An Invitation to Mathematics
Prof. Sankaran Vishwanath
Institute of Mathematical Sciences, Chennai

Unit
Combinatorics
Lecture - 10
Combinations with repetition, and counting monomials

Hey welcome. So, today we are going to talk about combinations with repetition.

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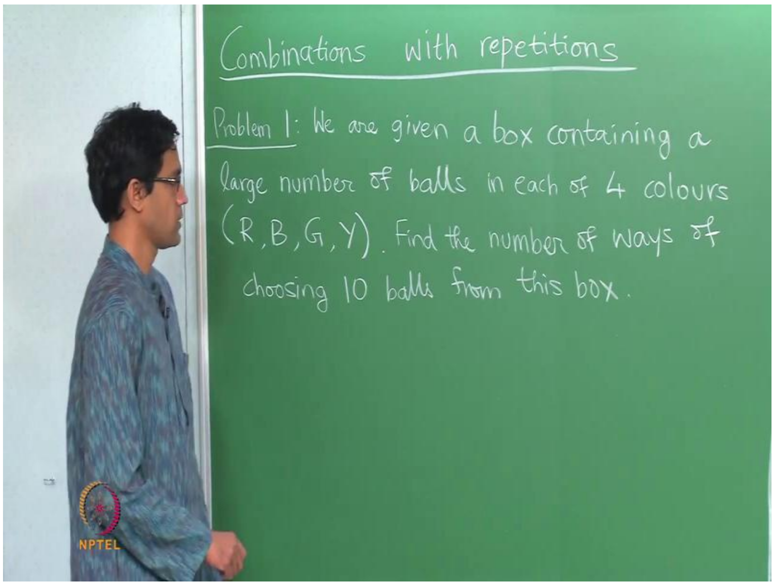

So, here is typical problem. We are given a box which contains, let us assume very large number of balls in each of four colors, which are red, blue, green and yellow. Now, the question is find the number of ways in which one can choose ten balls from this box. So, for instance let us see, what are various examples of choices.

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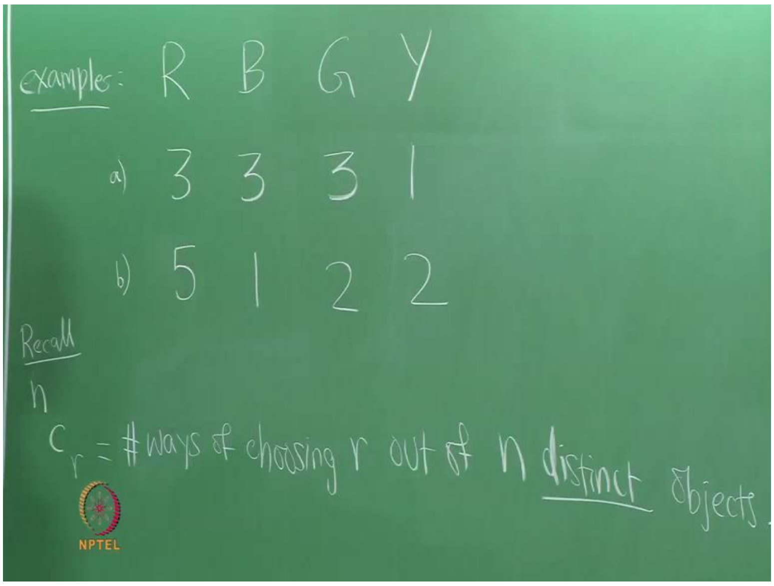

Let us look at a few example. So, how would we encode the choice? We could keep track of number of red balls, blue balls, green balls and yellow balls. So, for instance one choice could be we may pick three red, three blue, three green and one yellow, maybe we choose five red. So, here is one possibility. Here is another choice you could pick five red, one blue say two each of green and yellow and so, on. So, you could imagine various such choices of sum number of red some other number of blue green and yellow. But of course, the total number of balls that you pick has to be 10 that is of course the key constraint. So, observe that this is different from the problem of combination that we have looked at so, far. Because the typical problem of combination says something like if got n distinct object and from those n objects you want to choose r of them. So, we recall that the other combination problem, typical combination problem that we have looked at so far, for which number of choices this is n c r is the number of ways of choosing r objects from n distinct objects. So, this is number of ways of choosing r out of n key word here being distinct of that object.

