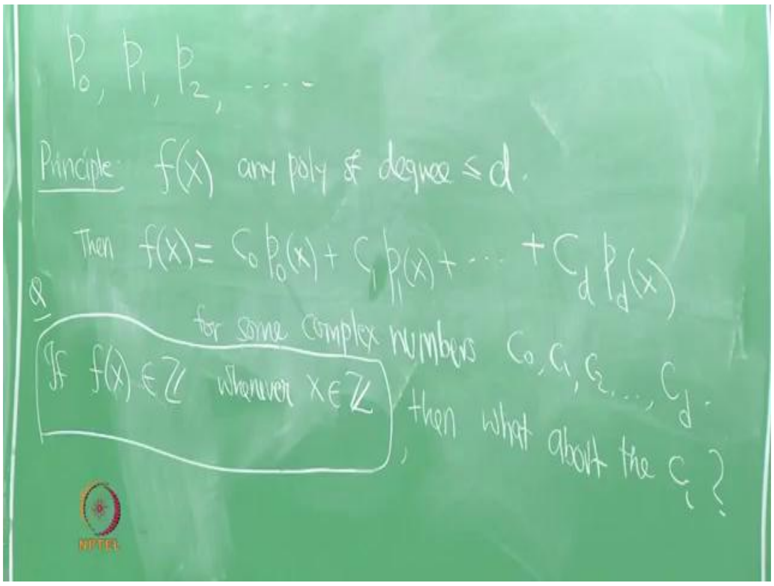

Now, here p r of x is again a polynomial of the same kinds, so the sequence of polynomial p 0, p 1, p 2 and so on, they have degree 0, 1, 2, 3 etcetera. So, the key principle that we talked about back then was the following that any polynomial can be written as a linear combination of such polynomials. So, recall the principle we talked about earlier that suppose f of x is any polynomial, it say of degree at most d.

Then, f can be written in the following fashion, then f of x can be return as some constant times p 0 of x plus some other constant possibly times p 1 of x and so on till some constant times p of x. For some constant which in most generally speaking, they could be some complex numbers, for some complex numbers c 0, c 1, c 2 and so on, because this follows from the fact that, the p i's are 1 in each degree.

Now, comes the key point suppose we want to look back on are earlier problem in polynomials, which is suppose f is a polynomial which takes integer values for integer choices of x. So, suppose we knew further the following that if f of x or let us call it if f of n, I have just use as if f of x as an integer, whenever x is an integer, let say that is the additional property of f, what we calling integer value polynomials.

Then, what can you say about these constants; that is now the question, if this property is known, then what about the c i's, what can be say about the c i's, does it force some nice property on the c i's. So, let us now compute the claim is in fact, the c i's will have to be integers, if f has this properties, so I am just going to calculate.

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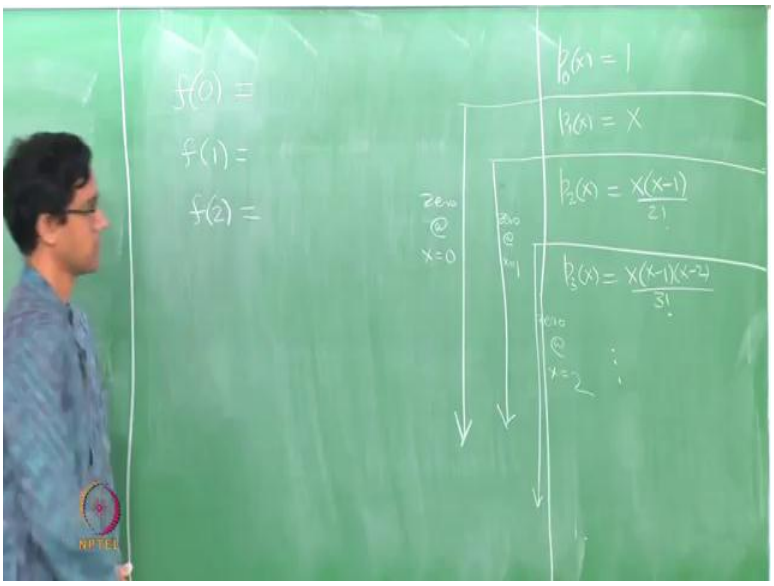

So, observe that here is the key calculation, if you plug in 0 for x, we you plug in 1 for x, we plug in 2 for x and so on and then, we see, what happens to the right hand side. So, let us imagine, what happens when you plug in let say x equal to 0. So, observe the p 0, so let says recall the first few polynomial p 0 of x was just the polynomial 1, p 1 of x by definition would be x, p 2 of x will have x, it will also have X minus 1, write just one more p here of x will have x, x minus 1, x minus 2 and so on.

So, if you plug in x equal to 0, what is going to happen is well all these polynomials from p 1, p 2, p 3 downwards, every one of them will vanish at x equal to 0, all these everything here staring from p 1 onwards, this will become 0 when x equal to 0, 0 at. Now, when you plug in x equal to 1, similarly p 2 onwards p 2 is 0, p 3 is 0 and so on, everything from p 2 onwards this is 0 at x equal to 1. Similarly, p 3 onwards will vanish at x equal to 2, so these will be 0 and so on, so there all 0 from some point onwards.

