An Invitation to Mathematics
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Unit
Combinatorics
Lecture - 14
The Sign of a Permutation, Composition of Permutations

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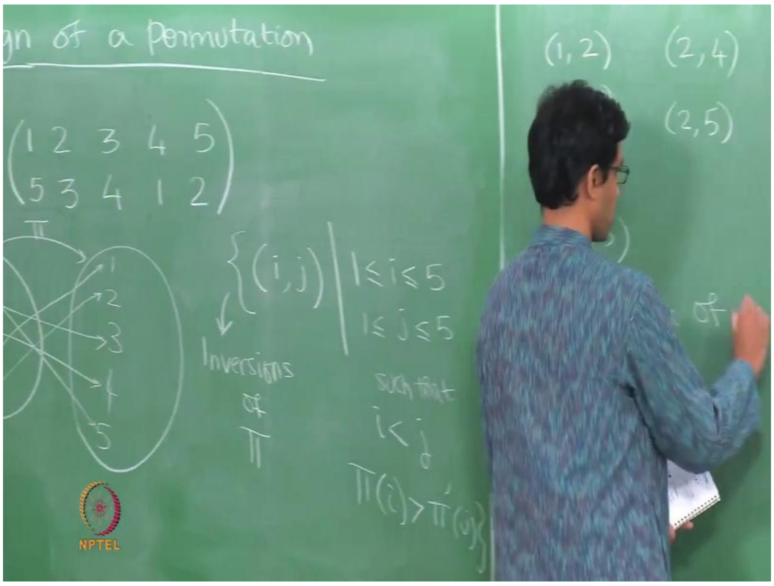

Welcome back, so what I want to talk about today is the notion of a Sign of a Permutation. So, again we will do this by example, so let us look at the following permutation written out in two line notation. So, now, look at the numbers 1 through 5 and here is a permutation which is 5 3 4 1 2. So, let us call this permutation pi, so again recall two line notation, it just means that these are just the entries this is I mean the bottom row is really the permutation of the numbers 1 through 5. So, often we know also think of it as functions, so we will write these entries as follows.

So, if you call this permutation as pi, we also say that pi of 1 is 5 for instance. So, recall the function way of thinking about it and this function is usually also denoted by the same letter pi. So, 1 goes to 5 pi of 2 is 3 and so on, pi of 3 is 4, pi of 4 is 1 and pi of 5 is 2. So, now, let us consider the following set, let us look at all these pairs i comma j now what are i and j well i is number it mean 1 and 5, j is number it mean 1 and 5, where i is smaller than j, where such that the following happens that i is number which is strictly smaller than j, but the image of i under this permutation pi of i is strictly bigger than pi of j.

So, given a permutation pi here is a set that one can naturally consider, which is you scan the list and you look for all i and j. So, entries for instance you look at in this example look at 1 and 2 for instance i is 1, j is 2 i is smaller than j, but pi of 1 which is 5 is strictly bigger than pi of 2 which is 3. So, in this example let us write down what are the i's and j's which satisfy this property you can have...

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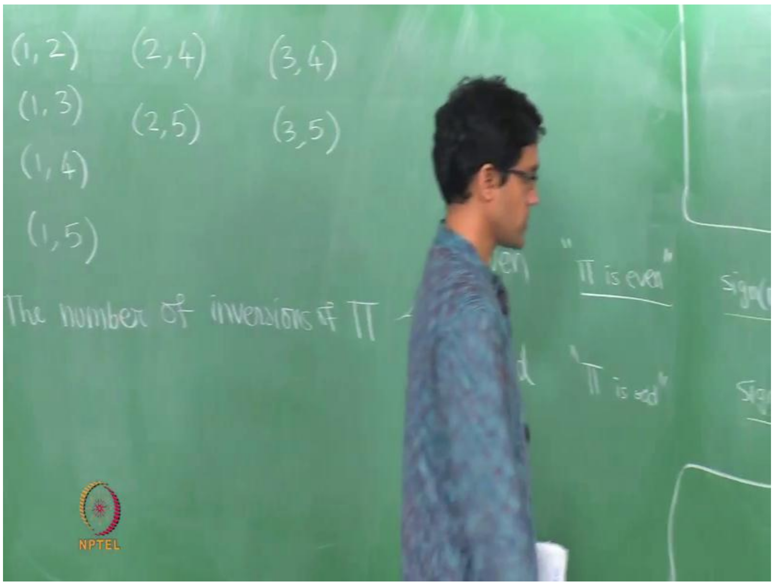

