Dynamic Data Assimilation
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Lecture - 06
Matrices continued

In the last lecture, we have been reviewing several of the properties of matrices special matrices operations and matrices. We are going to continue the coverage of other properties of matrices that are critical to power analysis.

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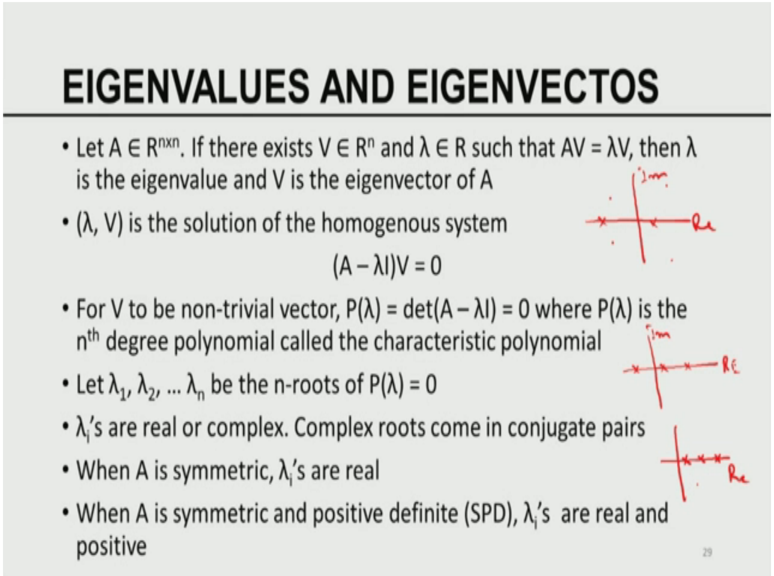

The first of the topic in that direction are going to be the notion of Eigenvalues and Eigenvectors of any real matrix let A be a n by n real matrix, if there exists a vector V belonging to R n at a constant lambda a real R, a complex constant R such that A V is equal to lambda V, then lambda is called the Eigenvalue and V is called Eigenvector of A.

From the definition, it follows that lambda V the pair the constant lambda and the vector V is a solution of homogeneous system that can be obtained from A V equals to lambda V for V to be a non if V is 0. This equation trivially satisfied V is equal to 0 is called the trivial solution, we are seeking non trivial vector; that means, a non-zero vector; for a non-zero vector to solve this equation, it is necessary that the determinant of the matrix A minus lambda, I must be 0. We have earlier seen one of the conditions necessary for the existence of solution of homogeneous system is the system must be singular, here the system matrix is a minus lambda I; the determinant of a minus lambda I must be 0, the determinant of lambda A minus lambda I elements of A are known elements of I are known lambda is a variable.

So, it becomes a polynomial of degree n this polynomial P lambda which is the determinant of a minus lambda I is called the characteristic polynomial in n th degree polynomial has the n roots. Let lambda 1, lambda 2, lambda n be the n roots of P lambda is equal to 0 from fundamental theorem of arithmetic, we all know that lambdas can be either real or complex; complex roots always come in conjugate pairs the reason complex roots come in conjugate pairs is that the elements of the matrix a are real this implies the coefficients of the polynomial P lambda are real and when you are trying to solve a polynomial with real coefficients the roots. If there is complex, it has to be complex conjugate that is for any general matrix for a special class of matrices when symmetric lambdas are real.

When A is symmetric and positive definite symmetric and positive matrices are called SPDs for symmetry PD for positive different definiteness lambdas are real and positive; this means that for a general matrix the for a general matrix the Eigenvalues lie in a complex plane. This is the real axis this is the imaginary axis. So, for a general matrix the Eigenvalue can be anywhere if it is complex, it might occur in conjugate pairs, it could be real, it could be positive, it could be here.

So, that is a general distribution of Eigenvalue for any general matrix for symmetric matrices; the Eigenvalues are always real the Eigenvalues are real this is for symmetric matrix for a positive definite matrix the Eigenvalues are always real and positive. So, you can see the restriction how it can strange the distribution of Eigenvalues, it could be anywhere in the 2 dimensional complex plane for a general matrix. It is along the real line for symmetric matrices it is in the positive half of the real line for symmetric positive definite matrices, we will have lot more occasions to talk about symmetric positive definite matrices.

So, this Eigen structure of symmetric positive different matrices is an important property that we need to keep in mind.