Now, combination with reputation just means that we do not really have distinct object any more for instance here what we have is let say we are given a very large number let say hundred each of red balls, blue balls, green balls and yellow balls and from amongst these say four hundred balls you need to pick ten of them. But the thing here is the original object are not distinct anymore many of them are alike for instance the hundred red balls are for all alike the hundred green balls are all alike and so, on. So, the key distinction the really comes about in the assumption that the initial set of objects is distinct, this is the usual combination problem. Now, what we are looking at is what you would called combinations with reputations.

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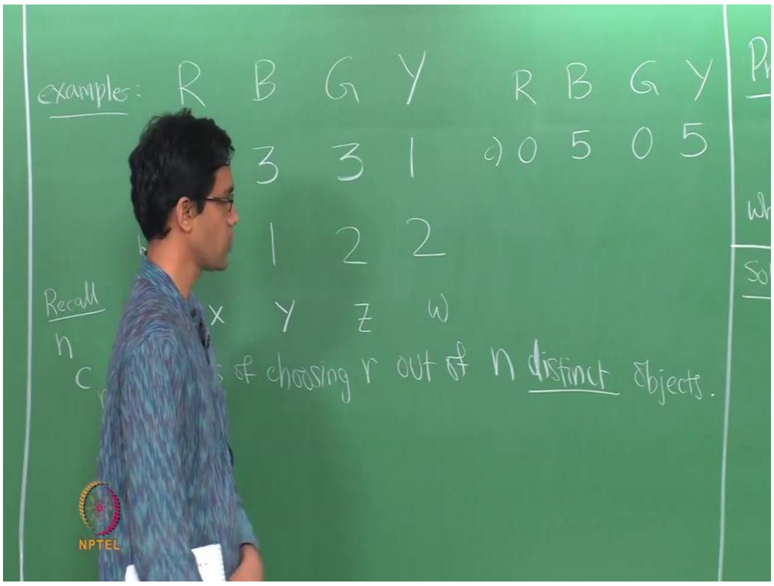

So, observe that this this sort of encoding here the number of red, blue, green, yellow balls. So, let say the most general encoding is let say that x red balls, y blue balls z green balls and say w yellow balls. Then what this is saying is in what are all the number of choices for non-negative integers x y z and w such that there sum is 10. So, observe here that, it is not necessary to choose a ball of given color for example, another choice here could be I could choose red blue green yellow you may choose zero red balls, zero green balls and may be five each blue yellow this is also valid choice. So, you are not required to choose one ball of each color at least you can choose zero balls for instance.

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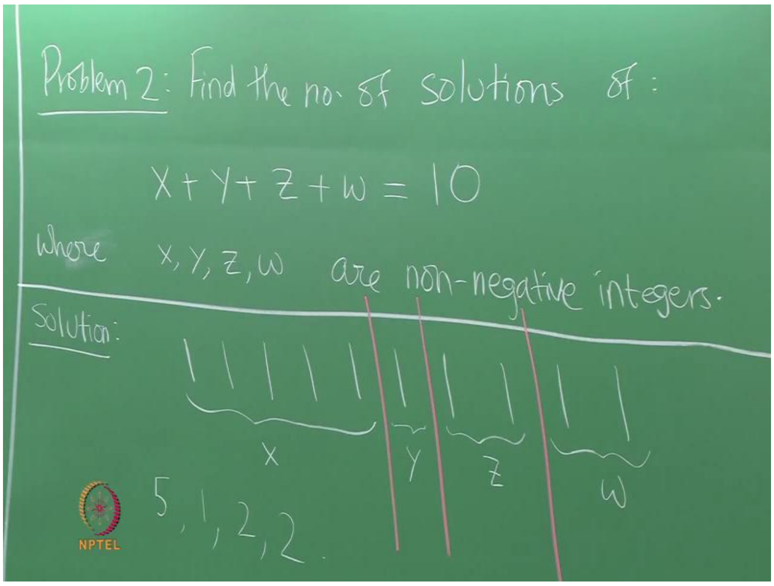

So, here is another equivalent reformulation. So, here is problem two which is really the same as problem one in some sense. Find the number of solutions to the equation, number of solutions of the following equation x plus y plus z plus w equals 10 with the constraint, with the requirement that x y z and w the non-negative integer. So, let just say this were x y z w are non-negative integers. These two problems are really equivalent to each other.