So, in this case this set consist of the following elements it has 1 and 2, because 1 maps to 5, 2 maps to 3, similarly it has 1 3, 1 4, 1 5 then it also has 2 4, 2 5, 3 4 and 3 5. So, let us just check if you look at 2 4 if i is 2 and j is 4 pi of 2 is 3, let us pi of 4 is 1. So, 2 is smaller than 4, but pi of 2 is strictly bigger than pi of 4 and similarly if I have 3 5 for instance 3 is smaller than 5, but what it maps to 4 is bigger than what 5 maps to which is 2.

So, this is the list of elements sometimes it is called the inversions, these are the places where you know if there were no permutation if everything map to itself, you could not have anything out of order. But, because pi in general reorders the numbers you will typically have pairs which are sort of out of order where in the wrong order in some sense. So, this is that full list of elements whose order is somehow change by the permutation pi.

And so let us just say something more, the set here the number of elements in this sets. So, that will be important for us. So, like a set sometimes calls the number of inversions.

So, the number of ((Refer Time: 04:50))... So, it say this is the set what we call the set of inversions of pi, the number of inversions of pi well there are only two possibilities it can be an even number or it can be an odd number.

And if the number of inversions is even, then we say that pi is an even permutation and if the number of inversions is odd, we say that pi is an odd permutation. So, in this case if the number of inversions of pi is even, we will say that pi is an even permutation. So, that is the terminology and if a number of inversions is odd, we would say that pi is an odd permutation and we also have a notion of sign which is another way of saying the same thing.

So, equivalently here is another thing we do, so pi is even that is one way of saying it, another equivalent way of saying this is to say that the sign of pi is plus 1 and if pi is odd we define something called the sign, we say the sign of pi is minus 1. Again these are all for now you can think of it as various different ways of saying the same thing, it will become somewhat clear or as we go along as to what the advantages are of thinking of it in terms of plus 1's and minus 1's. So, this is just the notion of even permutation and odd permutation. Now, here is a very nice pictorial point of view to keep in mind. So, let us pretty much draw the same picture out there, but maybe a little larger.

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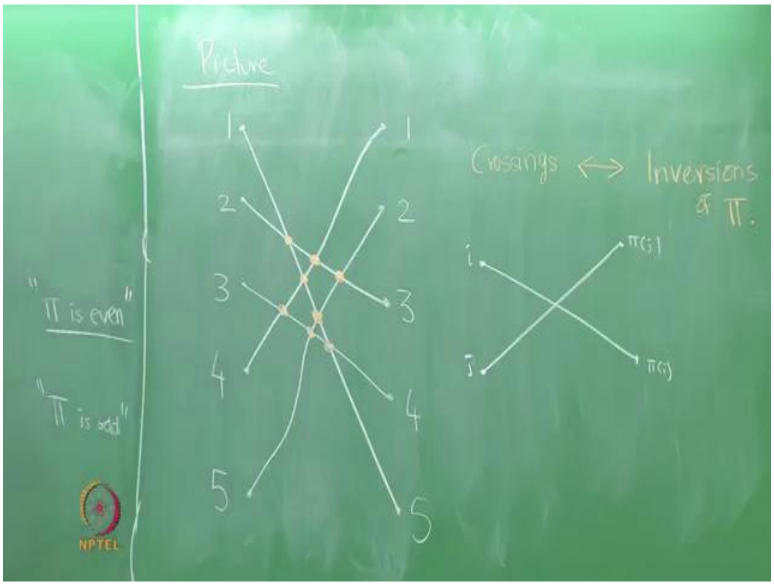

So, here is the picture again of pi as a function, so I have the numbers 1 2 3 4 and 5. So, try to draw them reasonably equally space and on the other side have the same numbers again 1 2. So, I am going to draw the function diagram what does pi do to each number.

So, pi maps if we just said it sense 1 to 5, so I will draw it by as close to a straight line as I can, so I will say pi maps 1 to 5 what is do to 2, it maps it 2 3. So, 2 goes to 3 it maps 3 to 4 it pairs 4 to 1 and it maps 5 to 2.

