An Invitation to Mathematics
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Unit - I
Polynomials
Lecture – 18
Real and Integer valued polynomials

Welcome back, first I want to talk about Real and Integer valued polynomials. So, what do we mean by this? So, let us begin by considering sort of the most general constants.

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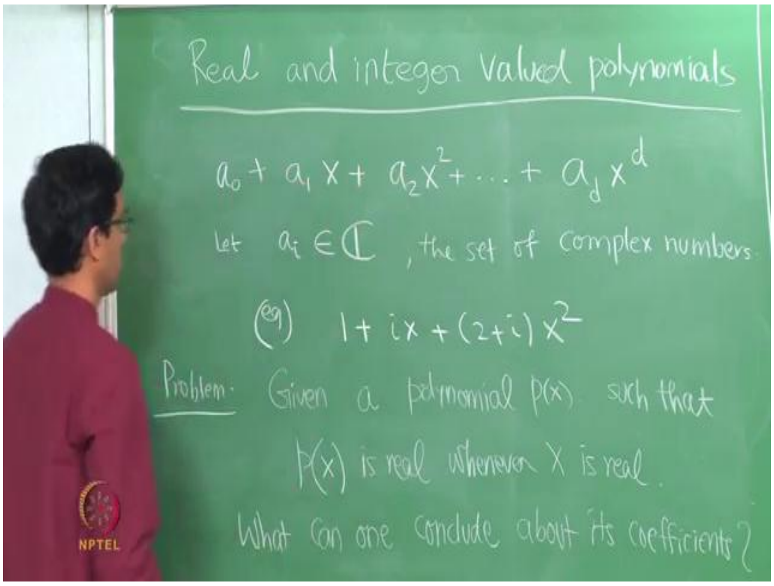

So, for as a polynomial is let say a 0, so recall a polynomial for us was an expression of the form a 0 plus a 1 x into x square and so on small d. Now, the most general thing, we could do for now is to allow the a i's, the coefficients to d complex numbers. So, let us assume, so let a i belong to C, which is the set of complex numbers. So, this is the set, so we will just call these complex polynomials, polynomials where all coefficients are complex.

So, what is a typical example you could take, so let see I have 1 plus i times x plus let say 2 plus i times x square, where i of course, is the complex unit the square root of minus 1. So, here is the polynomial, whose coefficients are 1 i and 2 plus i which are all complex numbers. Now, what we want, so when I said real and integer valued polynomials. Let we take the first case I want to look at real valued polynomials.

So, here is the problem given a polynomial as about p of x with complex coefficients, such that with an additional condition that p of x is real, whenever x is real. So, I want to look at is a polynomial, which has the property that whenever I plug in a real number x, then the answer p of x is also a real number. So, this is what I want to call real valued polynomials, they take real values and real numbers, then what can I say about the coefficients.

So, the question is this, given this polynomial, what can one conclude about the coefficients a i, so let say sort of look at the expression here p of x look like this. Now, one of the most obvious ways of constructing such a polynomial, how would be get an example of such a polynomial.

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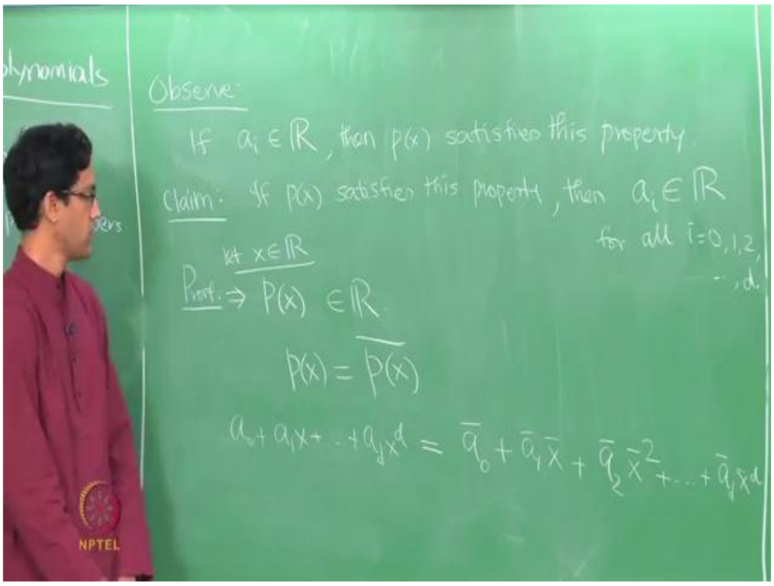

Observe, if all a i's are real numbers, so R is the set of real numbers. If all the coefficients are already real numbers, then p certainly satisfies this property, then p of x satisfies this property. So, if I do not have any of my i is appearing anywhere, if it is just 1 plus x plus 2 x square for instants, then if I plugged in x to be a real number of course, I would get a real answer.