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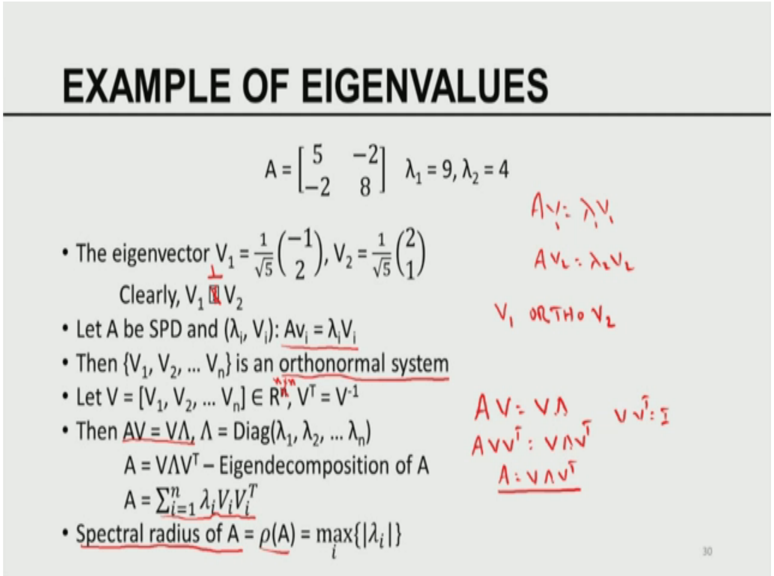

We are going to illustrate the computations of Eigen values and Eigenvectors; let A be a symmetric matrix by the previous claim, the Eigenvalues must be real, yes, 9 and 4; they are real, but by solving by solving the equation A V is equal to lambda V, A V 1, lambda 1, V 1 A V 2 equal to lambda 2 V 2, these are 2 equations corresponding to 2 distinct Eigen values. If you solve these linear equations, it can be found that V 1 is one Eigenvector V 2 is another Eigenvector; the Eigenvector, we are interested only in the direction of the Eigenvectors. So, we normalize it. So, V 1 is a normalized Eigen vector V 2 is a normalized Eigen vector it can be shown V 1 is this is not right, it is a perpendicular sign V 1 and V 2 are orthogonal V 1 V and V 2 V 1 orthogonal to V 2 orthogonal to V 2.

So, I would very much encourage the reader to be able to verify these computations. Now I am going to generalize this let a be a symmetric matrix, let lambda I V i be such that A V i is equal to lambda I V i for each i running from 1 to n, there are n such equations. So, we have a collection of vectors Eigen vectors without loss of generality as you mentioned Eigenvectors are going to be normalized. So, V 1, V 2, V n is a collection of mutually orthogonal and normalized Eigen vectors. So, it constitutes an orthonormal system, we have already seen the notion of orthonormality in the last class.

Now, I am going to construct a matrix V which consists of n columns the first column is the first Eigenvector the second column is second Eigenvector and the column is the n th

Eigen vector; this is a matrix there is a correction here, this is the matrix this n by n, this matrix is orthogonal. So, its transpose is equal to inverse. So, from the basic definition A V is equal to V lambda; this essentially tells you simultaneously all the equations that are summarized one for each i. So, this equation A V is equal to V lambda where lambda is the diagonal matrix. So, you can readily see a is the given matrix V is the matrix of n Eigenvectors lambda is a diagonal matrix of n corresponding n Eigenvalues look at the order lambda 1 lambda 2 lambda n V 1, V 2, V n, they are correspondence with each other since V transpose is equal to V inverse I can multiply on the right side by V transpose.

So, A V equal to V lambda. So, we can multiply A V V transpose is equal to V lambda V transpose, but V v transpose is equal to A is equal to i V V transpose is equal to i identity matrix. So, A is equal to V lambda V transpose; this is called the Eigen decomposition of a this Eigen decomposition of a can be expressed in element form. So, this is simply the sum of the product outer products of V i and V i transpose. So, V i V i transpose is a matrix each of these matrices are weighted by lambda i. So, a can be expressed as the weighted sum of rank one matrices each rank one metric corresponds to an Eigen vector the now we come to another important concept associated to this called spectral radius denoted by row of a spectral radius is equal to the maximum of the absolute value of the lambdas.

So, if A is a symmetric matrix lambdas are real, if A is a symmetric and positive rank lambdas are real and positive. So, the spectral radius of a symmetric matrix is given by the maximum of the absolute value of Eigenvalues.

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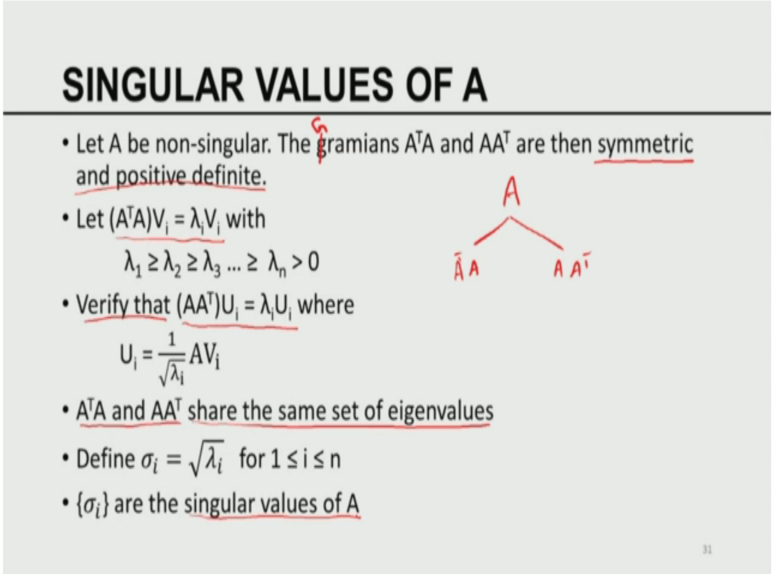

Now, we are going to introduce another related concept called singular values of A, let A be a non singular matrix the Gramian A transpose A and AA transpose R, then symmetric positive definite. In fact, there is a result here I would like you to think about the Gramian must be capital G because it is the name of the person capital G.

