**An Invitation to Mathematics**
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**Unit**
**Combinatorics**
**Lecture - 16**
**Signs and Cycle Decompositions**

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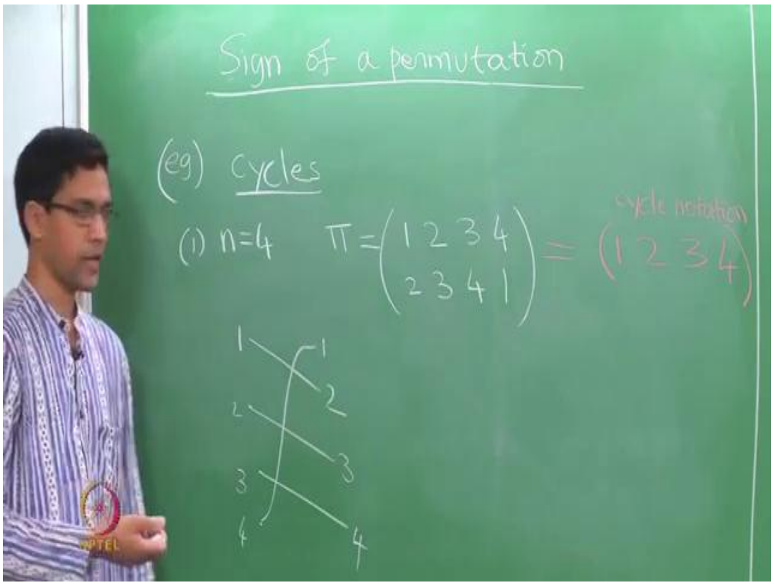


Welcome back, so what we wanted to, what we did last time is to talk about Signs of Permutations and this was define to be the… In some sense, you first take the number of crossings of the permutation which could be pictorially read of from the crossing diagram or tangle for the permutation. And then if it is even then you declare the sign to be plus 1 and if the number of crossings is odd, you declare the sign of the permutation to be a minus 1 also called even odd permutations. Now, I want to do a few more examples of these, so let us try and find science of cycles.

So, here is 2 examples, if you take n to be 4 and you take the permutation pi to be the cycle 1 2 3 4, so which I will write first in two line notation 1 going to 2, 2 going to 3, 4 1. So, this the, the cycle written in two line notation and also notice that, the other notation for permutations that we looked at is what we called cycle notation. So, in cycle notation the same permutation would be just represented by the four numbers 1 2 3 4 written in brackets.

This is to be thought of as meaning 1 goes 2 goes to 3 goes to 4 goes back to 1. So, what is the sign of this pi? So, that is a question, so let us draw the tangle diagram for pi. So, 1 maps to 2, 2 maps to 3, 3 maps to 4 and 4 maps to 1. So, here is the diagonal diagram for pi, this satisfies all the rules for drawing dangle diagrams correctly. So, the number of crossings is 1, 2 and 3.

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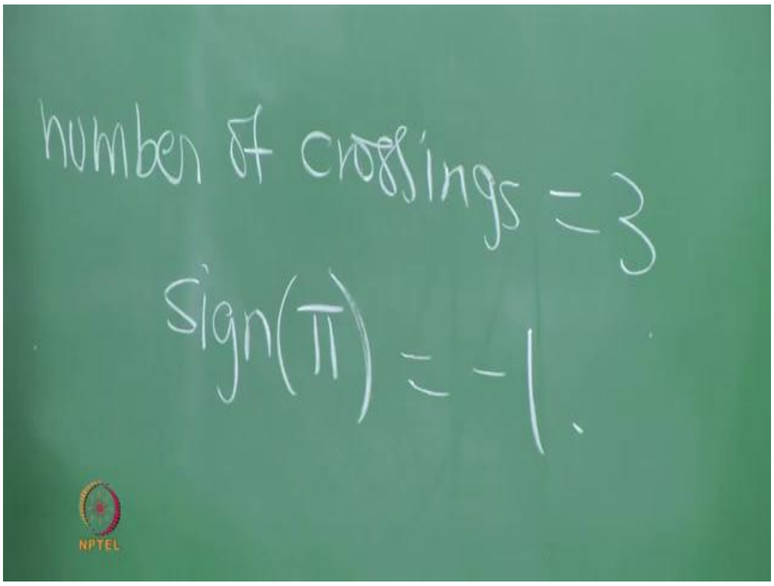


So, the number of crossings this case is 3 and so the permutation pi is odd.