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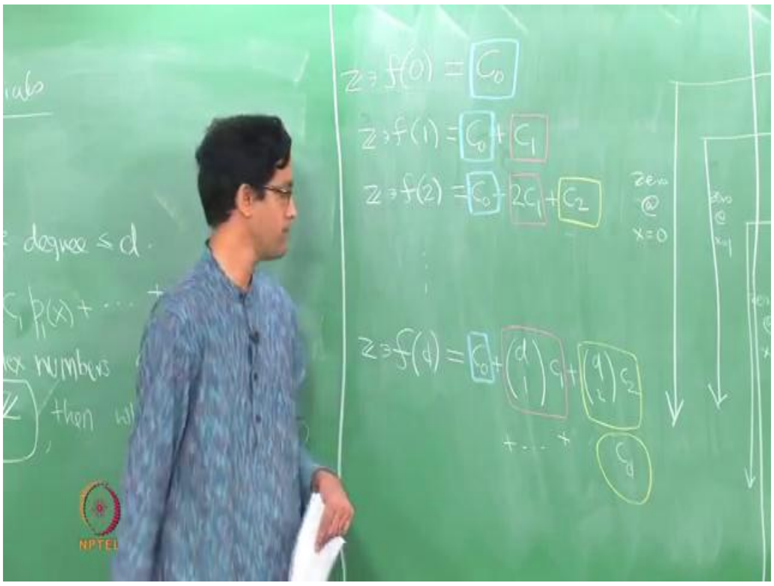

So, f of 0 for example is only going to be c 0 times p 0 of x, p 0 is just 1. Similarly, f of 1 is going to be the following, p 2 onwards, there is no contribution; there is only contribution from p 0, which is c 0 plus is contribution is c 1. So, put x equal to 1, this is just to 1. Similarly, if I put x equal 2, I will get c 0, 2 c 1 plus c 2 and so on. So, if you keep going down for plug in f d for x, well here is what it becomes, I get c 0 plus d choose 1 and now going to the other notation for combination d c 1 if you wish times c 1 plus d c 2 times c 2 and so on till the final coefficient will occur with.

So, the final c d occurs with coefficient d choose d what you get set of equations like this when you plug in 0, 1, 2, 3 and so on for x. Now, observe that 0, 1, 2, 3 are all integers, so f being an integer valued polynomial, when you plug in 0 this of course, f 0 an integer f 1 is an integer, f 2 is an integer so on. So, the left hand sides there all known to be integer; that is a given hypothesis on the function f, there all know to being z.

Now, from that let see what the conclusions are, since f of 0 is an integer, it tells you 0 of course an integer. So, let see c 0 is known to be known to be an integer. So, c 0 is f of 1 is an integer. So, c 0 plus c 1 is an integer, but a c 0 is already an integer, so I just box all occurrence of c 0, it already known to be integer. Now, c 0 plus c 1 is an integer that tells mean the c 1 is an also integer. So, that is known now in the second step that c 1 is an integer. So, c 1 is an integer of course, in 2 times c 1 is integer, so all of these occurrences of c 1 there all integers.

Now, in the next step this and this are integers, I mean the some of these three terms as an integer, the first two terms are integer. So, of course, the third terms also an integer. So, c 2 is also an integer for the same and so on, all occurrence of c 2 are integers. And you keep going in the next step; you can conclude the c 3 is an integer, c 4 as an integer and so on. In very last step d choose d is just a 1. So, this is a just a 1.

So, in this some all terms except the very last term all of them would have been proof to be integers until this step and this full sum is known to be integer. So, you will therefore, conclude that c d also integer is also an integer, this is sort of the inductive argument, you proceed once step at a time and conclude that all these constants are in fact, integer. So, here is the conclusion finally, so what can be say about the c x, so let us come back and answer this question.

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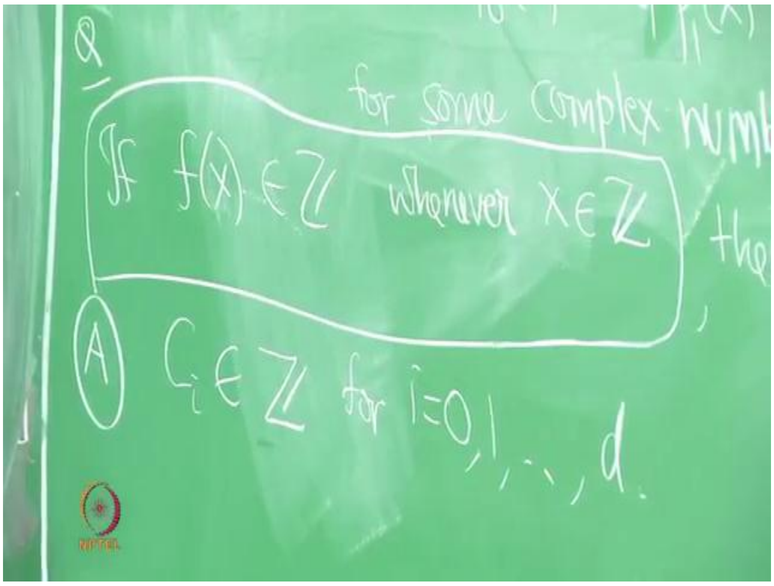

If f is known to be an integer value polynomial, then what do we know about these constant c i’s the conclusion is that the c i's are themselves integers. So, here is the full answer to the question, what can say about the integer valued polynomials, well there are integer combinations, integer linear combinations of these special polynomial p 0, p 1, p 2 and so on.

So, these polynomial p 0, p 1, p 2 may not have integer coefficients no doubt, they are somehow special polynomials. but, you where you write f as a combination of those polynomial the coefficient you end of a obtaining are all integers, what you called in integer linear combination of these polynomials p 0, p 1 and so on. So, next time will talk about combinations with reputations, which is just slightly more complicated then talking about the problem of combination itself.