So, here is the diagram which depicts what pi does to the numbers 1 through 5. Now, observe that the set of inversions has a very simple interpretation in terms of this diagram. So, let see what where the, what you know how many inversions did we have in this example, let start with that. So, if you look at the number of inversions we had 1 2 3 4 5 6 7 and 8, so there are eight inversions. Now, let us look at this picture and count the number of crossings, the number of places where these lines intersect. So, as 1 write there 2 3 4 5 is 1 here 6 7 and 8.

So, the number of crossing is exactly equal to 8 as well and this not sort of coincidence, this is pretty much just pictorial way of looking at inversions. So, observe that crossings in this diagram crossings or intersections really correspond to inversions of pi and why is that, it is a very simple observation in a what is an inversion of pi mean, it means that you have two numbers i and j, this is the definition that i is smaller than j.

So, let me draw it in this diagram, let say i is this and j is something larger than i. So, i could be 2 and j could be 4 for instance, but pi of i which means what i maps to under the transformation pi and pi of j what j maps to they are the opposite order. So, if pi of i is say something here, pi of j is going to be something larger than that. So, this is pi of i and this is pi of j, this is exactly what an inversion means.

So, every time there is an inversion, there is a pair i j which gives you an inversion of pi, then in this diagram what it would gives the crossing, those two lines would have to cross somewhere. So, counting the number crossings in these kinds of diagrams is a same as counting the total number of inversions of pi. Now, there just one little thing one have to worry about when drawing these diagrams, one will short of have to take care to ensure that you do not draw it in such a way that three lines or four lines or more than two lines intersect at a point. So, I will drawing these diagrams should avoid drawing them such that more than two things, more than two lines meet at a point.

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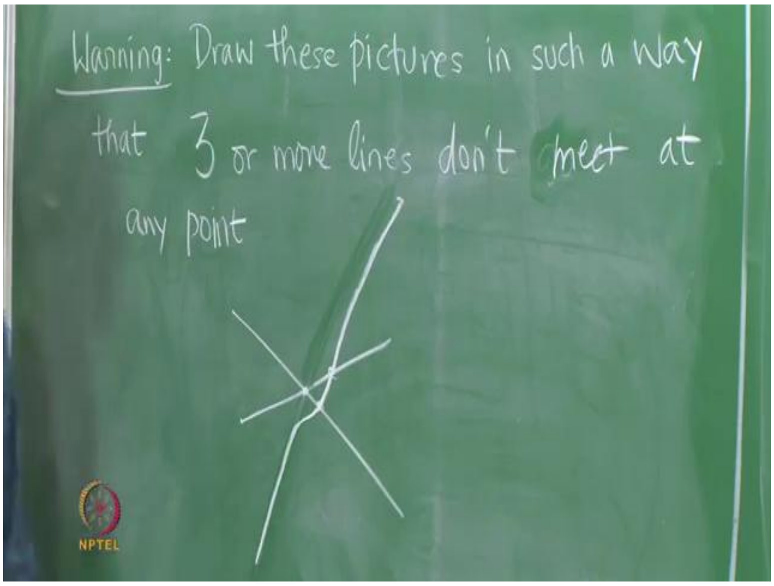

So, just a little warning well drawing these diagrams is draw these pictures in such a way that at most 2 lines meet at a point at most well meaning that then the different way of re facing it that 3 or more line. So, in such a way that we do not have 3 or more lines meeting at a point, so for instance if I had say something like this, so I have say for instance is and say if you do the third one also going through the same point, then counting the crossings becomes somewhat trickier. Well, technically what is meant here is that these two lines meet, these two lines meet and the first and the third lines also meet at that point.

So, technically speaking this is a point at which this really counts as 3 crossings, there are sort of 3 pairs of lines which sort of go transfers to meet each other. But, if you just drawing it as a single point, then it is not so easy to do the counting when one is trying to really make sense sort of it. So, what one should do when faced with the prospect of drawing such diagrams is and this something that I did violet do the previous diagram is to sort of just make the lines go gently around.

So, if it seems that I like to go through that just sort of slightly quick them. So, that now you will see in this diagram that there are 3 crossings. So, there is this crossing here and because we move the line a little bit this other to crossing is also clear enough to see. So, these the only thing to keep in mind when trying to draw these in such a diagrams, but this a very nice thing to have is pictorial way of counting inversions or crossing.