Now, let us look at the typical method of solution. It requires slightly different way of thinking rather than the usual combinations problem. So, how do we solve this problem? So, here is the typical solution how do we find the number of solutions to this equations. So, let us encoded in the in the following way what we need is…. let us think about it in the formulation of the problem two. What we want is the following let me write down 3,4,5,6,7,8,9,10 there are ten ones now on the board now what will do is the following. If you are given a solution x y z w for instance suppose I have x equals…. so, let us take the solution that we already had for instance if you take x equals 5, y equals 1, z equals 2 and w equals 2. So, 5, 1, 2, 2 we will think of it as follows x is 5. So, let me see there are 5 ones this much is x, y was 1. So, that is y z is 2 and w is a 2. So, we must mark these off in the list of ones and then we do the following, we sort of draw dividers between them. So, here is a divider, drawn in red which separates the x from the y now this, once I move to z and I put another divider and between z and w I have yet another divider. So, what I have done is written as string of ten one’s with three dividers in between. Now, this entire picture here represents the solution x equals 5, y equals 1, z equals 2 and w equals 2 this picture here represents a solution 5, 1, 2, 2. And let us sort of look at a few more examples. So, let me take the next one 3, 3, 1 for instance.

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So, here is the solution 3, 3, 3, 1 would be represented similarly 3, 4, 5, 6, 7, 8, 9, 10. So, that is the full string there is a divider after three, there is another divider after 6 ones is another divider after 9 ones, this picture now represents solution 3, 3, 3, 1. Let us take another example 0, 5 0, 5 that is yet another solution to this equation now how would that be represented. So, zero means the following that, the first divider occurs right in the beginning of this list. So, the number of ones to the left of the first divider is a 0 now I need a 5. So, 1, 2, 3, 4, 5 ones later I have another divider now the next 0, which means that the third divider is placed at the same place more or less as a second divider. So, this picture, now which has these ten ones with three dividers placed in these position the first divider right in the beginning and the second and third divider placed at sort of between the fifth and sixth ones. This picture encodes the solution 0, 5 0, 5. So, now, sort of from these examples I hope it is clear that one can actually encode So, each solution to that equation.

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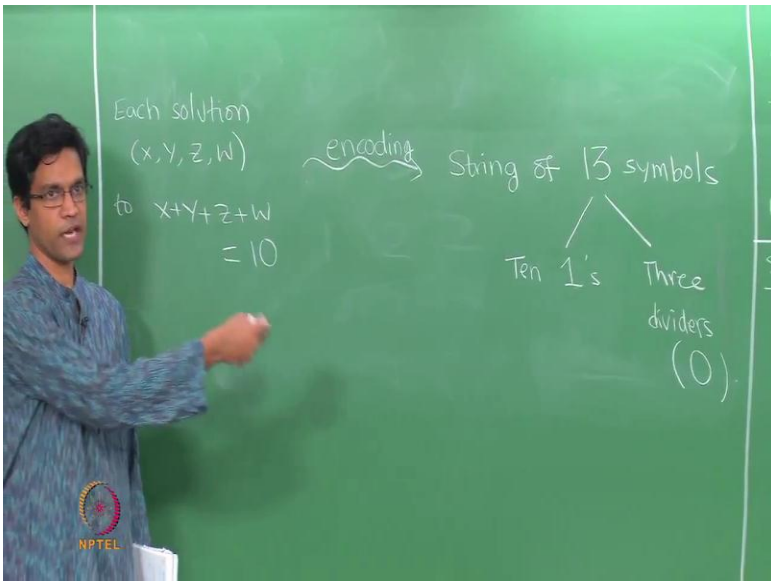

So, each solution x, y, z, w to the equation x plus y plus w equals 10. Each such solution can be encoded. So, what we are doing is, we coming up with an encoding we want to thing of the same thing in a different way which will be easier to count it is encoded by the following thing is encoded by a string. So, sequence of well it is a string of what? Well of thirteen symbols, hence what are these thirteen symbols. Well ten of these symbols are just one's or ten one's and then there are three dividers three of these line so, there are three dividers often instead of the divider itself which is sort of a hard to write often in when your other symbol is a one. So, to sort of make it uniform we replace the divider by a zero. So, we could as well say instead of looking at string of one's and dividers you might as well just look at as a string one's and zero's.

So, for instance let me do it here this string that we wrote out in the end which had divider in the beginning and so, on would be the following. It would be a zero followed by 5 one's followed by two dividers which means 2 zeros followed by 5 more one's. So, this string now of thirteen symbols ten one's and three zeros encodes the solution 0, 5 0, 5 to that original equation. So, we might as well just think of it like that it is a string of thirteen symbols with ten one's and three dividers. So, instead of dividers being this red line here I will just think of the dividers as being written out as zero. So, now, the question is well we want at the count the number solutions to these equations. It is the same as counting the number of strings of that form now this is something that we have already looked at.

