An Invitation to Mathematics
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Unit
Combinatorics
Lecture - 11
Combinations with Restrictions, Recurrence Relations

Welcome back, today we will talk about Combinations with Restrictions. So, so far we have talked about trying to enumerate the number of combinations of various kinds.

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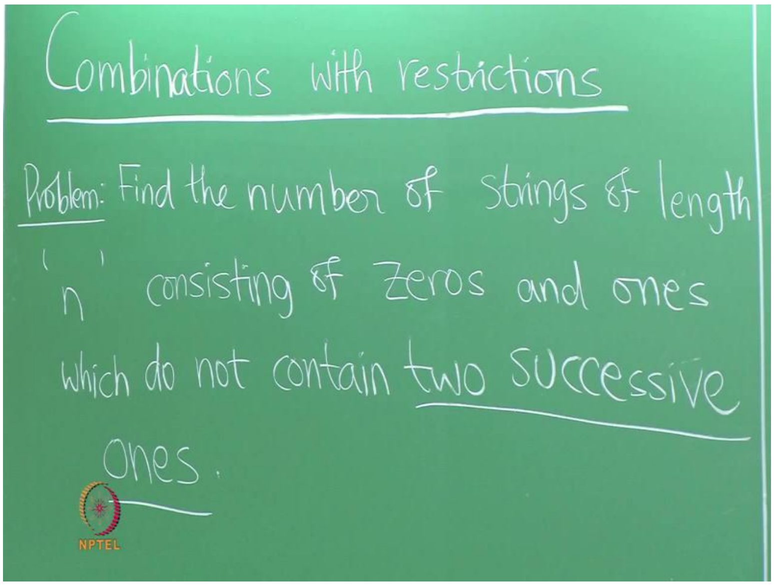

This one is slightly different in that, it imposes certain restriction on the allowed combinations. So, here is a proto difficult problem which says find the number of strings of length n consisting of zeros and ones in which you do not have two successive ones. So, let us try and write down a few such examples, now what are such strings.

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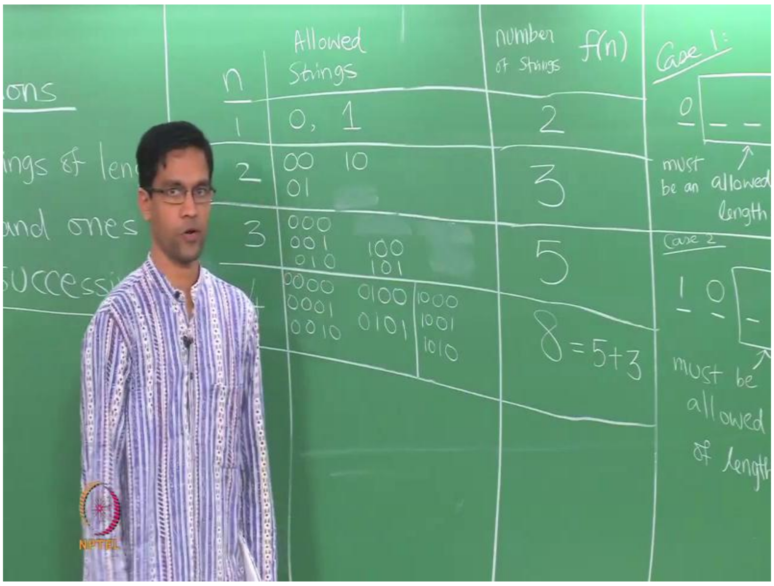

So, let us say what is n here n of course, is the length of the string, so the length of the string could be 1, 2, 3 and so on. So, let us do the following, let us write down a table 4 and so on and for each value of n, let us first enumerate the strings which are allowed in the sense it satisfy the required restriction. So, if I only have a string of length 1 consisting of a 0 or a 1, well there are two possibilities I can just take this string which are just has a 0 or I can take the string which just has a 1. So, both of these are allowed strings, because you do not have two successive ones appearing.

So, let us do the case n equals 2, so I have string of length 2, so I could have 0 0 or I could have 0 1 or I could have 1 0. Now, the fourth possibility which is 1 1 is no longer allowed in this case, because you have two successive ones. Now, let us do may be one more 3, string of length 3 you can have all three 0s, you can have 0 0 1 let us see I can have 0 1 0. Now, let us may be I will write all of them down and we will get read of the 1s that do not satisfy the constraint. So, 0 1 1 and then I have 1 0 0, 1 0 1, 1 1 1, so I also have 1 1 0 and then I have 1 1 1.