So, here is one way of constructing such polynomials, you can and take all the coefficients to be real numbers, but is this necessary is this be the only way of obtaining a real valued polynomial. So, we seen just a minute the answer is, yes that you really do need to have all coefficients to be real, only then can this property be true. So, here is the claim, if that in fact, the converse is true; that if p of x has it is property, then the coefficients have to be real numbers, there is no other way out.

So, let us try and prove this, the proof itself is rather you see. So, what is that mean to say that p of x is a real number, whenever x is the real number. So, it says that p of x bar, so let x be a real number, then we are given the p of x is real number. So, this implies that p of x is real. So, let me write in this, now a real number…So, now we need to recall a little bit about complex numbers. Remember, we have a notion of conjugation, complex conjugation in the set of complex numbers, it maps i in some sense to minus i. So, the conjugate of 1 plus i would be 1 minus i.

So, here is the property if p of x is real number, then that implies that p of x is the same as it is conjugate. So, in other words, so let us compute the conjugate. So, if I take 0 plus a 1 x a d x power d that is p of x, so that is the left hand side, the right hand side is the conjugate of this expressions, so I need to think of this as being a conjugate. But, again the well known properties of conjugation say, that if you have sum of complex numbers and you want to take their conjugate, it is a conjugate of each individual term.

So, this is just nothing but the conjugate of a 0. So, remember a 0, a 1, a 2 are potentially complex numbers for now. So, it is the conjugate of a 0 plus the conjugate of a 1 times the conjugate of x plus the conjugate of a 2 times the conjugate of x squared and so on, conjugate of a d, conjugate of x times power d. So, that is what you get when you take conjugate on the right hand side, but observe I said that the x is real number, which means that x has it is own conjugate. So, real numbers have their own conjugates so in fact, x bar is a same as x, because of the assumption that x is real.

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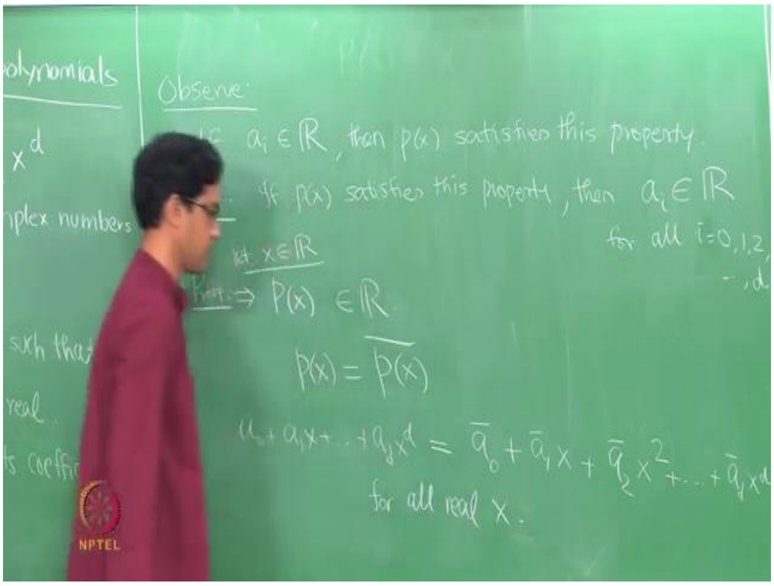

So, I can just remove the bar on the x's. So, here is the conclusion that a 0 plus a 1 x plus a d x d is the same as a 0 bar plus a 1 bar x plus for all real numbers x, for all real x. But, now observe that the left hand side, you can think of as, you know here is a polynomial of degree d, a 0 plus a 1 plus a d x d. The right hand side is another polynomial of degree d, which is a 0 bar plus a 1 bar x plus a 2 bar x squared and so on.