So, now, what would have odd permutation mean, well how do you check it something is odd you will draw this diagram of pi and then count the number of crossings. So, the number of crossing is odd then it is an odd permutations, if it is a need if the number of crossing is even it is a even permutation.

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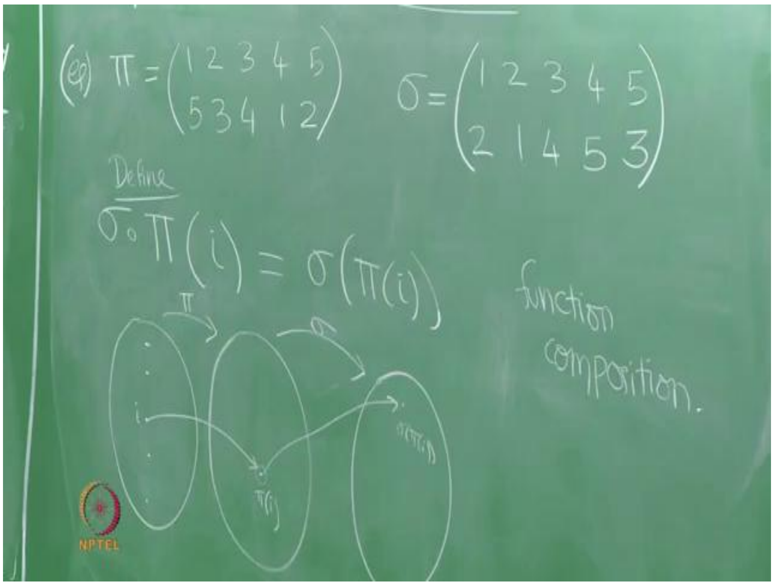

Now, this hole business of sign is especially interesting when one studies what is called composition of permutations. So, there is a notion of composition of permutations, so what is composition mean, well it means the usual notion of composition for functions. So, let us again do this by example, suppose I have two permutations pi is say again an two line notation it is 1 2 3 4 5, 5 3 4 1 2 and I have another permutation called sigma which is the numbers 1 2 3 4 and 5 or map to 2 1 4 5 3.

So, these are main two permutations pi and sigma, now what is sigma composition pi. So, here is the new operation it is a way of generating yet another permutation, so let us call it sigma circle pi or sigma composition pi. So, what is this defines us well it is a new function which is just the composition of these two functions. So, sigma composition pi is defined as follows. So, define a new permutation as follows how do you figure out what this does to pi, well this is just sigma of pi of i. So, this is just the usual notion of function composition.

So, if you think in terms of functions, so here is the function pi, so you first perform pi and once you have done something to pi something through pi then you follow it up pi sigma. So, for instance I have the numbers 1 2 3 4 and 5, so I let say have is a some number i here I first figure out what pi does to it, it may be max to set some other number and then whatever that number is I see what sigma does to it is, so may be sigma map to this number.

So, these two steps one after the other, that tell me what i finally, maps to it goes to sigma evaluated on pi of i. So, this is the number pi of i and this final answer is the sigma evaluated on pi of i. So, composing to functions is just done sort of in this pictorial set up. So, let us actually do it for these examples, because we already have one of these pictures of on the board. So, here what I have is already the picture for pi.

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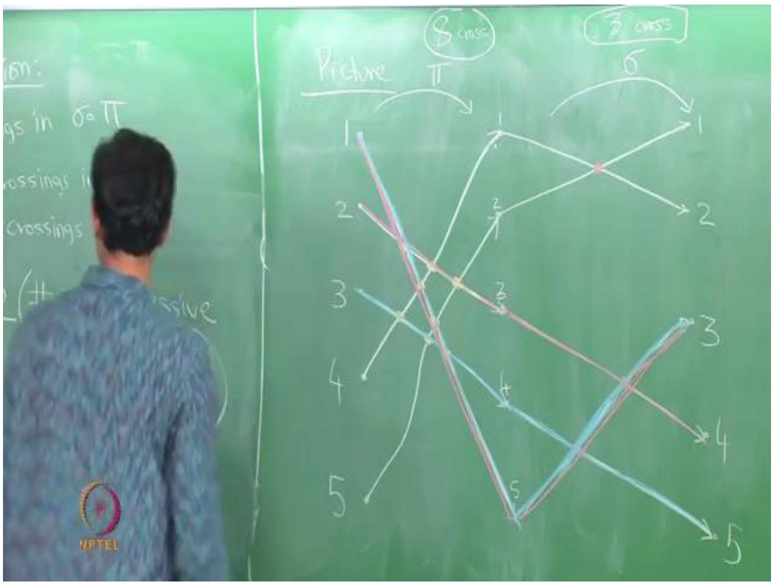