So, let see we got 6 plus 2 8 that is the total number, out of which we need to disallow 0 1 1, 1 1 0 and 1 1 1, they are not they contain two successive 1s. And so let us also do the following, let us count the total number of strings. So, let us say the number of such strings of allowed strings, so let us call this number something let us call this f of n. So, remember the number of allowed strings depends of n of course. So, let us call this number f of n, so here it is 2 f of 2 is 3 there are three allowed strings f of 3 is 1, 2, 3, 4, 5 and so on. So, so far we have figured out what happens in the initial cases.

Now, to really understand what is going on let us try and do the case n equals 4. So, as you can see enumerating these strings gets harder and harder. But, let us sort of see the pattern that can be obtained from this. So, if you take n equals 4, what you are looking for is a string of length 4, so I may do it here. So, I need a string of length 4 with 0s and 1s, so of course, the very first entry in this string is either a 0 or a 1. So, there is a bunch of strings which start with a 0, and then here is the other possibility.

So, let we write down case 1 and case 2, strings which start with 0 and strings which start with a 1. Now, suppose the strings start with a 0 what does it tell you about the remaining three places. So, what you would do with the remaining three slots, well observe that the condition says you cannot have two successive ones. But, well the first entry being a 0 that says that the next entry can be pretty much anything you want could be a 0 or 1 for a instance, and then you have another entry and another entry.

So, observe that these three entries have the following property that this must be an allowed string meaning it should not contain… So, this must be an allowed string of well of one smaller length of length 3. So, this is case 1 if your strings starts with the 0 then what follows it must be an allowed string of length 3 without two successive 1s. Now, case 2 is the following, if the strings starts with the 1 then what can one say about what follows it. Let us look at the constraint; it says that you cannot have two successive 1s in your string.

So, since the first entry is a 1 this has better be a 0. So, the first two entries get determined and now observe that what follows, so now, that I have a 0 the next entry can be a 0 or a 1. So, what we have in these two slots at better b and allowed string of length 2. So, this tells you how to go about enumerating all these strings, what you would do is the following. In order to write down all strings of length 4, you start with the allowed strings of length 3.

So, let us get read of the ones that are not allowed, so here are the 5 allowed strings of length 3 and to each of them you will adjoin a 0 in front. So, let us take all these guys and prefix each of them with a 0. So, I have 0 0 0, 0 0 1, 0 1 0, so I will first write the ones of the length 3 and prefix each of them with the 0. So, these are five strings of length 4 which begin with the 0.

And in addition you also have the ones which begin with the 1 and in fact, those must begin with the 1 0 as we just saw. So, what you now do for these guys in case 2 you go one level lower, you look at the allowed strings of length 2 and you prefix each of them with the 1 0. So, I had these five then let me write strings of length to 0 0, 0 1, 1 0 and I prefix each of them with the 1 0. So, what you get are essentially these two cases and that makes for easy enumeration.

So, what is the total number here, well observe it is the number of allowed strings in the one preceding level plus the number of guys in two steps before it. So, this number is actually 8 and how did we obtain this, well it is the one preceding number plus the number which came two steps before it.

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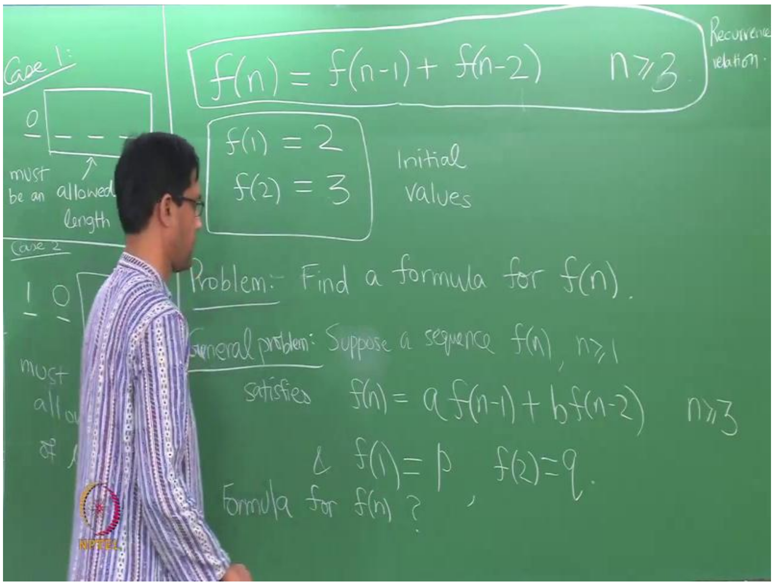