So, A is non singular A transpose A and AA transpose are symmetric matrix, if A is non singular, it is said to be full rank, if A is non singular and full rank, then AA transpose A transpose A are both symmetric and positive definite this is a very fundamental result with respect to with respect to the symmetric positive definite matrices and its relation to Gramian.

So, if A is non singular A transpose A is symmetric; therefore, I can do a symmetric decomposition Eigenvalue analysis A transpose A V i is equal to a lambda I V i. This is the same as we have done for A. Now what we did for A, I am redoing for A transpose A here lambda 1, lambda 2, lambda n are the Eigen values because A transpose A is positive that even the least Eigenvalue is positive we are going to order Eigenvalues the largest is called lambda 1. The next largest is called lambda 2 the least largest is called lambda n in the least largest is also positive; that means, everybody else is positive.

Now, I would like to relate the Eigenvalues Eigen vectors of a given a matrix A, there are 2 Gramians A transpose AA; A transpose both are symmetric and positive definite, I am now going to argue, if you know the Eigenvalues and Eigenvectors of one of the

Gramians; we also can infer the Eigenvalues and Eigen vectors or the other Gramian to that end, I am giving it a homework problem to verify it is very simple A transpose A times U i is equal to a lambda I U i where U i is different by 1 over square root of lambda I A V i. So, if you, but if i know a i know A transpose A; if I know A transpose A i know lambda I V i. So, if know A i know V i; I know lambda i. So, using A V and lambda you define a new vector. So, new vector U i is simply a linear transformation of the vector V i scaled by 1 over square root of the Eigenvalue.

So, this if I define U i this way, it can be verified that A transpose of AA transpose U i is equal to lambda I U i. So, this essentially tells you lambda I is simultaneously Eigenvalue of AA transpose as well as A transpose A; they both share the same Eigenvalue the Eigenvectors V i and U i are related by this. So, here is a summary A transpose A and AA transpose share the same Eigenvalues and the Eigen vectors are also related you can essentially see U i is related to the V i. Now if I define sigma i to be square root of lambda I know please remember lambda Is are Eigenvalues the symmetric positive definite matrix they are all positive. So, square root of that exists and square root of that is real.

So, I am now going to define the positive square root of lambda I equal to sigma i and the sigma. So, for each lambda I there is a sigma i there are n such sigma i sigma i by definition are called singular values of A. So, the Eigenvalues of A transpose A are called the Eigen the square root of the Eigenvalues of A transpose A are called the singular values of A. So, singular value decomposition singular values Eigenvalues Eigen decomposition these are all the related concept that we are seeing in this part of the talk.

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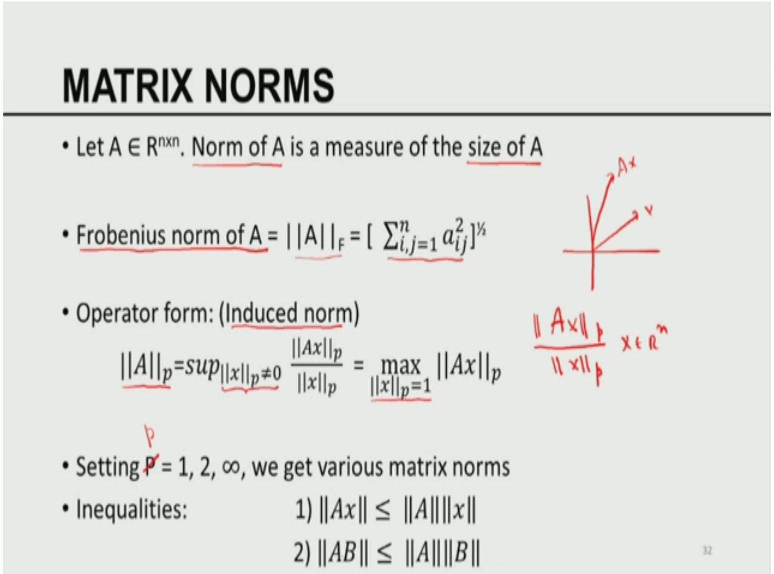

Now, we move on to another interesting concept relating to matrices just like vectors have a size just like the size of a vector is captured by the notion of a norm of a vector matrix is also an object every object can be endowed with the definition of its size. The size of a vector is measured by the norm of a vector the size of a matrix is also going to be defined by a norm of a matrix. So, I am now going to define the notion of what is called norm of a matrix A, it is the measure of the size of the a just like in the vector case we had various norms 2 norm one norm infinity norm in Kovelsky's norm energy norm; in the case of matrices also we have quite a variety of norms to talk about I am not going to talk about all the possible norms.