So, recall that the first appear, this first map is already the picture for pi, so maybe I should draw arrows. So, 1 2 3 4 and 5 and pi get map to what are they get map to, so we drawn the crossing diagram, now let us look at the function sigma. So, we need to do this one more step, so let us look at the function sigma and see what sigma does to these five numbers. So, let us do this let me is the write the numbers on top is let us a 1 2 3 4 5, so let us look at sigma what is it due to the number 1, well sigma maps in 2 to the number 2. So, first let us again write down 1 2 3 4 5 on the other side.

So, what is sigma do to the number 1 valid maps it to the number 2 that is according to, so 1 maps to 2, what is sigma due to the number 2 valid it maps to the number 1. So, I draw this diagram, 3 maps to 4, 4 maps to 5 and 5 maps to 3. So, what we have done now is also drawn the diagram for the picture for sigma. And let us again do the same thing let us mark out the crossings, see you some other calling here. So, how many crossings just the diagram for sigma alone have and here is a one crossing, here is another crossing, here is the third crossing. So, that is seems to be there are 3 crossings.

So, we call it pi had 8 crossings and sigma seems to have 3 crossings, now the question is what is the composition b. So, pi here, so let we write down the number of crossings, so pi has 8 crossings and sigma has 3 crossings. Now, we just talked about... So, of course, what are the odd and the even is... So, 8 crossing means pi is a even permutation and 3 crossings would make sigma and odd permutation.

Now, let us look at the composition that is really where these diagrams become very interesting. As we said the composition is just you take the first diagram and follow it by the second diagram. So, let us draw the composition diagram, so I will take these two things and I will compose them. So, let me saw the composition looks like.

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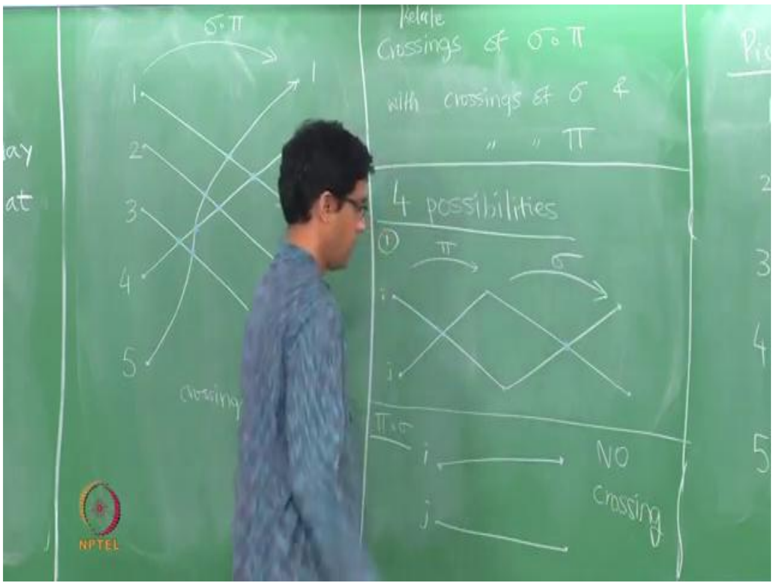

So, again is write down numbers is on and let just compose these two things. So, for instance 1 maps to 5 which in turn maps to 3, so 1 should go to a 3 under the composition. So, this is I am now going to draw the diagram for the composition of sigma pi. So, 1 maps to 3, so I am just using the definition you should just check that this is in fact, correct by actually using the definition.

Now, 2 maps to 3 which maps to 4, so 2 maps to 4 similarly 3 maps to 4 maps 5. So, 3 maps to 5, 4 maps to 1 which maps to 2. So, 4 maps to 2 under the composition, and finally 5 maps to 1, so again here I need to keep in mind that warning that I said about not making three things pass through the same points. So, I will sort of just tweak my line in a little bit, so has to make it go like this.