So, this analysis in fact, tells us the following observe what have we just shown, we have realize the following that f of n is just the sum of the two preceding numbers and well of course, this only make sense if n is at least, let see n is at least a 3. So, in order to compute f of 3, f of 4 and so on all you need to do is just take the sum of the two preceding things. And in addition what else do we have, we also know the first two values of f, for instance f of 1 we knows the 2 and f of 2 we knows the 3.

So, these two are often called the initial conditions, the initial values if you wish we have the initial values and this relation here valid appeared several times before in our lectures. So, we have add a case n to encounter these guys when we talked about for instance the Chebyshev polynomials where of course, all these guys are actually polynomials in a variable x, here there are a numbers. These also appeared when we were doing the problem of counting monomials in polynomials of many variables, counting monomials of a given degree those where again given by a occurrence relations.

So, this equation here is what is usually called a recurrence relation, which determines certain term of the sequence in terms of the preceding terms. Now, here is the problem that we would really like to understand, how do we find a formula for a f of n. So, find a formula for f of n, it is easy to write down the initial values you just keep adding itself whatever 2 3 5 8 8 plus 5 13 21 34 and so on, but what we want is given n of formula which tells you what f of n is.

So, here is a slightly more general problem which basically concerns recurrence relations. If you are given arbitrary recurrence relations with, say even coefficients in front one can still use similar approach to solve it. So, let me post some more general problem, so here some more general problem. So, given a recurrence relation, so given a sequence of numbers f of n and greater than equal to 1, which satisfies, suppose a sequence f of n satisfies a recurrence relation f of n is given by some a times f of n minus 1 plus b times f of n minus 2.

So, this is for n at least 3 and what are a and b, a and b can be arbitrary real numbers or complex numbers and so on some constants and with some initial conditions. And of course, I need to tell you what f of 1 and f of 2 are. So, f of 1 is given to be some number p, f of 2 is given to be some number q and the problem now is to find a formula for f of n, so find a formula for f of n, so let us try and do this.

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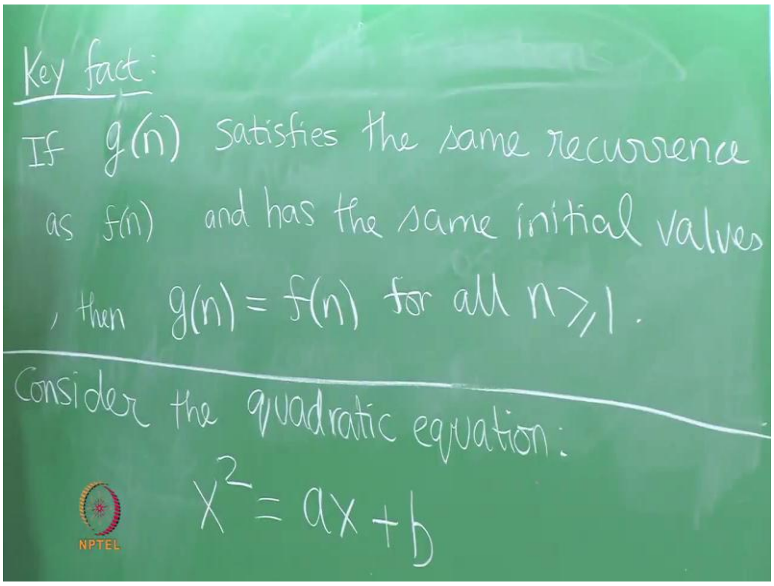

So, the key observation is the following, well one of the key observations, so here is the key fact concerning sequences given by recurrences. If suppose you have another sequence f of n, yes I am calling it capital F may be I call it g of n. So, suppose I have another sequence g of n which satisfies the same recurrence on the same initial conditions, satisfies the same recurrence as f and has the same initial values as f, then g and f are in fact equal.

