An Invitation to Mathematics
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Unit
Combinatorics
Lecture - 15
Rules for drawing Tangle Diagrams

Welcome back, so today we will talk about Tangle Diagrams. So, this is just the terminology that sometimes used to describe the pictures that have been drawing for a permutation.

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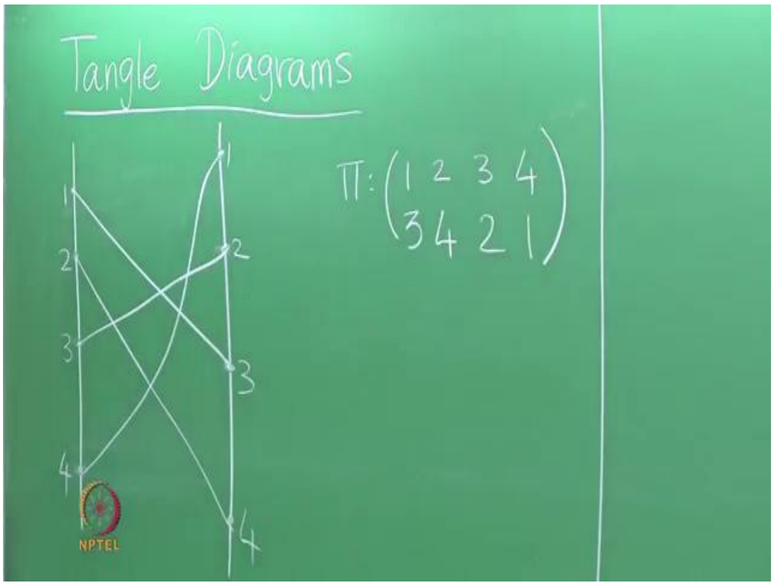

So, here is roughly what we have been doing, so if we want to depict a permutation, say let say we are looking at a permutation of 4. So, we draw four points, label 1 2 3 4 on either side, 4. And so suppose I want the permutation pi which is let say. So, let me write it in two line notation 1 2 3 4 maps to let say 3 4 2 1. So, 1 maps to 3, so we join 1 and 3,2 maps to 4 which means we join 2 and 4, 3 maps to 2 and 4 maps to 1.

And so, the question of course is, you know what are the rules for drawing these diagrams. Because it seem, so one rule we looked at already, we said it is best to avoid getting three or more lines to pass through a single point, because that makes counting crossings somewhat harder. But, we also need to understand, what are all the admissible ways of drawing these diagrams?

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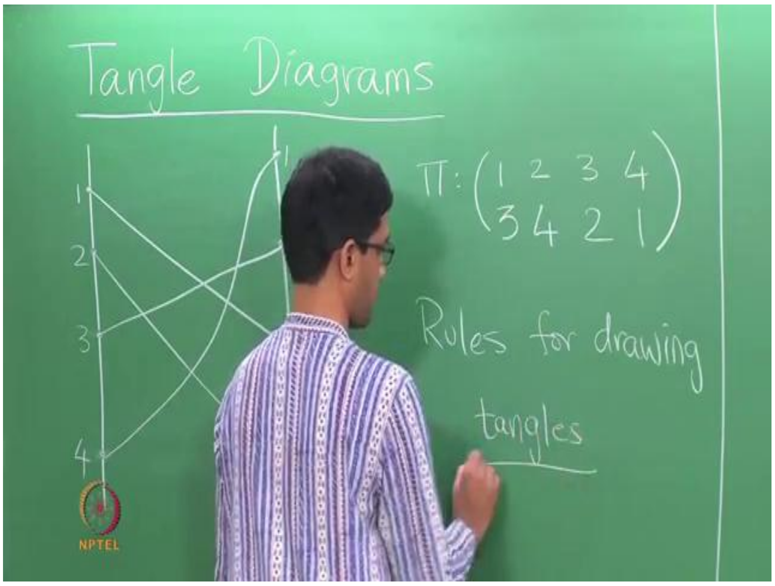

So, what are the rules for these diagrams? So, the question is what are the rules for drawing such a diagram? So, this is sometimes called a tangle diagrams. So, what are the rules for drawing tangles?