I am going to talk about some of the simple norms which are often used in analysis the first of those norms is called the Forbenius norm Forbenius norm of a is simply an extension of the Euclidian norm for the matrix A; the Forbeanius norm is denoted by this symbol the norm sign with a subscript f and what is it you take the sum of the squares of all the elements of the matrix take the square root of it. This is exactly the way we are defined the Euclidean norm. The Euclidean norm of a vector is equal to the square root of the sum of the squares, here it is a square root of the sum other of the squares of all the n square elements in the matrix and that is one measure of the size of the norm there is another norm called induce norm. These induced norm are defined using the notion of an operator.

So, let A be a matrix that corresponds to a linear operator or a linear transformation the p th norm of a defined by the norm symbol a with a subscript p that is defined by the suprimum taken over all x that is not 0 of the ratio A x p norm divided by x p norm. So, you can essentially see the following given a given a pick any arbitrary vector x A x is a vector computes its p norm x has also its p norm compute this ratio this ratio varies a is fixed x varies you vary x; x belongs to R of n there are infinitely many x s as you vary x this ratio varies as this ratio varies, I am interested in the maximum value. So, suprimum is like you can think of supreme is a very technical term I do not want to get into the technicality for practical purposes you can assume it is a maximum value of the ratio of the p norm of A x to the p norm of x.

So, what does this tell to the following a 2 dimensional analogy is like this here the vector x here is the vector a f x the a f x has. So, the numerator tells you the p th norm of a f x the denominator tells you the p th norm of x if this ratio is larger than 1 A x is larger than x; that means, there is a magnification if this A x if the numerator is less than the denominator then there is a shrink.

So, a linear operator can either along need a vector or a shrink vector the maximum of this ratio the magnification factor is called the p th norm off off of the operator a or a linear transformation a equalently, we can also compute the p th norm p th norm of f where x is constrained by this relation in other words you can consider all those vectors that whose p th norm is 1.

So, you reduce the range of values of n which is which is which is equivalent to this definition. So, this is how you define the p th norm of a matrix by setting p is equal to by setting I, it should be lower case p by setting p is equal to 1 to infinity, we get various matrix norm, you can get 1 on 2 norm infinity norm and so on given the matrix. Now that we have a matrix norm, we have a vector norm there are standard inequalities which are of great interest improving several results in analysis. So, then norm of a transformed vector. So, x is a vector A x is another vector A x is a transformation of x by a.

So, what does the left hand sides say in the inequality one the norm of the transformed vector is less than or equal to the product of the norm of the operator a and the norm of the vector x likewise the norm of the product of 2 matrices a and b is less than the product of the norm of the operator a and the operator b these are 2 fundamental inequalities. Now please realize in this inequality I did not specify the nature of the norm these inequalities true for any and every type of norm you can pick a 2 norm; 1 norm infinity norm or any other norm for all of these norms these inequalities all good these are fundamental inequalities and these inequalities are very similar to several inequalities we have seen for the vectors.

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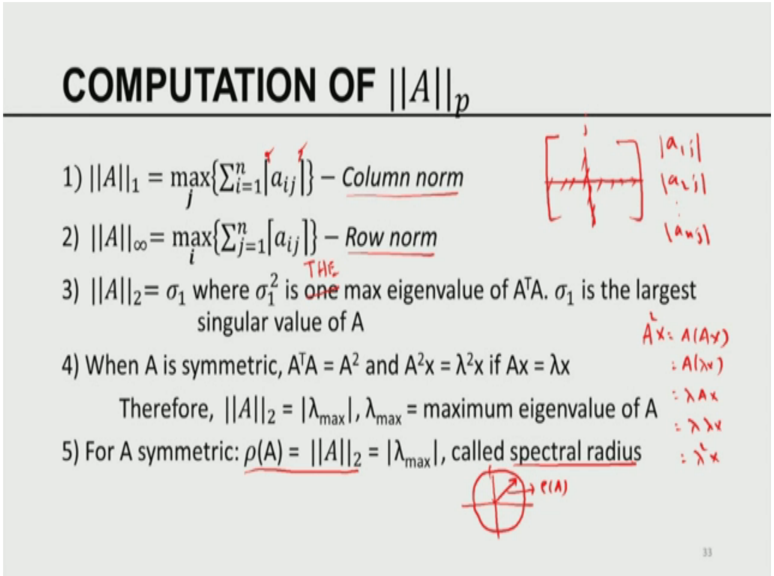

Now, we have defined the norm, but the whole question is how do I compute these p norms how do you compute in other words; how do I compute these various norms for matrices here is an example of the computation if a is matrix the one norm of a it can be proven that is equal to the maximum over j of summation i equals one turn a i j. So, let us talk about this. Now I have a matrix A i have a matrix AA has different columns. So, let us consider the j th column of a the elements of the j th column are going to be a one j a 2 j and a n j.