So, what it says is the recurrence relation together with the initial conditions uniquely determines the sequence f and it is sort of easy to see why. Because, the first two values are the same for both g and f and how you form the successive values is also the same, it is given by the same prescription. So, for instance in this example that we just talked about, if g of n is the sum of the two preceding values and the first two values are 1 and 2 or let us say 2 and 3, then there is only one way in which you can determine the values of g.

So, we have 2 and 3 of the starting values, so the next value must be a 5, the next value must be an 8, the next value must be a 13 and so on. So, it is force there is pretty much no choice at all here. So, this is one elementary, but very useful observation and so here is another somewhat more surprising observation. So, not, so obvious at first, so here is an approach to solving this recurrence, consider so here is the very interesting ingredient here, consider the quadratic equation.

So, observe we assume that f satisfies the recurrence f of n equals a times f of n minus 1 plus b times f of n minus 2. Now, here is the following thing that we do, we take those same numbers a and b, and then form a quadratic equation write x square equals a x plus b. So, observe this just another way of x square minus a x minus b 0.

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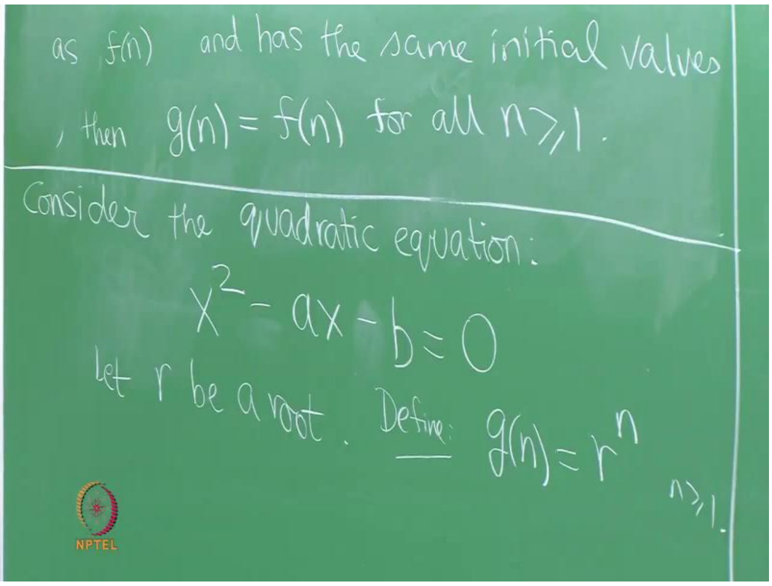

So, may be we could do that let us do let us write this as x square minus a x minus b equals 0 and let us pick a root of these equation. So, quadratic equation has two roots of course, the roots could be complex numbers, but never mind. So, let r be a root of these equation let r be a root, so pick one of the two roots, then I am considered define a sequence g of n as follows, just take powers of r just call it r power n for all n at least 1.

So, what we have done so far is to take of the quadratic equation and you form a sequence now of course, the sequence is a sequence in general of complex numbers, but never mind.

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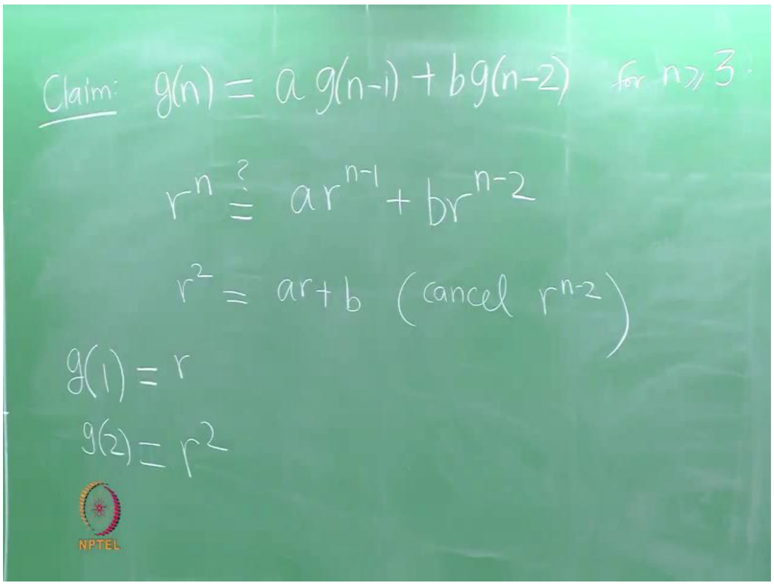