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Now, similarly of course, if you drew a permutation, the diagram for the five cycle, so I take pi to be… Now, I just use cycle notation 1 goes to 3 goes to 4 goes to 5 goes to 1 and again imagine drawing the… So, here is the tangle for this, 2 goes to 3, 3 goes to 4, 4 goes to 5 and 5 goes back to 1 and now, the number of crossings is 1 2 3 4. So, there are four crossings, and therefore this permutation now is even, because there are four crossings here.

So, from this it is kind of clear, how cycles behave if you have cycle of even length. So, 1 2 3 going on till some even number, then that permutation would have sign minus 1. And if you have a permutation cycle with odd length, say going 1 through 5, then that would have even number of crossings, and therefore it would be an n even permutation.

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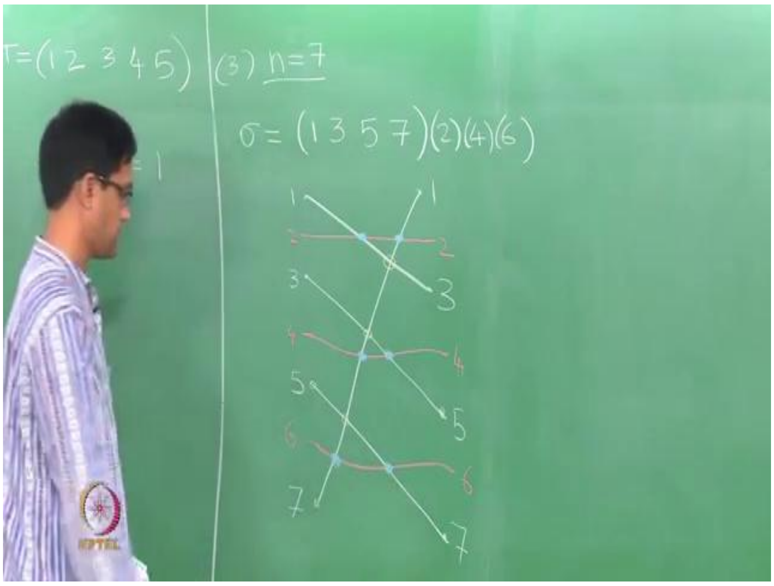


Now, here is slightly more general example, we will still look for a 4 crossing, but we will do this with the… Let us say n equals, so what is it mean. So, let me take the example, so let me take sigma to be the following, it is a four cycle with signs 1 to 3 to 5 to 7 and then it leaves a remaining numbers as it is 2 4 6. So, here is the permutation sigma written out in cycle notation. So, recall what is this really mean, so let us draw the triangle diagram for this.

So, let me first draw 1 3 5 and 7, so here is 1 3 5 7, so what this permutation does is, well it is a four cycle if you think of only these numbers 1 3 5 and 7. So, 5 goes to 7 and 7 maps to 1, so that is the diagram. If you only looked at the numbers 1 3 5 and 7 and the number of crossings here is of course, the same as the number of crossing in the other 4 cycles that we do. So, there will be 1 2 3, so there are exactly three crossings as far as these 4 numbers are concerned.

But of course, we still need to through in the other numbers, so for instance there is 2 4 and 6. Now, what this permutation does is, it maps 2 to 2, so I just need to draw straight line which joins 2 and 2 is taking care to ensure that the rules for tangles are present. So, for instance I do not want to have 3 things passing through a single point. So, when I join 4 to 4, it is need to take a little care to ensure that it does not pass through the same point.

So, similarly 6 to 6, so here are the three etcetera curves that you need to through in 2 going to 2, 4 going to 4 and 6 going to 6. So, now, in addition to the three crossings that you already had, amongst the lines which joined 1 3 5 and 7, what we now have are well a few more lines or few more crossings. So, let us count the number of crossings, so observe that the red line which joins 2 to 2, it accounts for two crossings, because it needs the line joining 1 to 3 as well as the line joining 7 to 1.