So, what is that we are now going to be looking for we are going to be looking for the absolute value of each of these and I am going to take this sum of the absolute value this must be absolute value sum of the absolute value of the elements and take the maximum over j. So, one is called the column norm another is called the row norm. So, the maximum is taken over j for the column norm because j is the column index i is the row index. So, now, look at this now. So, for one you sum along the row and for another one you sum along the column.

So, one is called the first one the one norm is called the column norm the infinity norm is called the row norm. It can be shown that one norm can be easily computed by this infinity norm can be in easily computed by this these are computational algorithms for quantifying the values of these norms the 2 norm of a matrix is where is can be simply stated as sigma 1 where sigma one square is the is the maximum Eigenvalue this must be the maximum Eigenvalue the maximum Eigenvalue of A transpose A sigma one is also called the largest singular value we simply introduce the notion of a singular the singular value in the last couple of slides.

So, given A; you compute A transpose AA transpose A is symmetric and positive definite if A is it is non singular; it is symmetric and positive definite. So, all the Eigenvalues are real and positive the square root of these Eigenvalues are called the singular values, the maximum of those singular values; it is called the 2 norm of its call the 2 norm of when A is symmetric; it transposes the A.

So, A transpose A is a square a square x is equal to lambda square x of a is equal to lambda x what is that a square x is equal to a times a of x this is a times lambda of x this is equal to lambda times a of x there is equal to lambda times lambda of x that is equal to lambda square of x therefore, if lambda is an Eigenvalue of a lambda square is an Eigenvalue of a square.

If lambda is an Eigenvalue lambda to the power k is Eigenvalue of a to the power k. So, that is how the Eigenvalue square itself when you square the matrices therefore, by combining 3 and 4 we can see the 2 norm of a is simply the maximum of the Eigenvalue I do not have to even put the absolute value sign because A transpose A is positive symmetric and positive definite.

So, sigma one is always positive, but for safety sake one can introduce with our loss of generality and lambda x is the maximum Eigenvalue and we can also recall that the maximum Eigenvalue is called the spectral radius. Therefore, we can conclude the 2 normof a symmetric positive definite matrix of asymmetric positive definite matrix is given by the spectral radius spectral radius.

So, what is it what a spectral radius means if you consider a circle with centre or origin and diameter as I am sorry that the radius as rho of a all the Eigenvalues of the matrix a lie within that circle.

So, that is the notion of the spectral radius of this matrix A. So, we talked about matrices there are norms; we have studied various properties of norms these are the computational procedures for computing the values of different norms of interest in analysis.

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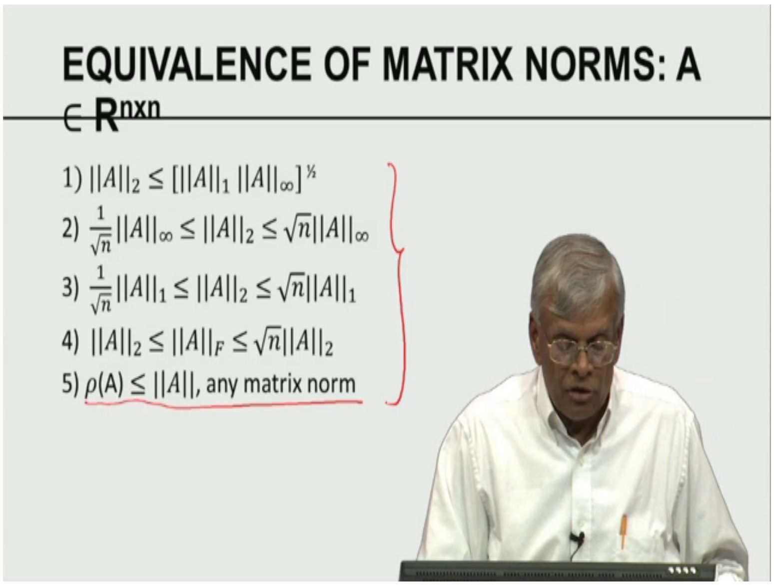

Just like we had talked about equivalence between the one norm 2 norm infinity norm for vectors here also, I have a set of inequalities that relate to the behaviour of various norms. So, you can show given a matrix A the 2 norm is less than or equal to the product of the square root of the product of one norm and infinity norm.

The infinity norm and the 2 norm, I can sandwich the 2 norm using the infinity norm I can sandwich the 2 norm by one norm, I can sandwich the Forbenius norm by 2 norm, we also know that another result which is a fundamental important the spectral radius is less than I am sorry the spectral radius the spectral radius, I want to highlight this the spectral radius of a matrix is less than any matrix norm equality happens when the metric system is a symmetric.

So, these are some of the interrelations between the 2 norm the one norm the infinity norm the Forbenius norm of matrices in the case of matrices. In the case of matrices, the Eigenvalues play definitely role in the definition of norms especially for the 2 norm and this is a very nice summary of the various properties of norms of matrices.