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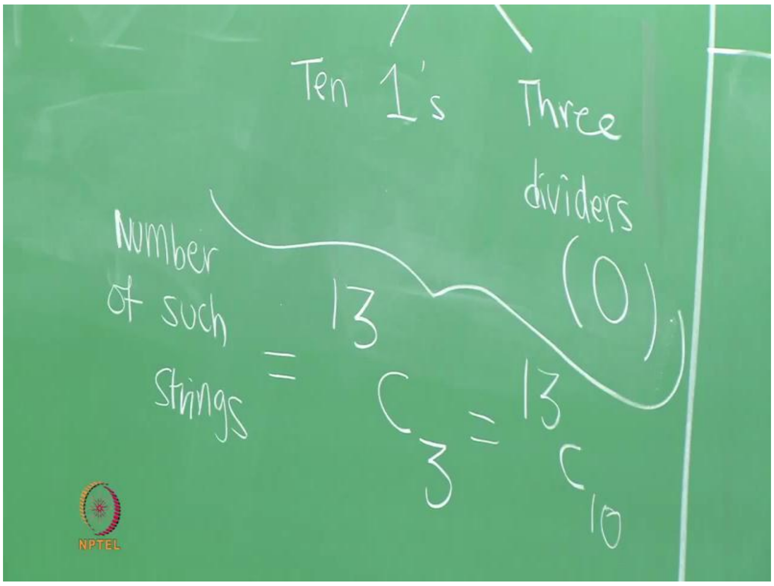

So, this can be counted the number of such strings. So, recall when we in one other earlier lectures we talked about the writing about string is a a's and b's we said the number of strings of length and which contain r is and n minus r b's is just n choose r because you just have to find the position where the a's occur. Now, this is similar we have a string of thirteen symbols and you pretty much only need to choose the positions where the let us say either the 10 one's occur or the 3 zeros occur. So, the number of such string is nothing, but 13 choose 3 for instance or 13 choose 10 which are of course, both the same that is the solution to the problem. Now of course, it is clear the general solution will look like. So, let us write out general version of problem two.

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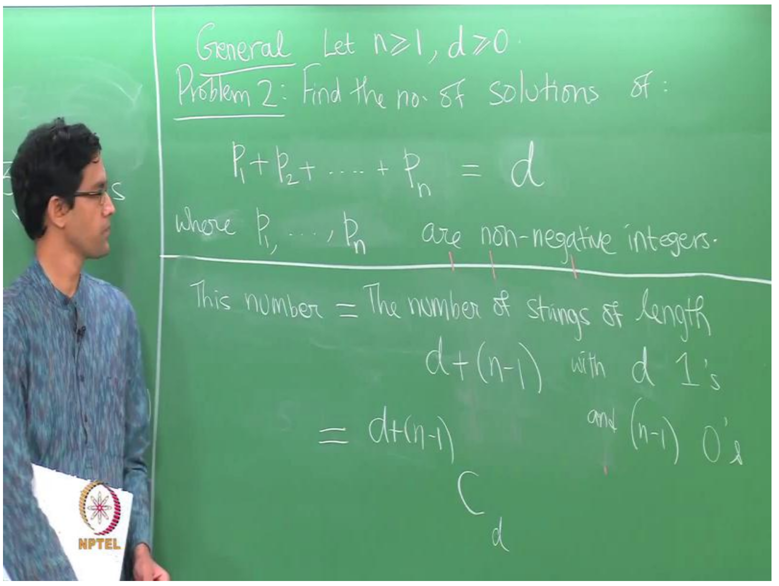