So, what I have drawn now is the diagram for the composition and again let us count the number of crossings. So, here is 1, here is another, here is another 4 5 6 7, so here the number of crossings seems to be... So, the number of crossings is 7, so what we now have on the board are both diagrams one hand we have the diagrams for sigma and pi. And we notice they had a 8 and 3 crossings and on the other hand we have the diagram for sigma composition pi, where we notice they have 7 crossings.

Now, here is the question how do we a relate the number of crossings of sigma pi and sigma composition pi. So, I want to do the following I want to relate the number of crossings of sigma composition pi relate crossings of this with the number of crossings of each of the two pieces, crossings of sigma and number of crossings of pi. So, how do we figure out what happens. So, let just do it case by case let us look at what all the various possibilities would give us and then we will see if we can make sense out of this.

So, what are the 4 possibilities, so really there are only 4 possibilities. So, 4 possibilities for what well here is a typical what are the various possible diagrams. So, I first have pi and then I follow it by sigma, so this is what happening when my take the composition of these two permutations. Now, here is what could happen, I could have say two numbers i and j here which under pi give me a crossing. So, pi i maybe maps to something large j maps to something small, so therefore, the cross these two lines cross each other.

And now I have two result and numbers here which I need to further map and it sigma and may be sigma does same thing to them, sigma again leads to crossing it inverts the order of these two elements. So, here is one possibility that I have the numbers i j for this pair pi gives me a crossing and for the resulting numbers sigma again gives me a crossing, so this is two successive crossings.

Now, suppose such a thing happen then what could happen, so this is case 1, this is the very first possibility that I have two successive crossings. Now, if this happens then... So, I have you know each of them gives me 1. So, pi gives me 1 crossing, sigma gives me 1 crossing, but when you look at the composition of the 2. So, what happens when you look at pi composed with sigma, so this is the individual picture if I look at pi composed with sigma and here is what it does I have i j.

So, first the result pi of i is bigger than pi of j, but then sigma of pi of i is smaller than sigma of pi of j. So, when I compose the two i maps to something which is smaller than whatever j maps 2. In other words, if I have one crossing followed by another crossing then they undo the affect of the each other. The final answer in the composition of these two things sigma of pi of i would actually be you know you would be smaller than sigma of pi of j.

So, two successive crossings lead to know crossing in the composition, this leads to no crossing, let us actually see it in action. So, let us look at an example where we have two successive crossings, what would be a such a things. So, for instance we could look at the numbers, so for instance I have a crossing here it sigma. So, just 1 and 2 give raise to crossing, but of course, in the preceding step that is not a crossing.

So, let us look at another one, let us look at 4 and 5, so sigma I have the numbers 4 and 5, 4 maps to 5, 5 maps to 3. So, in sigma they give raise to crossing, now let us look at the one preceding step I have 1 and 3 which give raise to crossing again. So, let us write this down.

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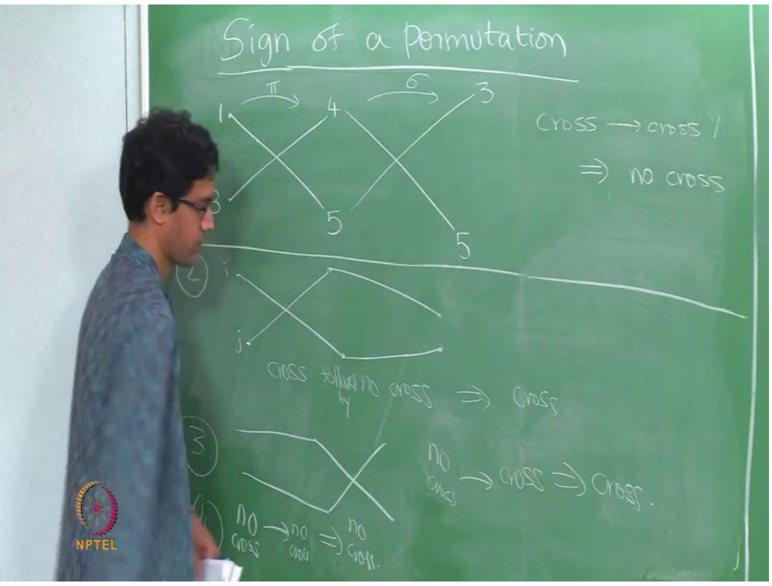

So, here is what we have in our example the 1 and the 3 map to in the first step the map 2 4 and 5 that is the crossing and the next step and the sigma 4 maps to 5 and 5 maps to 3. So, this is under pi, this is in the diagram for sigma I am not drawing the full diagram only the relevant part. So, here is an example where I have 2 crossings and in the final answer 1 maps to 3 that is the composition and 3 maps to 5. So, there is no crossing anymore in the final answer.