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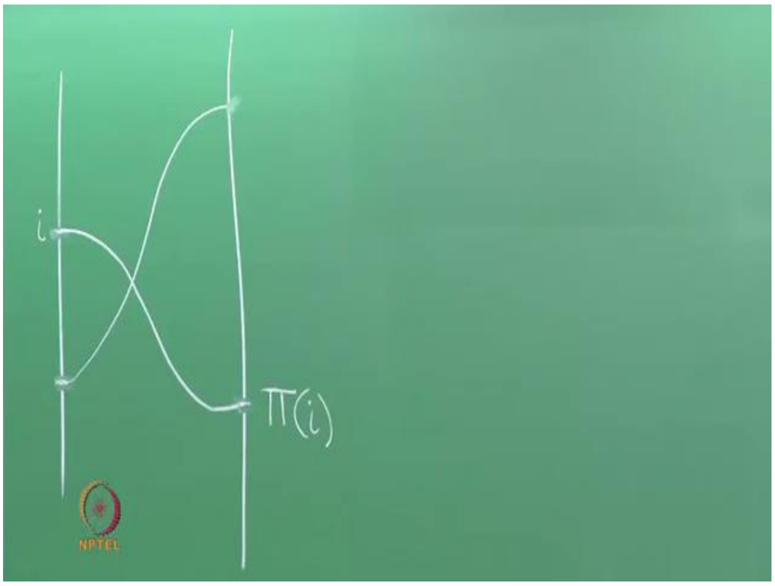

So, of course, the primary thing is of course, what is a tangle, it is for each i, so you draw two lines like this for each i. If you want to represent a permutation, you draw a curve which joins i and it is image under the permutation which we called pi of i. So, and similarly you pick a different value of i and you join it to whatever it maps to under the permutation and so on. So, it is nothing, but, a bunch of curves drawn between these lines.

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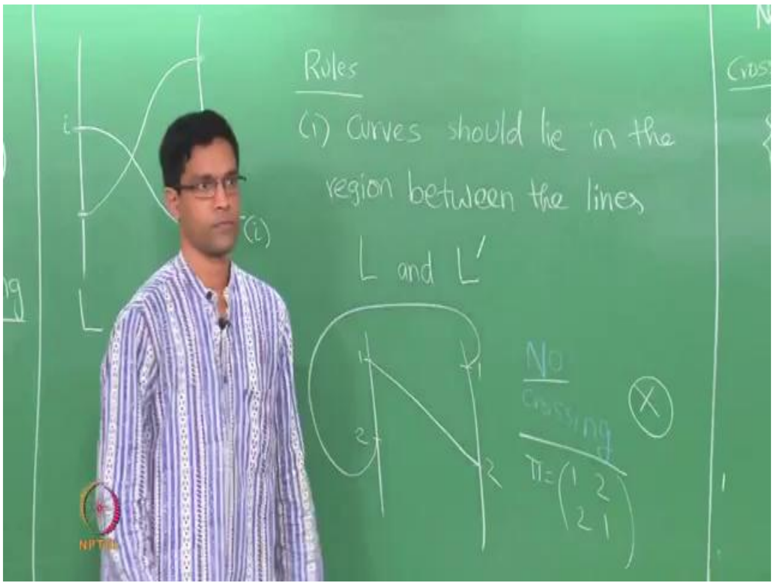

Now, what are the rules well for a start, this curve should lie between these two lines that are drawn here. So, let us call these lines as L and L dash, so what we want is the curves should lie in the region between these lines. So, if for instance we do not follow this rule, then here is what you might do. So, for instance if you wanted to depict a permutation, say which sense 1 to 2 and 2 to 1, so you join 1 to 2 in this way and suppose we join 2 to 1 like this.

Now, doing this would not produce a crossing. So, observe that if you did this, this does not lead to a crossing between these two lines, but recall that we do want to have a crossing in this case.