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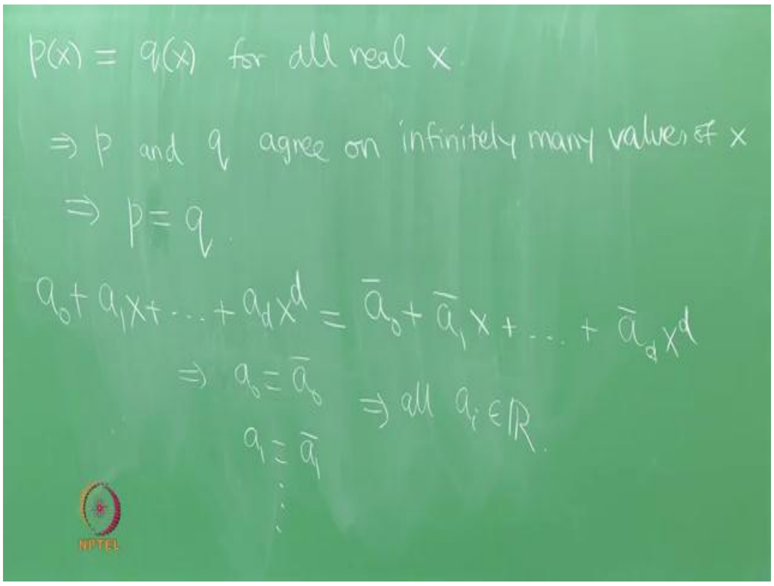

So, think of, so let us call that another polynomial, so maybe we will just call it q of x may be, let us call here is another polynomial of degree at most d. And what we are claiming is that p of x, the polynomial p and the polynomial q agree, they are equal for all real numbers x, for all real x.

Now, observe this principle that we talked about a few modules ago, which is, if I have a polynomial of degree d and anther polynomial degree d. And if they agree on d plus 1 points, they have the same value on some d plus one points. Then, they must have the same value of all points, being they are actually exactly the same polynomial. But, then observe for every real number x, I know that they are actually agree and there are infinitely many real numbers.

So, agreeing on infinitely many points will be in the surely agree on d plus 1 points. So, therefore, p and q, so the polynomials p and q agree on in fact on infinitely many points and all real numbers. Since, they have to at mostly this means that p and q are exactly the same polynomials, so p equals q, so let me write this. So, what is that mean? The polynomial a 0, so p is the polynomial a 0 plus a 1 x plus a x d is the same as 0 bar plus. So, these two are exactly the same polynomials.

Now, what is it mean to say they are the same polynomial, it means coefficients are all the same, a 0 as a same as a 0 bar, a 1 is a same as a 1 bar x and so on. So, these are identically equal polynomials. So, it sort of a little bit like this equality, but this equality is thought of as only holding for all real values x, this equality actually holds for all complex values of x as of your wish.

Since, they are identically the same polynomial. So, this means that a 0 is a same as a 0 bar, a 1 is a same as a 1 bar and so on. Every coefficient is equal, but then this means that a 0 is a real number, the next condition means a 1 is real number, the next condition means a 2 is a real number and so on. So, this in fact implies that all a i's are real, so that is in fact the proof of the converse.

The only way you can get a complex polynomial, you take real values on real numbers, if all the coefficients are themselves real numbers. Now, here is an interesting variant of this problem which is sort of the second part of the title, what if you wanted integer values.

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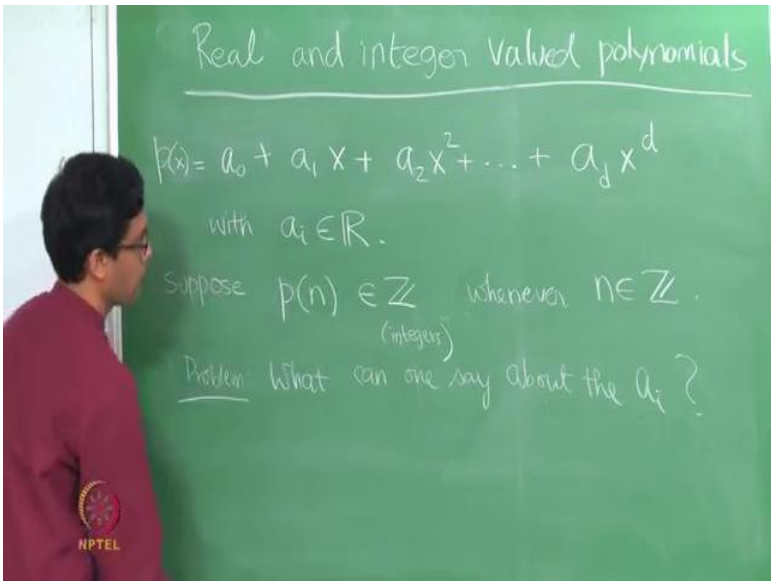

So, let p of x be like this with a i's, so let us for now consider real valued polynomial for it is, I mean exactly coefficients being real with a i's in R. Now, I want these two have integer values on integers. Suppose, I tell you the following that p of x a, so let we call it p of n is an integer. So, Z is denotes a set of integers, p of n is an integer, whenever n is an integer for so when.