So, let us look at that as well, so I have 1 mapping to 3 and 3 mapping to 5, so these two lines 1 going to 3 and 3 going to 5 will of course, not crossing each other. So, the key moral of this story is that if I have two successive crossings, then in the n product there is in any crossing, there are no crossings any more. So, this is somehow the important case all the others are quite straight forward. So, what are the other possibilities is well you could have one crossing. So, here is case 2 if I look at pi.

So, I have my numbers i and j say that in the diagram for pi there was a crossing, but in the diagram for sigma they do not cross. So, maybe this cross here and that goes somebody. So, it is a crossing and followed by a no crossing, so if I have a crossing followed by a no crossing, then in the composition there will still be a crossing, because this i would map to something in the end, which is in fact, still bigger than whatever j maps 2. So, cross followed by no cross, let say followed by no cross leads to a crossing again. So, this is cross followed by cross leads to no cross.

And similarly, you should work out the other situation yourself. So, if there is no crossing to start with, but then sigma leads to a crossing, then again there still be a crossing. So, no crossing followed by a crossing, old lead to a crossing finally, the fourth case where you do not have a crossing and I mean both pi and sigma do not give you crossings and that happens you do not have a crossing at the end. So, that is the forth case which is again no cross followed by no crossing of course, leads to no crossing.

So, the key case really is the very first case that we looked at which is that when you have to successive crossings, in the end there is in any crossing. In all the other cases, if one of them is a cross and the other is not a cross, then you do have one crossing and if both of them are not crosses you do not have a crossing the end. So, putting this discussion together here is basically what we conclude.

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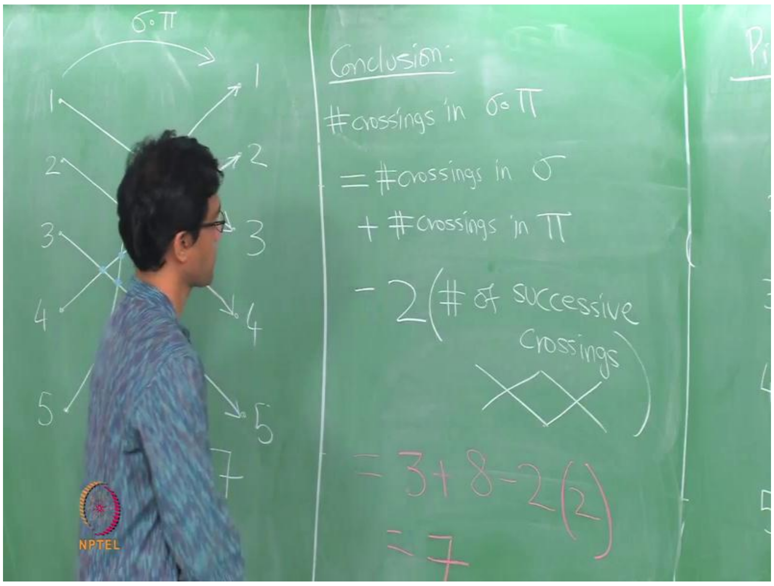

So, here is the very interesting conclusion of this hole discussion, if I want to count the total number of crossings. So, the number of crossings in the composition of sigma pi is valid you can obtain it using the following formula, you first count the number of crossings in sigma to which you add the number of crossings in pi. So, you add all the crossings in sigma to the number of crossings in pi, except when there are successive crossings, if sigma if there is a crossing in pi followed by crossing in sigma, then as we saw in the end those two sort of undo the effect of each other, the end of the day there is in the crossing in the composition.

So, this plus this minus 2 times the number of what we will called successive crossings by successive crossing we mean a picture of this kind. So, this is the way of counting the total number of crossings in sigma composition pi. And as you can see the proof is moralize what we have already said that there are these cases and except in the case where... So, observe if say this where the configuration, if there is a successive crossing then this the crossing here would be a crossing for pi. So, that would contribute one for a crossing of pi, this guy here would contribute when you count the crossings of sigma.