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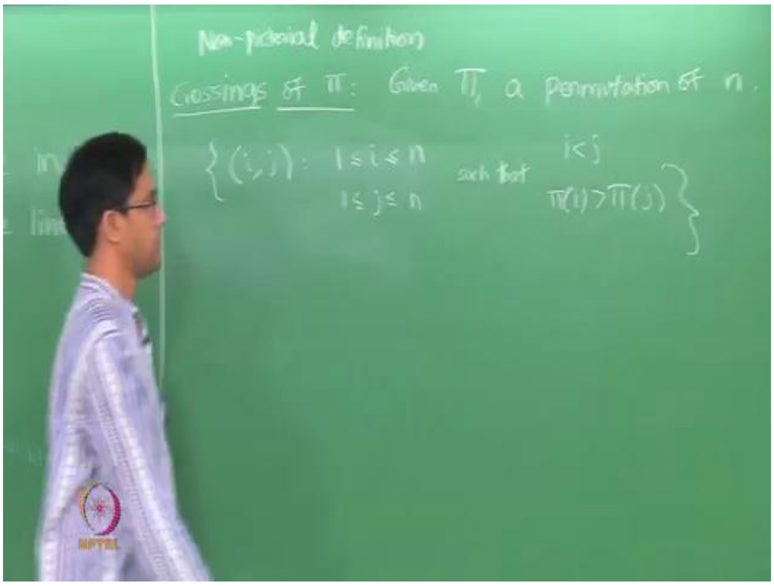

So, recall what does crossing, the sort of the non pictorial definition of a crossing mean. So, what are the crossings of a permutation pi, this is the sort of more formal definition, this is the non pictorial definition of crossings. So, recall from one of a previous lectures that, this just means you pick all pairs, ordered pairs i comma j were, well what are i and j, they are numbers between 1 and whatever n, where n is a… So, let say pi is a permutation of n, so given pi a permutation of n.

You look at all pairs I j, where i and j are numbers between 1 and n, satisfying the following property such that i is smaller than j, but the image of i is greater than the image of j under the permutation. So, this is exactly what we would want to understand by crossings or inversions, but when we draw pictures we want the pictures to correctly capture this notion. So, if you allowed for the curves to go outside this region r, then here is a perfectly legitimate picture, but it fails to show a crossing between these lines.

Well, in fact, there is a crossing because this permutation pi here is just a permutation which seems 1 to 2 and 2 to 1. So, of course, i could be 1 and j is 2 and that would satisfy the formal definition of a crossing. So, this is, this picture is bad for the reason that it does not show crossing when in fact, there is 1, so we do not want to allow this, so that is one.

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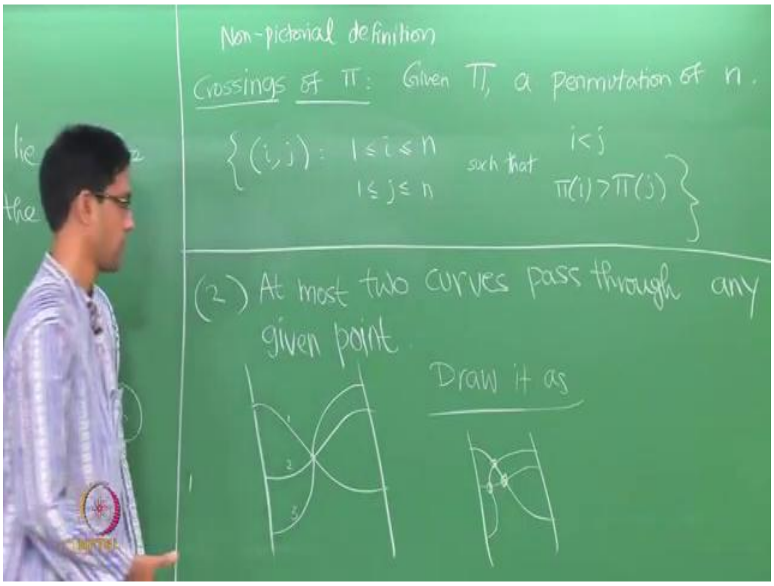

So, let us look at rule number 2, so this something that we have already looked at. You do not want too many curves to pass through the same point, so at most two curves pass through any given point. And the reason for this is, while this is not two important the rule, but the reasoning behind is this that it makes for counting crossings much easier. So, for instants if you allowed say three curves to go through the same point, in some sense what this means is that there are three crossings at this place.