So, here is the condition that is given that p takes integer values, when you plug in an integer for x, then the question is, what can you conclude about the coefficients. So, again the same question, so problem what can one say about the a i, the coefficients. The observe a similar thing can serve as a starting point, if all the a i is an integers, so if for incidents we are polynomial is just a 1 plus 2o x plus 3 x squared and so on or you know any other integer choices for a i's. Then, plugging in x equals an integer will of course give you an integer also.

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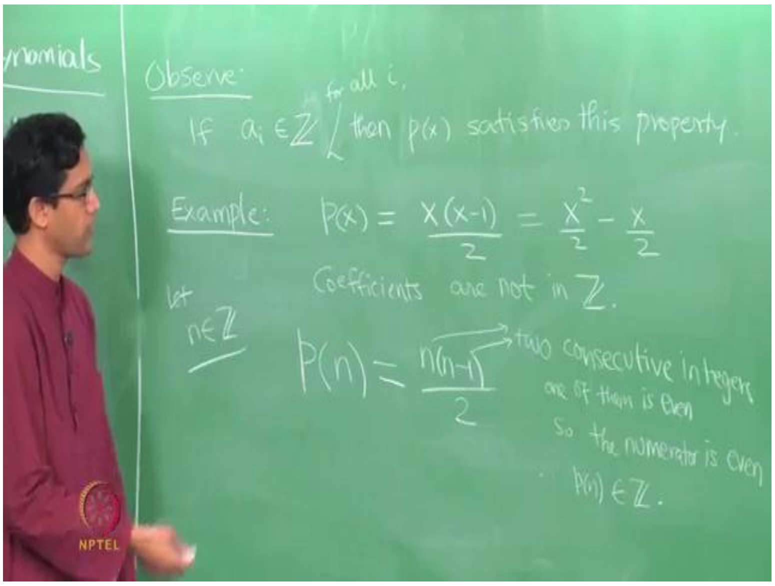

So, observe similarly just like before that, so I am just going to modify right here, if a i's are integers a i is an integer for every i. So, for all i, then p of x certainly satisfies this property. Now, what we really want to worry about is sort of the converse is it true that is the only possibility. Is it possible that a polynomial takes integer values on integers, but the polynomial does not necessarily have integer coefficients.

So, it terms out that the converse is actually not true in this case. So, here is an example which will short of demonstrate this fact, let us define p of x to be the polynomial x into x minus 1 by 2. So, here is the polynomial who is degree is 2, it is x squared minus x divided by 2. So, let us write it out for each x square divided by 2 minus x divided by 2. So, it is a degree 2 as x square by 2 minus x divided by 2. So, it is a degree to polynomial, it does not have integer coefficients, because the coefficients are in fact, half and minus half.

So, the coefficients are clearly not integers not all coefficients are integers, what you needed, general here in fact both coefficients are not integers. So, now the question is, what do we know about integer values for x. So, let us compute p of n, when n is an integer. So, let n be an integer, let us calculate p of n p of n terms out to be n, n into n minus 1 by 2 by the function.

Now, n is an integer n minus 1 is the one preceding integer, so this is the product of two successive integers and then, you are dividing the answer by 2. Now, observe that if I have two successive integers, so n and n minus 1 are two consecutive integers. Now, when I have two consecutive integers, one of them will be odd and other will be even t. So, which implies one of them is odd the other is even, so I really only care about even part.

So, one of them is an even number, which means that it could be either them could be n or n minus 1. But, whichever them it is when it divided by 2, what I will get will be, will be a whole number; will be an integer again, so therefore, it is divisible by 2. So, the numerator is an even number being the product of even and odd number therefore, it is divisible by 2.

So, therefore, in other words p of n is in fact an integer. So, observe here is a polynomial which has the property that we wanted that whenever you substitute an integer for x. Then, what you get as an answer is again in integer, but the coefficients themselves are not integer. So, coefficients are half and half, in spite of that it turns out that this gives you integer answers.

So, it is it somewhat mysterious at this point, but so I am going to defer this for later, what I like you to do in the mean time; we will come back to this in a few modules from now in somewhat different contacts. But, for now, I have like you to think a little bit about this, and see what best you can do or for a start try and construct more such polynomials sum sort of by using the same idea. So, find the few more such polynomials and try and see if you can find out what the general situation ((Refer Time: 17:35)). So, next time what will do is to talk about the many variable polynomials.