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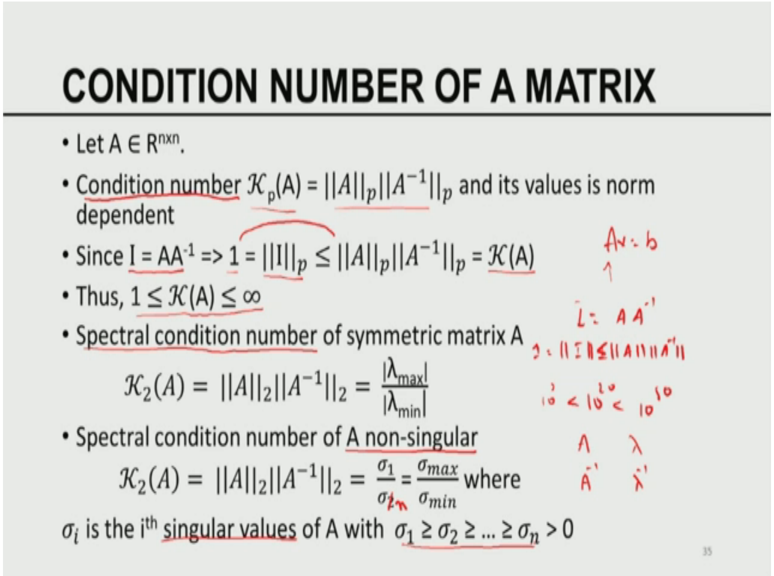

Why do we do norms for 2 reasons one is to be able to measure the size of the norm. Secondly, the notion of 2 norm is very useful in trying to quantify certain properties of matrices we say a matrix a singular we say a matrix is non singular we say a matrix is well conditioned we say a matrix is ill conditioned one of the conditions for the solution of A x is equal to b, we all know if I want to be able to solve A x is equal to b, we would like to be able to make sure a is non-singular.

We also know when. So, we have singular and non singular yes or no day and night, but in practice some matrices may be very close to being singular without being singular. So, such matrices are said to be ill conditioned. So, I need to be able to characterize the degree of non singularity how do you measure the degree of non singularity one way to be able to measure the degree of non singularity is through the notion of what is called a condition number of a matrix.

So, let A be n by n matrix this is the definition the condition number of a matrix is denoted by the symbol kappa of a the condition number is dependent on the definition of norm. So, this is the condition number using p norm of a matrix the condition number of A p norm of a matrix is simply the product of the p norm of A times the p norm of A inverse.

Therefore you can see the definition of the condition number is non independent. So, I can have norm one conditioning norm 2 conditioning norm infinity conditioning so on and so forth; this in general if you cannot solve the equation A x is equal to b, we throw the word or the matrix is ill conditioned. So, if something is ill conditioned then there must be a concept of well conditioned is something is singular non singular very nearly singular, these are all fuzzy characterizations of properties of matrices we would like to be able to quantify this fuzziness using certain measure of the properties of these matrices that is where the condition number comes into play; how the condition number is related to the well conditioning ill condition of the matrices that is what we are going to be talk talking about presently recall the standard identity i is equal to AA inverse the p norm of i is one for every p 1 to infinity.

So, by, but we know. So, if I is equal to AA inverse; if I took the norm of i the norm of I, it must be less than or equal to norm of a times norm of a inverse this is inequality that we saw in couple of pre slide the 3 slides, but the norm of identity matrix is 1; therefore, I get this inequality one is less than the product of the p norm of A p and A p inverse A p inverse by this definition the product of a p; A p inverse is the condition number. So, you can readily see the condition number of A is always greater than equal to 1 is always greater than equal to one. So, condition number is greater than equal to one condition and that is a positive number it can be very large. So, the range of the values of the condition number is one to infinity

So, in this scale when the condition number is closer to one we say it is well conditioned when the condition number is very large is ill conditioned again how large is large we will talk about that in a minute; how large is large depends on the computer machine position in a thirty bit the arithmetic well that is only a largest value you can measure therefore, if a condition number kappa of a matrix; let us say is 10 to the power of 20 or 10 to the power of 50 a matrix A 10 to the power of 50 is said to be more real conditioned than a matrix is 10 to the power of 20 which is more real condition than a matrix with 10 to the power of 3.

So, this ranking of the of the condition number helps you to in some sense quantify the degree of ill condition associated with the matrix . So, now, let us; I used the p norm, I am now going to specialize the discussion of the norm for a spectral condition number. So, let a be a symmetric matrix spectral condition number is related to the maximum Eigenvalue we also know the following if lambda is an Eigenvalue of a lambda inverse is the Eigen value of a inverse therefore, the 2 norm of a is the maximum Eigenvalue of a the 2 norm of a inverse is the minimum Eigen value of A. So, condition number for symmetric matrices is simply given by the ratio of the maximum Eigenvalue to the minimum Eigenvalues.