So, there are two crossings, the line the red line which join 4 to 4 again accounts for two crossings, the red line joining 6 to 6 again accounts for two crossings. So, the total number of crossings is, well how do we count the number of crossings.

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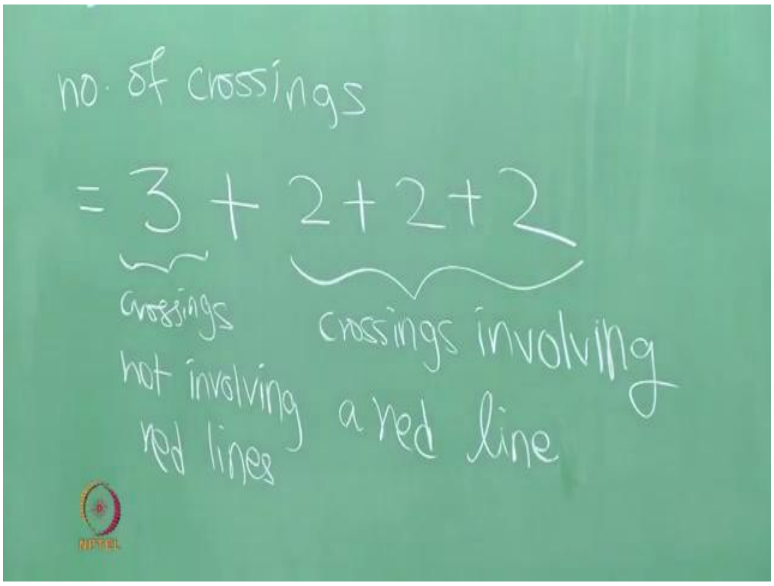

So, the number of crossing is therefore, the following there were three original crossings, plus the crossings that come from the red lines. Well, they look like 2 times, well each of those red lines contributes a 2. So, it is 2 plus 2 plus 2, one for each red line. So, this is the crossings involving the red lines. Can these are the… The crossings set only involve the white lines, so these are crossings that do not involve the red lines and not involving in the red lines.

So, as you say these are crossings that involve at least 1 red lines, so involving a red line. You count the crossings in these different ways, you look at the crossings in which at least 1 of the two curves which is a red line that counted by this. And then you look at the crossings in which you do not think about the red lines at all, only look at the white lines. And so since we are only interested in whether or not this, the number of crossings is even.

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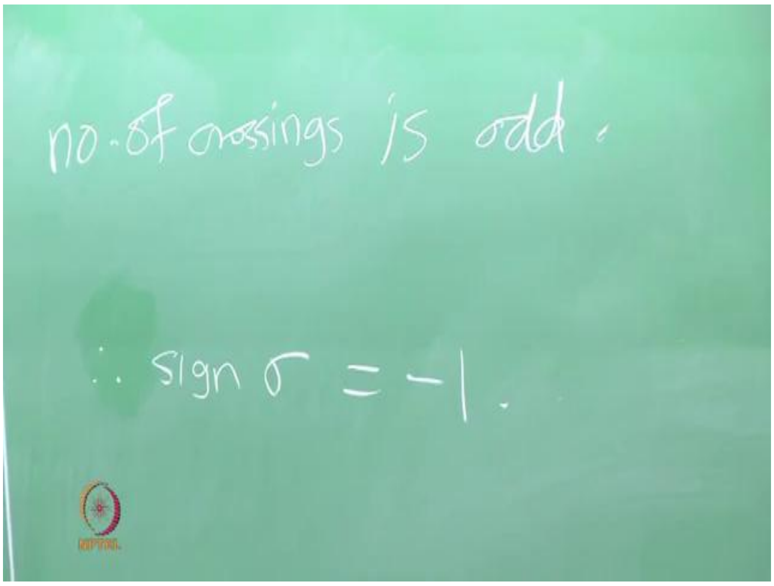


So, observe that the number of crossings in this case is still odd, because ((Refer Time: 08:37)) it really only depends on whether the original number of crossings was even or odd, the number of crossing not involve the red lines. So, this turns out to be the general phenomenon. So, if you wrote down a permutation like this, so our permutation like sigma here, ((Refer Time: 08:56)) you would often also call this a 4 cycle.