So, you have a sequence that claim is that in fact, g satisfies the recurrence of f. So, g of n is exactly a time g n minus 1 plus b times g n minus 2 and greater than equal to 3. So, here is a surprising fact that the sequence g of n that we have constructed actually satisfies the recurrence that we have began with and the prove is modules straight forward. So, observe this equation, so we just need to show that this equation holes. So, this equation is just the following equation that this is r power n, this guy's r power n minus 1 that r to the n minus 2.

And of course, you know dissolve we need to check is this true and just observe that if you cancel of an r to the n minus 2 from both sides, this equation just is the same as r square equals a r plus b how do you get this by cancelling of the common factor of r to the n minus 2 from both sides of the equation. And of course, that was the beginning assumptions on r where it is root of that equation.

So, of course the proof is a very simple once we know what to want to prove it is rather trivial to see that it is true. So, this is somewhat surprising that you can actually manufacture sequence with satisfies a given recurrence by just taking the root of this quadratic equation and just raising into various powers. Now, of course, in order... So, if suppose g satisfies the initial conditions if it had the same initial values as f then we would be done, because then this would just be g has to equal f according to what we just said.

So, this would pretty much gives us a formula for f of n. So, of course, it is too much to expect that the initial values of g equal the given initial values of f. So, observe what are the initial values of g in this case? So, if I put n equals 1 this is just r power 1 and if I put n equals 2 is going to give me r square, then as I said r and r square or in general some complex numbers r is the root of this equation. And of course, these may not be given initial conditions.

So, for instance f of 1 and f of 2 where given to be p and q this may fail to equal that, but this is very good beginning we already have at least way of ensuring that the recurrences is satisfied which only the initial conditions that require some words.

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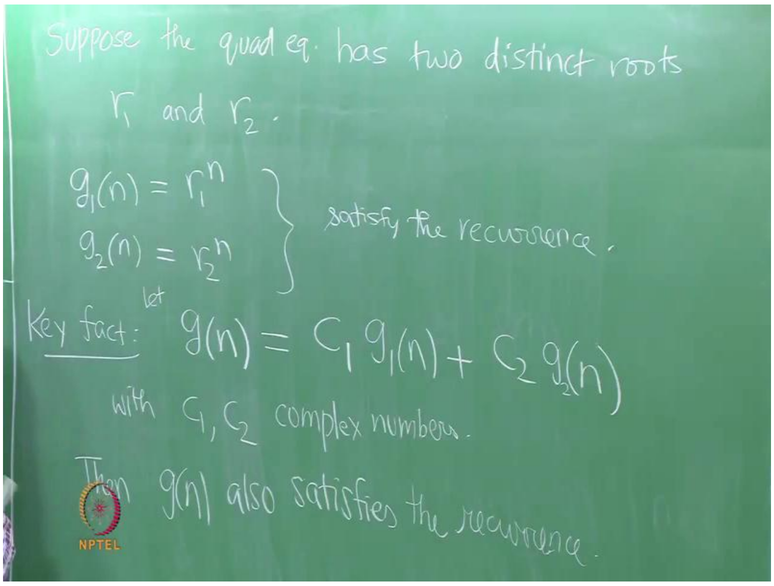

So, here is the further case now suppose let us assume the following, suppose this quadratic equation has two distinct roots, suppose that the quadratic equation that we wrote out has two distinct roots let us call them r 1 and r 2, then observe we have two different solutions. So, let us call it g 1 of n is r 1 power n hers the sequence that satisfies the recurrence, g 2 of n equals r 2 of n. So, both this sequences they satisfy at least the recurrence relation.

And so here is an another key fact which tells you how to manufacture many more sequences with satisfy the recurrence starting with two sub sequences, if you know two sequences with satisfy the recurrence the key fact is the following if you form any linear combination of these guys. So, you define g of n let g of n the some constant c 1 times the first sequence plus another constant times a second sequence this what we call ones before a linear combination of these two sequences, any linear combination will also satisfy the recurrence that is the key thing.