So, let me here say is general version of problem to find number of solution of well of what, here I only have 4 variables x y z and w in general I could assume there are n variable. So, the left hand side let we replace by the n variables for find the number of solution of the equations let us call the variables as p 1 plus p 2 plus. So, there are n variable p 1 p 3 p n and right hand side here was the number 10 in general I could assume it some non-negative integer d. So, what the assumption let n we at least one so, that at least one way variable in the left hand side let d b some non-negative integer and the question is find the number of solutions of the equation p 1 plus p 2 plus p 3 till p n equals d where all the p's are required to be non-negative integers. So, where the variables or require to be non-negative integers and again the solution methodology is the same what we want to do is to encode every sub solution as a string of zeros and ones. So, the answer now is, the number of solutions to these equations is the same as, this number is the same as the number of strings of well what as the total length of a string. So, observe again I need to have d ones they should be a string of d ones and the number of dividers should be n minus 1 because that is it is one less than the number of variables that you have right you only have if you have n minus one dividers it divides the string ones in the one parts those are the n values of the variable. So, the number of strings of length what is a total length d is a number of ones and n minus 1 is the number of zeros. So, total length is which contain with d ones and n minus 1 zeros. So, these are the dividers. So, this list of strings really counts the number of solutions to this equation and of course, we now that this is nothing, but d plus n minus one choose d or if you wish d plus n minus one choose n minus 1. So, that completely solves the problem.

So, of course, the same thing can be phrased as you in terms as balls for instance there are balls of n colors and you have very large number of balls of each color and this problem is just another way of saying in how many different ways can I pick d balls from such a box.

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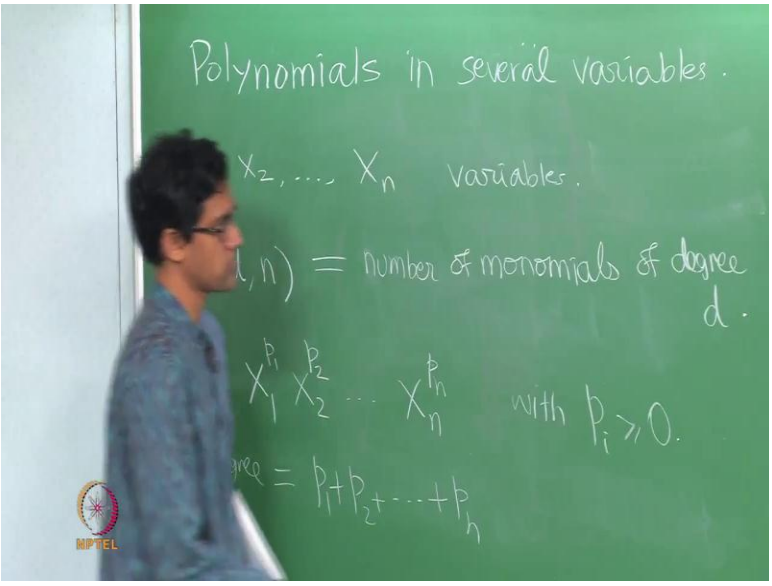

Now let us sort of look at another problem going back to something that we have talked about write in the beginning. So, recall we talked about polynomials in more than one variable right and there we post a counting problem. So, we had the following question we want to do the following suppose there are n variables let us call the n variables x1 x2 and x n, these are the variables. So, we looking at polynomial in n variables and what we want to do is the following thing we said.. We in fact, gave it a name event let us call this number d comma n. So, we defined for each d positive a number called a d n which was define to be the number of monomials. So, this is the number of monomials of degree d in these n variables of course. Now we sort of looked at some patterns we wrote down the first the few values of d we obtained recurrence relation for of d and so, on.

Now recall what is monomial? Monomial is something other the following form it is x 1 raise to some number x 2 power something and so, on till x n raise to something. So, let us call these exponents is something let us call it x 1 power p 1, x 2 power p 2, x n power p n. So, what is a typical monomial look like that is the monomial where what are the pis, pis being non-negative integers for all I and what is the degree of such a monomial. The degree of the monomial is just the sum of all the powers right. So, the problem that we looked at earlier of trying to count the number of monomials of degree d translates into following. Find the number of solutions to the equation p 1 plus p 2 plus p 3 till p n equals d where all the pis have to be non-negative integers. So, observe that example exactly the problem that we that we talked about, that is a general problem two.

So, from the discussion so, far here is what we conclude that this number a d n that we studied while ago is just the same as well what is it d plus n minus one choose because a d n is just nothing, but the number of solutions. So, a d n. So, as I said is number of solution to the equations p 1 plus p 2 plus p n equals d where the pis restricted to be non-negative integer observe that we have solved exactly this problem that was pursed. So, so what I like you to do is to go back do to that lecture where we tabulated the values is of a d n and where we obtain that nice recurrence relation and. So, on and check that this. In fact, does match. So, check. So, here is a nice exercise do check that the this. In fact, matches the values in the table this matches our table from before one of the past lectures. So, problem on counting monomials turns out to be just a problem of doing some combinations with reputations being along. So, next time will we will take about something more on permutations.