It really only permute these 4 numbers in a cyclical fashion, it does not do anything to the other numbers. So, when you have a permutation like this, the sign of this permutation is just the same as the sign of the underline 4 cycle, which is it is an odd permutation. Or if say, this where a five cycle and these were all being map to themselves, the sign of this would just be the sign of that five cycle portion.

So, this observation here is sort of it is very useful, so the number of crossings is odd therefore, the sign of sigma is in fact, minus 1. So, let us just do more complicated example now, which tells you how to put two different cycles together.

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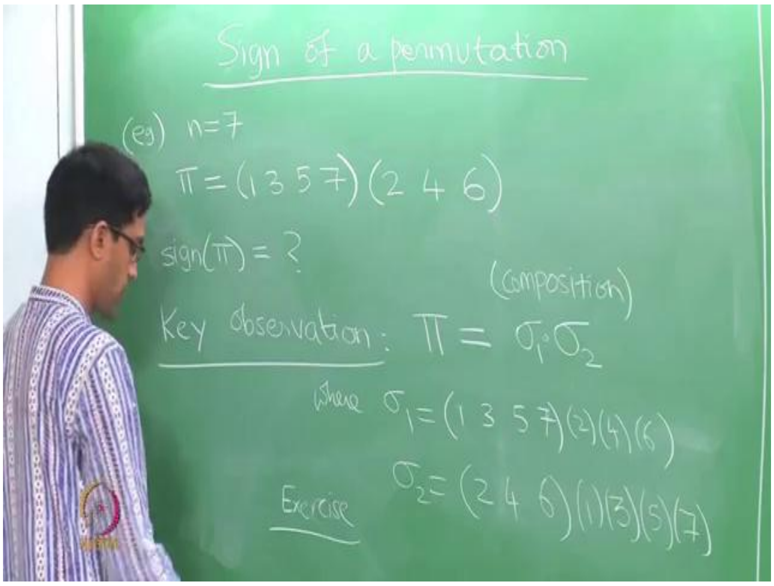


So, let us take this example now, again if we take n equals 7, take a permutation which maps 1 3 5 7 to themselves, but let us now do 2 4 6 also map cyclically announce themselves. So, it is a, this notation here remember means that 1 3 5 and 7 form a four cycle and then 2 4 6 among themselves form a 3 cycles. Now, the question is what is the sign of such a permutation? Of course, conceivably what one could do is, to try and count the total number of crossings.

We would look at you know draw the same sort of diagram in which you would have a four cycle and then tangle diagram for a 3 cycle, but then they would all be crash crossing each other in rather complicated fashion. So, instead of sort of just counting it by a group force, here is something that, here is the question. So, here is the key observation that you will used to simplify this calculation, so here is a key observation.

You can think of pi as a really mean the following, pi can we thought of as being the composition of two permutations. So, remember we talked about the composition of permutations, so this is sigma 1 composed with sigma 2. So, where well what sigma 1? Sigma 1 is the permutation that we just wrote out 1 3 5 7 the four cycle, in which the remaining three are unchanged and sigma 2 is the three cycle 2 4 6 with 1 3 5 7 unchanged.

So, the key observation is that, you can now use this notion of composition that we talked about and obtain the given permutation pi really has a composition of two simpler pieces, one being sigma 1 and other being sigma 2. So, I leave this as an exercise for you to check that in fact, the composition of these two permutations does give you back to the original permutation pi. So, now, we go back to this whole business of sign and the relationship of sign to composition and so on.