But, when you compose them these two sort of cancel out the effect of each other. So, they give you no crossings, so what that formula is doing is really counting this as a 1 counting this as a 1, but then subtracting two times such double crossings or successive crossings. So, what you get is that formula there, now what we do here is just check that this works fine in our example.

So, let us just look at the diagram and try and figure out how many configurations. So, what we need to do, we know the number of crossings in sigma is 3, the number of crossings in pi is 8. So, 8 plus 3 is 11 let us count the number of successive crossings in this diagram ((Refer Time: 32:34)). So, where you can get an successive crossings well here is 1, so I look at this configuration here in 4 5 and 5 3. So, that is and the thing here 1 going to 5 and 3 mapping to 4, here is one successive crossing highlighted in this diagram.

So, as one of them and there is another which is let see the 1 which seems 3 to 4 and 5 to 3. So, let us also try and highlight this, so diagram is going to be something of the mess, but we are 3 4 and 5 3 that is another cross and what receives that is 2 3 and the same thing 1 5. So, what is marked in the diagram in the blue lines give you one successive crossing and the red lines give you another successive crossings. So, there are two successive crossings in this diagram ((Refer Time: 34:05)).

And so this formula here in this example at least gives you the following, it says there are 3 crossings in sigma, 8 crossings in pi and 2 successive crossings., so minus 2 times 2, so there should be 11 minus 4 which is exactly 7 crossings. So, the number of crossings in the composition is exactly given by this formula. Now, what you know the main consequence of this is really for the sign.

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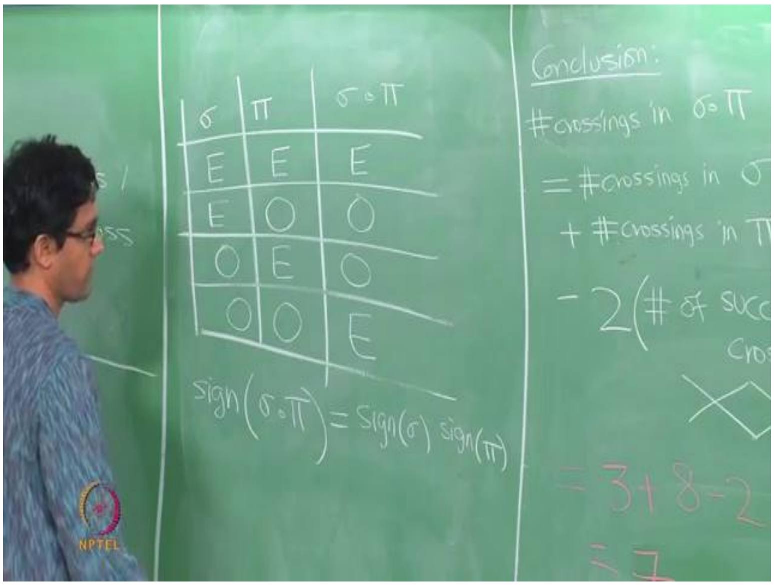

So, observe that here is the nice corollary that if here is a nice table we can make a sigma pi and the composition. So, if sigma and pi are both even permutations by E I mean even;

that means, that the number of crossings in sigma is even, the number of crossings in pi is even. The sum of two even numbers is even and this thing you are subtracting out is always a even number, because it is two times of something. So, if both sigma and pi are even numbers then of course, the right hand side is even number.

Similarly, or let us see the one another is even, the other is odd the number of crossings. So, this is even that is odd, even plus odd is odd from which you subtracting even this still get an odd number. So, one of them is odd, the other is even, the answer is odd if both are odd, the answer is even. So, if you compose two even permutations or two odd permutations, the answer is always an even permutation if you compose one odd and one even permutation, then the answer is an odd permutation.

So, sometimes a very easy way of writing this out this is to say, the sign of the composition. So, remember the sign was defined to be plus 1 or minus 1 depending on whether they are even or odd. So, sign of the composition is just the product of the sign, so other way of saying if both are plus 1, the answer is plus 1, if both are minus 1 the answer is still plus 1. But, one of them is plus and the other is minus the answer is a minus 1.

So, this is sometimes called the multiplicativity of the sign function and what we have done is really try to understand it in terms of the inversions and sort of a little more pictorially in terms of these crossings in the diagram itself. So, next time we look at still more properties of permutations and sings and so on.