So, the first curve here and the second curve cross each other, the second and the third curves cross each other and the first and the third curves cross each other. So, this actually counts for three crossings, but that much harder to actually count you know when you are doing pictorially. So, what you want to do is to avoid having more than two curves pass through the same point. So, whenever we see a configuration like this, we would ideally want to draw it like.

So, we would want to draw this particular tangle as follows, you would say you want this to go here. You would want the second fellow to go here and when you want to draw the third guy, you draw it for instance slightly shifted. So, that now there are in fact, three crossings, so each of these three here shows up as a crossing. So, this picture on the left is not allowed, but what you would ideally want to do is draw it like the picture on the right.

So, that is the second rule for drawing tangles. So, number 1 you should stay within the regions defined by the two lines, number 2 you should avoid having more than two fellows crossing at the same point, but that still allows for a wide range of ways of drawing tangles.

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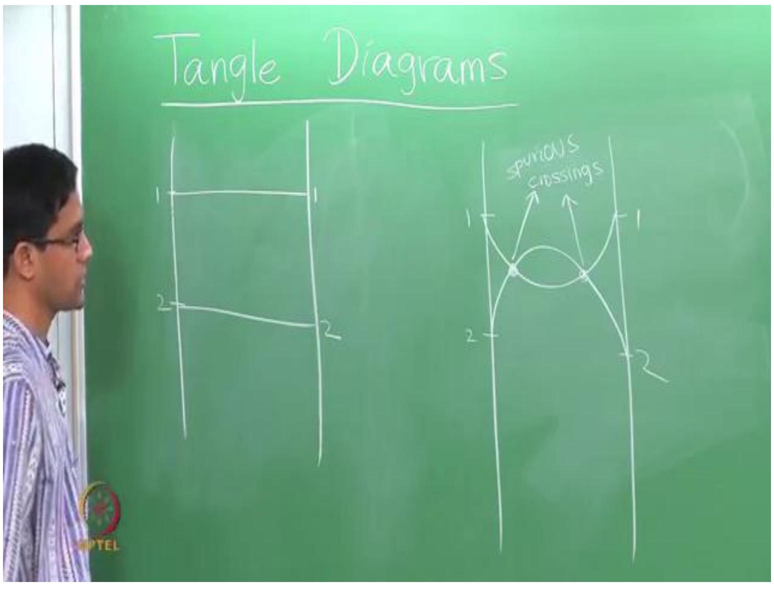

For example, suppose we just had a simple permutation, say 1 goes to 1, 2 goes to 2.

Now, just allowing for rules 1 and 2, you could very well draw this as follows, you might have drawn this by saying, well I need my curves to stay within the lines. So, let me send 1 to 1 and let me send 2 to 2. So, this of course, satisfies both rules 1 and 2, but what this does is, it produces two spurious crossings.

So, here and here are well, they are not really crossings, they should in be crossings, but what one has done by virtual of drawing the curves in this way is to have introduce two spurious crossings, which should in really be there, you do not want to think of them as crossings. Because, for instance this is really the correct diagram for the permutation and it does not have any crossings, 1 goes to 1, 2 goes to 2. If you apply the formal definition of a crossing, this does not have any crossings. So, you do not want to allow things like this and well, there are other things you could have done, you could have even looped. So, for instance what is another way of producing a spurious crossing?

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Similarly, if I had just 1, let say I even had a permutation of a single number 1 going to 1, I might have drawn it like this, 1 goes to 1 which again ends up producing a spurious crossing here. So, you really do not want to have configurations like that.