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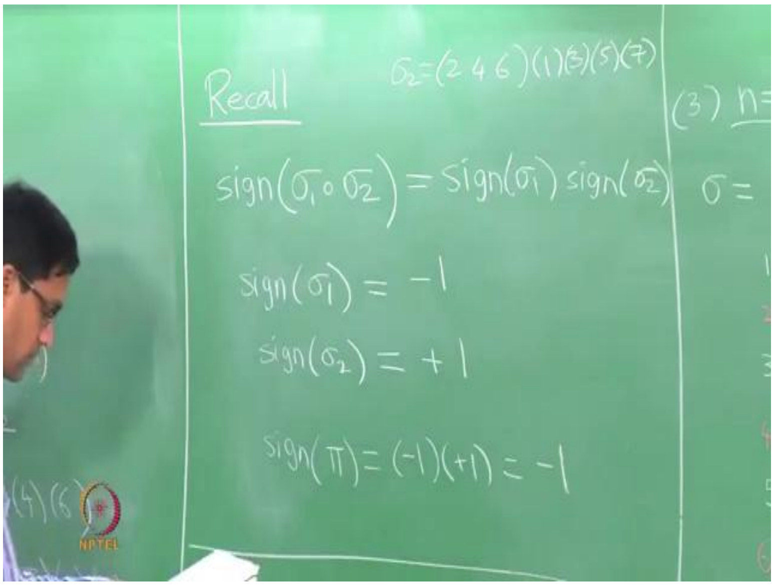


So, recall from before, the sign function is really multiplicative with the respective composition. So, if I add sigma 1 compose sigma 2, the sign of composition is just the product of the 2 signs. So, this is really the, you know the important reason why compositions are especially suited when you want to try and understand signs. If you can write your original permutation as the composition, then the sign is just the product of signs of the individual consequence.

So, here it is just a sign of sigma 1 and sign of sigma 2, but observe we have already talked about, how to understand the sign of sigma 1 and that exactly the thing that we wrote out here. Sigma 1 is really the permutation which cyclically permutes 1 3 5 7 and does nothing to 2 4 6. So, it is the sign of sigma is really just the sign of whatever this four cycle part is. The things which map to themselves the red lines, so this to speak always contribute an even number of crossings. So, they do not really contribute to the sign of the permutation.

So, the sign of sigma 1 would really just be the sign of the four cycle, so which is minus 1 as we said. And similarly, the sign of sigma 2, so notice sigma 2 in our definition was just the three cycles 2 4 6 and in which the remaining just map to themselves. So, they are sort of like the red lines in our picture, they will contribute an even number of crossings and we only really need to worry about the number of crossings between which only involve the lines joining 2 4 and 6.

So, here it is a sign of the three cycle 2 4 6, which is a plus 1 and so putting these together the sign of the original permutation pi is just the product of these two signs therefore, it is a minus 1. So, here is a way of really understanding signs by making full use of this whole of composition as an operation.

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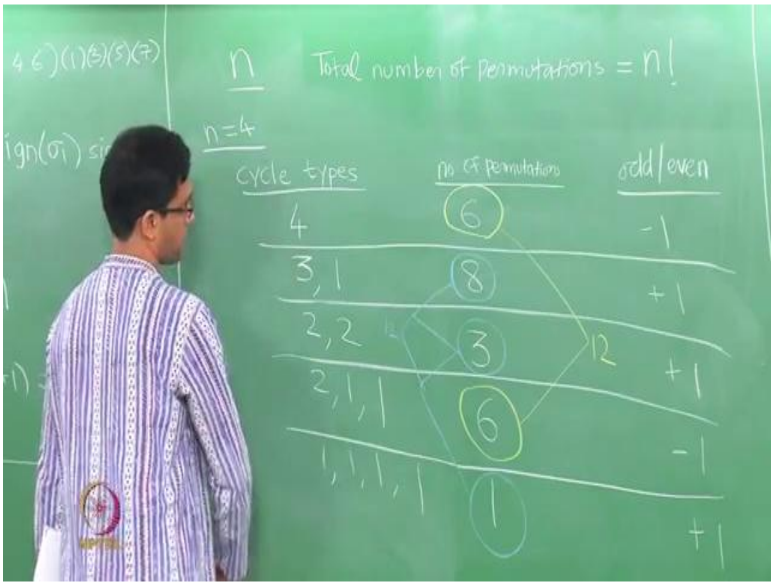


So, and finally, using this business here is asking that would be nice to do in this business. If you have n, let say a natural number let us do the following, total number of permutations of n, remember it is just n factorial. So, for n equals 4 for instance, so recall this whole table that we talked about, the notion of cycle types or cycle structure. So, the various cycle types that we talked about, you can have four cycle or you can have a three cycle and a one cycle, you can have two cycle and other two cycle, two cycle and two one cycles or four one cycles.