Therefore the spectral condition number so, for a symmetric matrix this for a general non symmetric, but non singular matrices the condition number is simply given by sigma one by sigma 2 where sigma one is the largest I am sorry this must be sigma n sorry this means be sigma n, this is the ratio of the largest to the smallest singular values where sigma i is the i th the singular value and sigma the signal values are counted like sigma one is greater than equal to sigma 2 there than equal to sigma n and sigma n is positive.

Now, for this slide provides you a new a concept of associating a number with the matrix called the condition number the value of the condition number is very indicative of the difficulties that one will have in computational process.

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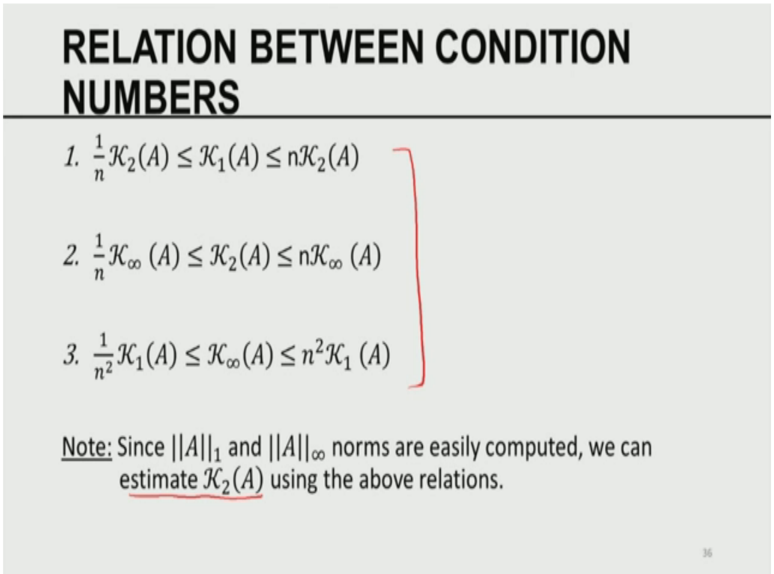

Before going further I also want to be able to relate the various properties of condition numbers again matrices are related the Eigenvalues the matrices these singular values are related the norms are related they. So, there is a relation between all the condition numbers themselves because condition numbers are defined in terms of norms.

If norms are related if condition numbers are related to norms condition numbers also must whole certain relations among themselves. So, the 2 condition number one condition number infinity condition number and 2 condition number infinity condition number and one condition number you can say they are all interposed. So, what does this mean if a matrix is well conditioned in one norm, it is well conditioned in every norm if a matrix is ill conditioned one norm, it is ill condition in every norm. So, what does this tell you can pick any now that suits you computationally and do the analysis without having to worry about the choice of the norms? So, that gives you provides that provides you a lot of freedom.

But among all the norms or the one norm and infinity norm are is very easily computed when is the column norm another the row. Now, therefore, from a computational perspective one may want to be able to use one norm or infinity norm, but in mathematical analysis theoretical analysis, they generally often use the 2 condition number 2 now because 2 condition about 2 norm is intimately associated with the Eigen structure spectral radius and so on that is a very appealing very appealing property.

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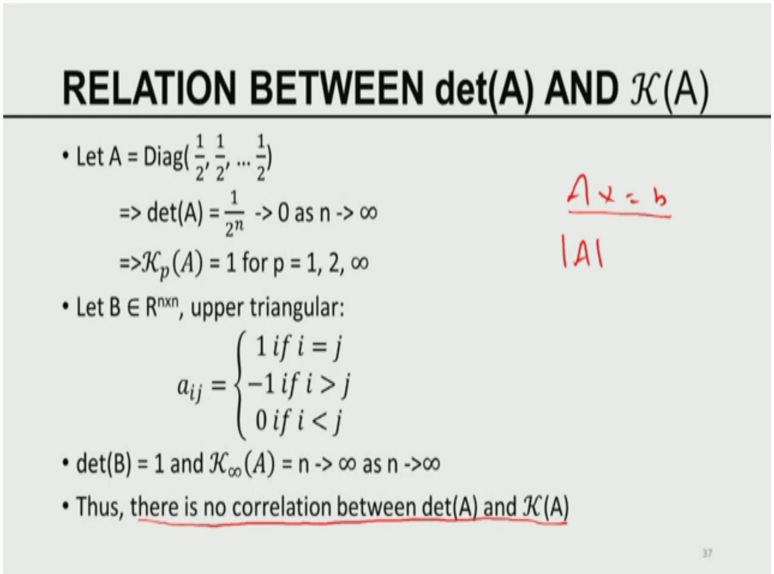

In the first course, in linear algebra, we are generally told the ill condition of the matrix is decided by the value of the determinant, but I am going to give you a counter example to show, it is not the case in other words what we are told in a first course in linear algebra if I have difficulty in solving A x is equal to b if I am; if I have difficulty in solving A x is equal to b if the determinant of a is very large or very small then they will simply tell that that you will have numerical difficulty yes you may have numerical difficulty, but the ill conditioning are the well conditioning of a matrix is not determined by the determinant of a matrix as might often be given to understand in the first course.