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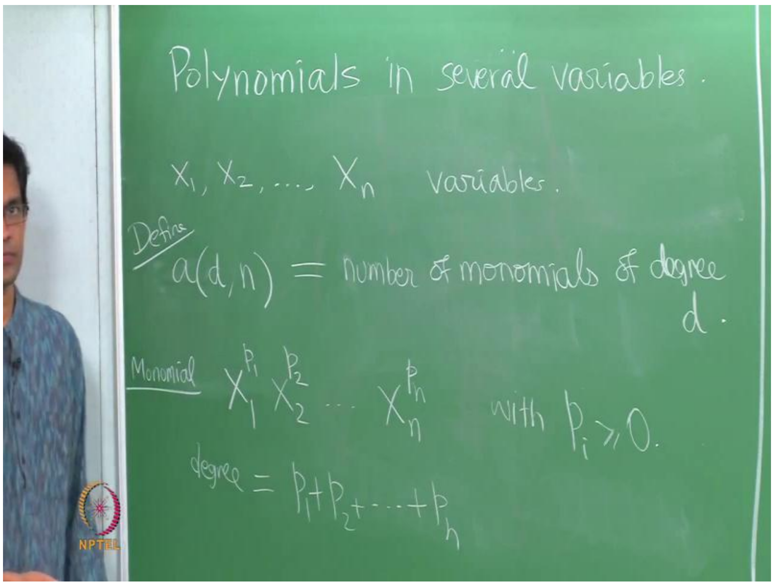

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in these n variables of course, now and we sort of looked at some patterns we wrote down the first the few values of d we obtain recurrence relation for of d and. So, on now recall what some monomial monomial is something other the following form it is x one raise to some number x two power something and. So, on till x n raise to something and. So, let us call these components is something let us call it x one power p one x two power p two x n power p n. So, what is a typical monomial look like that is the monomial where what are the with pis being non negative integers for all I and what is the degree of the such a monomial the degree of the monomial is just the sum of the all the powers right. So, the problem that we looked at earlier of trying to count the number of monomials of degree d translates into following find the number of solutions to the equation p one plus p two plus p three till p n equals d where all the pIs have to be non-negative integers. So, observe that example exactly the problem that we that we talked about is a general problem two.

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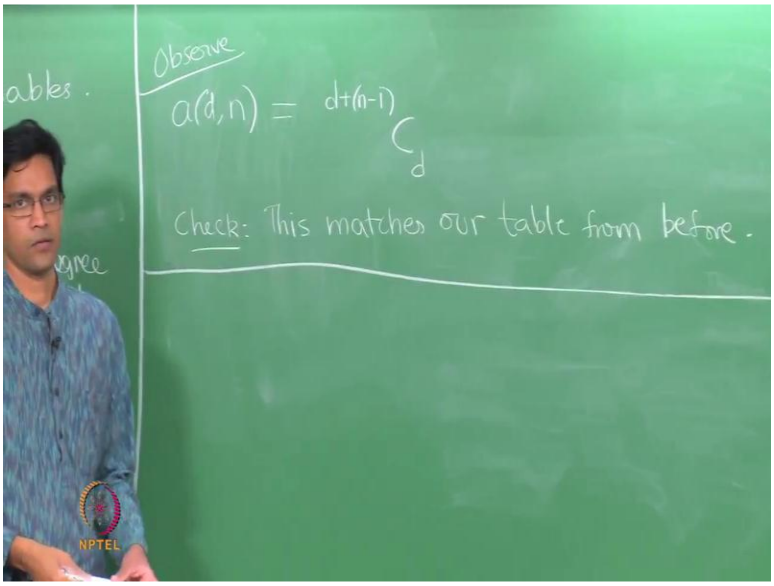

So, from the discussion so, far here is what we conclude that this number a d n that we studied while ago is just the same as well what is it d plus n minus 1 choose because a d n is just nothing, but the number of solutions. So, a d n, as I said is number of solution to the equations p 1 plus p 2 plus p n equals d where the pis restricted to be non-negative integer. Observe that we have solved exactly this problem that was posed some time ago. So, what I like you to do is to go back do to that lecture where we tabulated the values is of a d n and where we obtain that nice recurrence relation and so, on and check that this in fact, does match, so, check. So, here is a nice exercise, do check that this in fact, matches the values in the table this matches our table from before one of the past lectures. So, problem on counting monomials turns out to be just a problem of doing some combinations with reputations being along. So, next time will we will take about something more on permutations.