So, let us g of n be define like this with what are c 1 and c 2 they can in fact, be a any complex numbers, then g of n also satisfies the recurrence. So, now, what this gives us is a way of manufacturing dumps and dumps of sequences with satisfy the given recurrence, because we know r 1 r 2 a distinct. So, here are two distinct solutions and here is a prescription which allows you to construct many more solutions starting with two solutions.

Now, that already gives us some hope that may be one of these guys will in fact, have the required the initial values. So, if we can somehow arrange these constants c 1 and c 2 such that g of n has the required initial values then we would have formula for the required sequence f of n.

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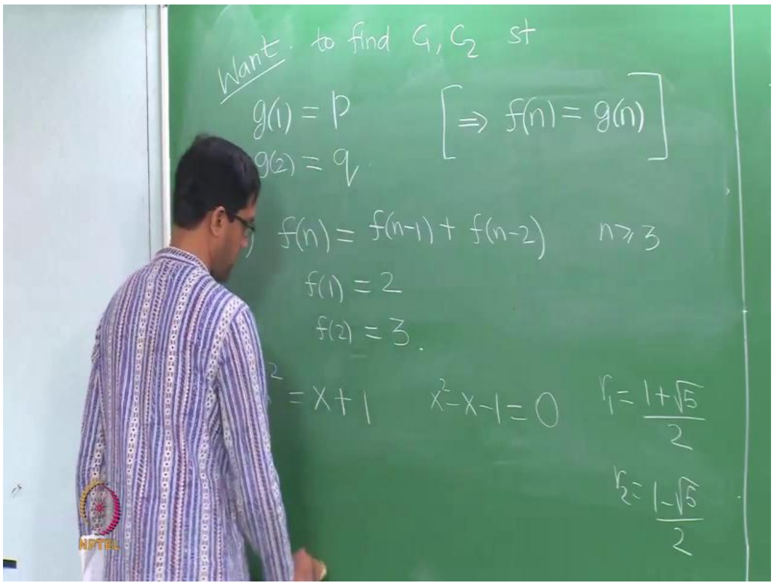

So, now, the problem really reduces to the following what we would like to do we want to find constants such that g of 1 exactly gives you the required initial condition p and g of 2 gives you the required initial condition q. If we can do this then f and g would automatically b equals. So, observe if we can do this then this would imply that sequence f that we would want must in fact, equal the sequence g and that would give you a formula for f of n. Because g of n is of course, just given by a formula, it is given by c 1 r 1 power n plus c 2 r 2 power n.

So, let just see this in the example that we want, so the case of the sequence f of n that we started out with. So, observe here where f of n was f of n minus 1 plus f of n minus 2 this is the same f that we started out with the number of 0 1 sequences of length, then which do not contain two successive ones. So, this is for n at least 3 and we had the following initial conditions f of 1 is 2, f of 2 is 3.

So, here is our procedure, we start with the quadratic equation, so sometimes is called the auxiliary equation. So, it is the following quadratic equation here it is x square equals x plus 1 we solve this equation. So, how do we solve this well we use the usual quadratic formula. So, here is the quadratic equation and here there are two roots, so let us write those two roots down r 1 equals 1 plus root 5 by 2 and r 2 is 1 minus square root of 5 by

2. So, there are two distinct roots in this case and so what does are prescription say it says you look for solutions of the form.

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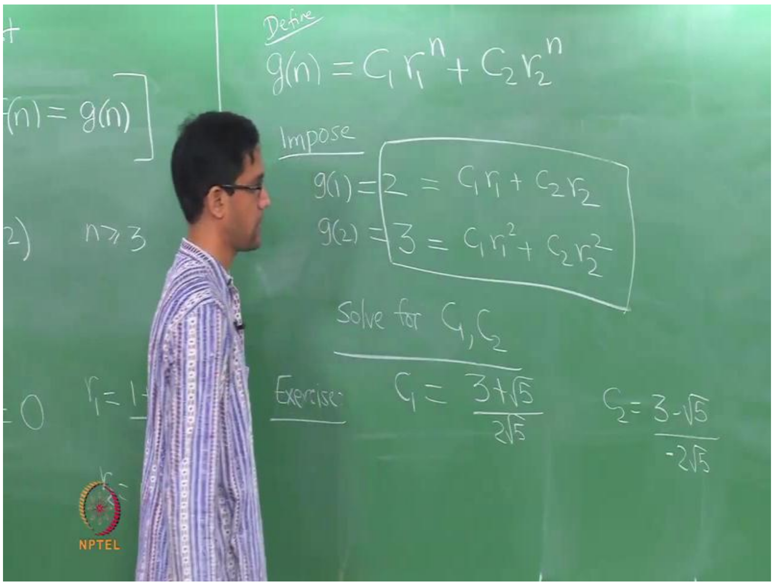