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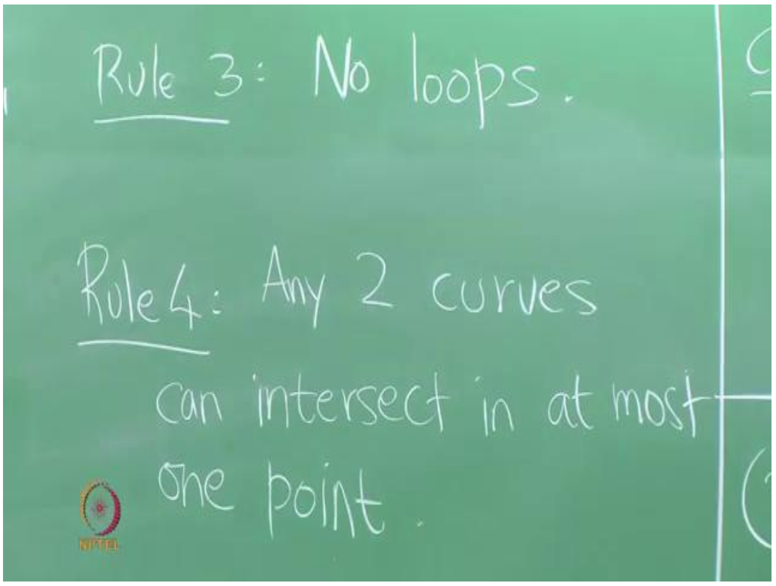

So, what we really want to do is to write down a couple of rules which will prevent these sorts of things. So, rule number 3 says there should be no loops, you are not allowed to have loops in a curve, it should in intersect itself and rule number 4 says, when I have two different curves, for instance the configuration that we just root out here. If you have two different curves intersecting in two points in this way, if that happens again it means there is a spurious crossing.

So, rule number 4 says any two curves can intersect in at most one point, so each pair of curves intersects at most ones. So, if you put that in place, then the configuration that we drew with two curves sort of, you know hitting each other along a loop like that would it be allowed any more.

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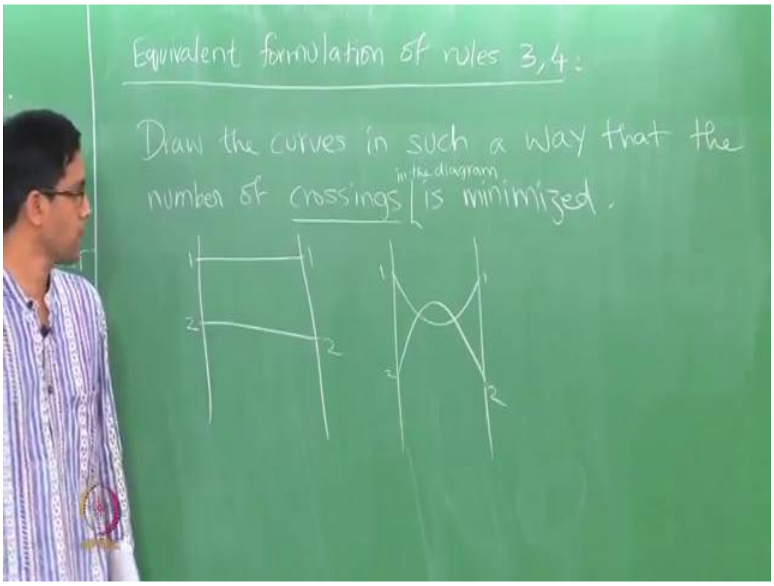

So, another way of formulating these rules 3 and 4 is to say, you just draw the tangle in such a way as to minimize the number of crossings that you produce. So, here is some equivalent formulation of rules 3 and 4, it is to say you draw it in a such way ask to minimize the number of crossings, draw the curves in such a way that. Set that the number of crossings that you produce in the diagram is minimized.

So, if you keep this in mind, then for instance the way to draw just this permutation 1 going to 1, 2 going to 2 would be this which produces no causing's at all. Now, if you did anything funny like we depicted in our earlier diagram, it still is going to send 1 to 1 and 2 to 2, but this way of drawing the diagram produces two crossings. But, by sort of moving those two curves apart, you can further minimize the number of crossings.

You can just, you know get zero crossings as suppose to 2 crossings in that way of drawing the diagram. I should, it is a sort of emphasize the number of crossings in the diagram, so here when I say crossings I am using it in the sense of place where these two curves cross not in the formal definitions sense of a crossings for a permutation. So, draw the curves in such a way that the number of crossings in the diagram is minimized.

So, that is sort of the one easy rule to keep in mind when drawing these diagrams. But, even with all these rules, in terms out there are many different ways of drawing the same tangle. And one should check that in fact, no matter which way one draws it, the number of crossings that you finally produce is always the same.