These are the various possible cycle types, when you trying to permute four numbers and the number of permutations of each type. So, the number of permutations of each of these cycle types, we are worked out before is 6, 8, 3, 6 and 1. And finally, since we talked about signs right now, it is natural to wonder which of these permutations are odd and which are even. So, we could ask which of these permutations are odd and which are even and we already have the techniques to answer these questions.

For instance, a four cycle if you have a permutation which is just a four cycle, we just talked about signs of cycles. A four cycles is always odd, yes I just write minus 1 for odd. Now, if I have a three cycle together with the one cycle, so we again just did this. When you have, say a cycle the composition of the products of two cycles, it is you can think of it as a composition of two permutations and so it is just the product of the signs of each of these case.

So, a three cycle would be even and a one cycle would be even, the product of two evens is again even. So, this is a sign plus 1, similarly here we do the same thing here, there is a two cycle composed with the other two cycle. Two cycle is odd and other two cycle is therefore, also odd. So, product of two odds, odd permutation is any one permutation. Now, here a two cycle is odd, one cycles are both, so odd times even times even, we still give you an odd permutation.

And finally, each of these ones is really an even permutation, so this is the full list which of given a cycle type, is it odd or even and the answers are right there. So, let us count the total number of odds and total number of even permutations. So, which of these are the odds, well the first row there are 6 odd permutations with cycle structure 4, there are another 6 odd permutations with cycle structure 2 1 1. So, between these there are 12 permutations with 12 odd permutations and the even permutations are there are 8 of them with cycle structure 3 1, 3 of them with cycle structure 2 2 and finally, 1 with cycle structure 1 1 1.

So, these three together again gives you 12, so there are half the number of odd and half the number of even permutations. And this is in fact, statement which is true in general, that n factorial divided by two permutations should be even and the remaining n factorial by two permutations should be odd. So, it is probably a nice exercise to try and proof this fact that in general, half the number of permutations are odd and the other half are even.

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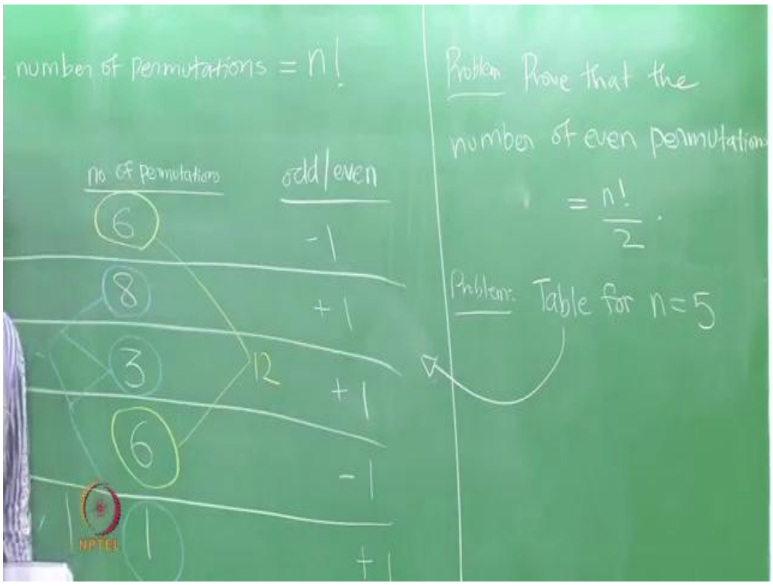


So, let me just state this as a problem, problem prove that the number of even permutation is n factorial by 2. And other exercise is to try and work out this table that I just it for n equals 5, so here is another exercise problem that I like you to try out is to work out this table. So, write down the entries of this table for n equals 5. So, the same table that we just true, write out all the possible cycle structures for n equals 5.

And for each of them, work out the number of permutations with that cycle structure and also work out whether that permutation would be an odd permutation or an even permutation. And from that conclude that in at least, see that half of n factorial. So, 5 factorials in this case is 120, so you should get 60 even permutations and 60 odd permutations in order. So, next time we will talk about little bit more about permutations.