So, let us form the sequence g of n, so let us define g to be a linear combination c 1 r 1 power n plus c 2 r 2 power n. So, we define g in this way, this guy certainly satisfy the recurrence relation and all we want is for it to satisfy the initial conditions. So, now, let us impose our conditions, so let us impose the following conditions g of 1 we would like it to be 2 and g of 2 we would like it to be 3. So, these are the two conditions that we would like to be satisfied, so what do these amount to g of 1 is just c 1 r 1 plus c 2 r 2 g of 2 is c 1 r 1 square plus c 2 r 2 square. So, what you now have r 2 equations for the two unknowns c 1 and c 2.

So, all we have to do is just solve these equations, so solve for the constant c 1 and c 2. So, I will just give you the solution here, so exercise check that here are the choices that give you the required initial conditions. So, c 1 in this case is 3 plus root 5 by 2 times the square root of 5 and c 2 is 3 minus square root of 5 by minus 2 times square root of 5. So, here are the constants which you can just get by solving these to linear equations for the constant c 1 and c 2. So, what is that give you finally, well it gives you solution to the to the original problem, it says that the number of 0 1 sequences which have no two successive ones is in fact, given by the formula c 1 times.

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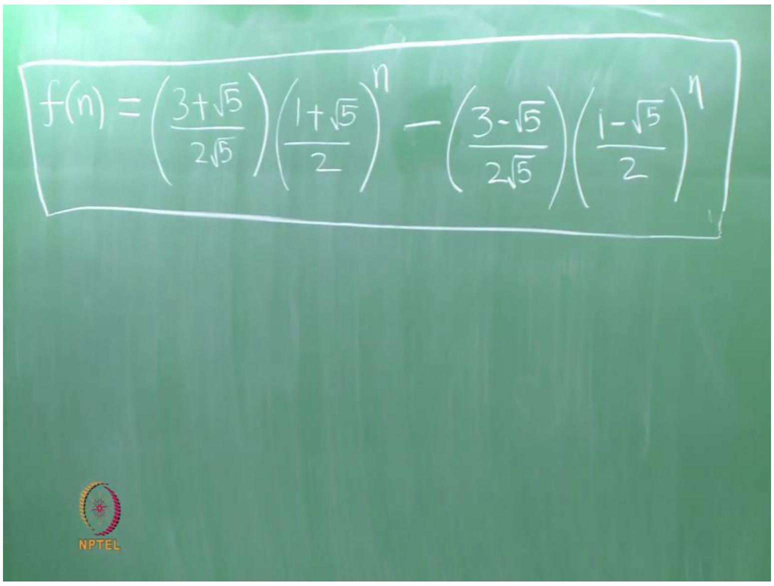

So, what does this imply finally, he is our conclusion the number of 0 1 sequences with no two successive ones must in fact, be the same as g of n, because it has the same initial conditions and the same recurrence. So, this is just if you as 3 plus root 5 by 2 5 that is the constant c 1 times r 1 power n r 1 is plus the constant c 2. So, the constant c 2 in this case is 3 minus root 5, the minus 2 root 5 to a term times 1 minus root 5 by 2. So, here is a formula for the function f of n that we wanted.

And it is other surprising that you know that this will always work out to be positive integer as it should. Because, f of n after all counts the certain number of configurations of a kind and it is rather strange at with all these square root 5 is floating around here that the end of the day this always works out to be a positive integer. So, check for certain values of n you just plug in various values of n to see that this in fact, gives us at least the first few term correctly.

So, next time will talk a little bit more about these very same sequences and may be where else they appear in they appear in a few other problems. So, we will talk a little bit more about that next time.