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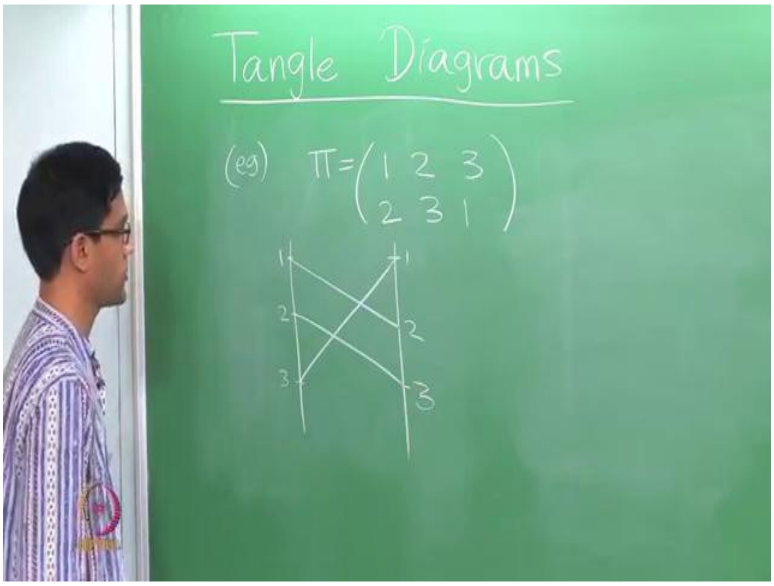

So, you can still draw rather complicated looking tangles. So, example if you took the three cycle, if you take a permutation pi you just be 1 goes to 2, 2 goes to 3, 3 goes to 1, then you could depict it like this. So, 1 maps to 2, 2 maps to 3, 3 maps to 1, here is the tangle diagram for the permutation pi.

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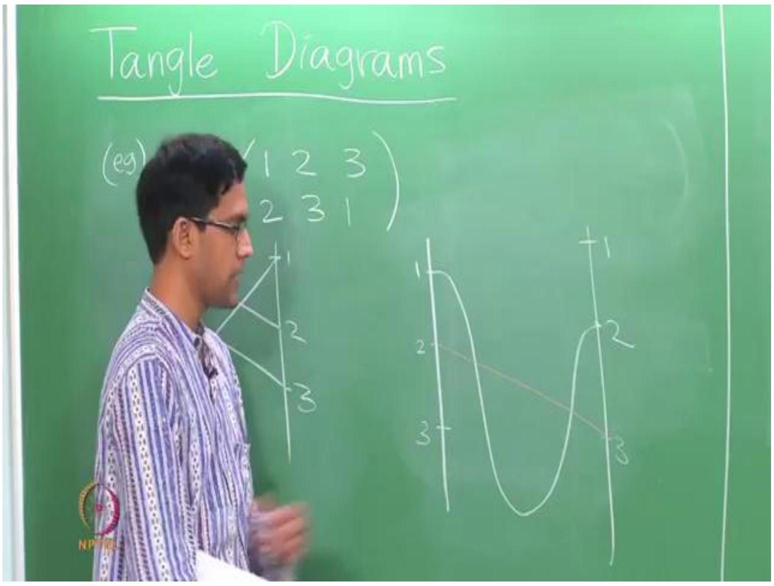

But of course, you could still do things like. So, you just want 1 to go to 2, so let see, so maybe I do this, 1 maps to 2. So, now, I need to look at how to get 2 to go to 3, now and then finally, I need to make 3 to go to 1. So, when 2 maps to 3 I should, so if I sort of make it go across, so here is one obvious thing one might of thought of. So, let us just try and make 2 go to 3 like this, but that would violate rule number 4, because these two curves now, the first curve that we drew and the curve in red, they intersect in two points. So, we are not allowing double intersections.

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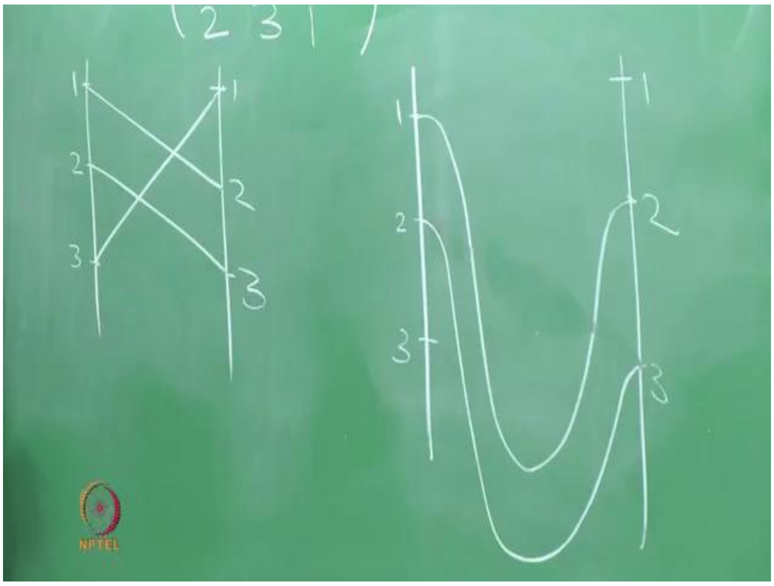

So, what one could want to do is to redraw this curve, so we modify in the following way. So, what would we want to do? We would want to make it, this is the only thing you can do. If you make the two of them intersect once, then they will also have to intersect the second time, there is no way of making it intersect exactly once. No matter, so you can try drawing this in various ways.

If you try to join 2 and 3 by drawing any curve which remains within the two lines, if you hit this first curve once, then you sort of have to hit it once more in order to try and reach the point 3. Since, you are not allowed double intersection, the only thing you can do is to have no intersection at all. So, you have to go for instance all the way down. So, as you can see that, you know the usual way of drawing it is the straight line, we joins 1 to 2 and 2 to 3, these two lines of course, do not cross. So, there is no crossing between those two lines and that is what happens even if you draw the curve in a much more complicated way. And finally, you want to draw join 3 to 1.

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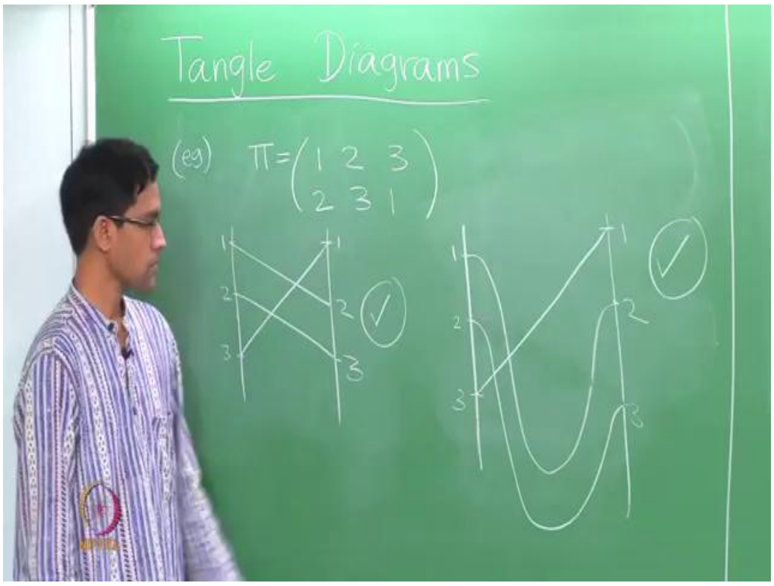

So, joining 3 to 1 again you know the simplest thing would be to just do this and that only has 1 crossing each with each of these two curves. So, that is a perfectly legitimately of a problem. So, this is also, so this each of these is a perfectly valid tangle diagram which depicts the permutation, the three cycle 2 3 1 and you can sort of play with these things a little bit more. It is good to have the flexibility to draw it in various ways you know not always having to draw it by straight lines.

There will be situations where you might want to see some proves pictorially and there, it is useful to have more flexibility in drawing these tangle diagrams. So, these are the various rules to keep in mind when you encounter tangle diagrams or when you want to draw tangle diagram yourselves. So, we will try and use this little bit more next time to try and understand the notion of sign and so on.