So, here there are a couple of very good examples let a be a diagonal matrix of all halves the determinant of the determinant of a is 1 over 2 to the power of n you can readily see the determinant of a goes to 0 as n goes to infinity, but the condition number of a is one for all p. So, the determinant and condition number they do not have much of a relationship as another example let b be a n by n matrix consider an upper triangular matrix given by this we can readily see the determinant of b is one, but the condition number of a infinity condition number is n and go that goes to infinity as n goes to infinity.

So, what does it mean I can have matrices where the determinant goes to 0, but the condition of the remains constant I can have matrices where the determinant remains constant, but the condition about can go to infinity.

So, this essentially tells you there is no intrinsic correlation between determinants and condition number even though we simply say a matrix must be non-singular; that means, the determinant that should not vanish for being able to solve A x is equal to b the appropriate way to describe the properties of solution one obtains from solving a linear equation one has to relate it to the condition number of a kappa. So, kappa is much more important than the determinant why kappa is more important.

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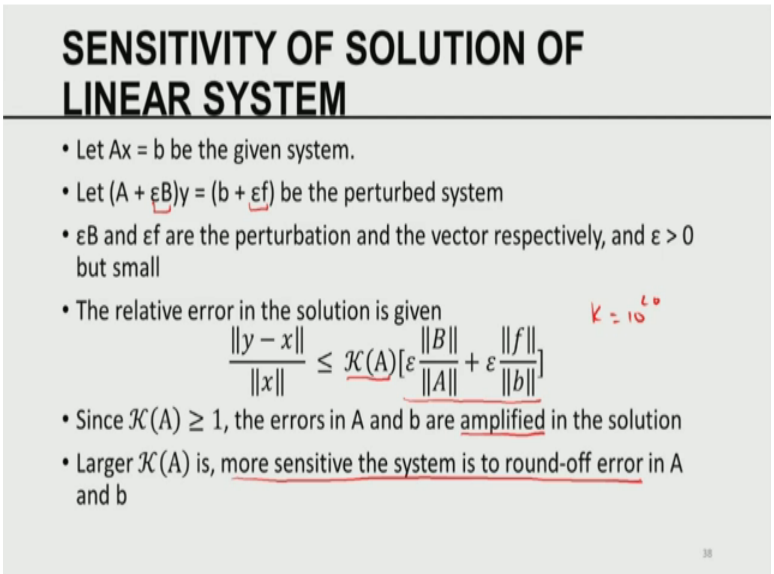

Now, I am going to give you another result that will force the importance of kappa the condition number within the context of solving linear systems; let us suppose I want to solve A x is equal to b.

Now, let us think of the possible way suppose you want to enter the number 1 over 3 and you press the key 1 over 3. So, 1 over 3 is supposed to be stored in your machine, but 1 over 3 can never be stored correctly its point 3 3 3; what is the problem 1 over 3 does not have a terminating decimal expansion only numbers that have terminating decimal expansion will be able to one can hope to be able to represent them correctly.

So, 1 over 3, 1 over 7; these numbers once you store them to start with there is an error only rational numbers have terminating fraction a rash a general real numbers may not have terminating fraction when you do arithmetic you cannot confine yourself simply to rational arithmetic we are supposed to have real arithmetic. So, when you try to store a real number in a finite precision machine there is always error in representation; that means, you start with your left foot.

So, when you think you have solving A x is equal to b you are not actually solving A x is equal to b you are solving a plus epsilon b y equal to b plus epsilon f. So, what does it mean the epsilon b is the error in the matrix a epsilon f is the error in b there are 2 kinds of errors a may be obtained from experimental that that could be an inherent error in the experimental measurements a the number is told them storage error. So, epsilon b in this case I am simply I am not worrying about other errors that arise out of finite precision arithmetic.

So, epsilon b; b is the matrix the epsilon is a small number. So, if I am thinking I am storing a; you are not storing a you actually are storing a plus epsilon b, you do not know what epsilon b, but you know that there is an error epsilon f is likewise an error. So, why is the system; you are solved; why is the solution system you are solving and you are pretending why is x this is the game we all play, that is nature of business. So, epsilon b and epsilon f are the perturbations of the matrix and the vectors respectively.

But we are epsilon is greater than 0, but small. So, if y is not equal to x there is an error I am not going to consider the relative error in y. So, y is a vector x is the true solution y minus x is the error vector in the solution I am going to take the norm of the error divided by the norm of the true solution. So, what is that call the left hand side is called they are real I am sorry relative error in the computed solution I am not going to show the derivation the derivation I generally do it in my class, but it you take us too much into the outside of this scope of these lectures it can be shown that this relative error is bounded about by the product of condition number times the epsilon divided f times b by a plus epsilon times f by b.

Now, let us talk about b what is b? B is the error matrix that corrupts AA is the real matrix. So, this is the relative error in a; this is the relative error in b epsilons are the are the are the multiplying factors the same epsilon in here. So, the right hand side is the constant multiple of epsilon times the sum of the relative errors in the matrix and on the right hand side now the computer precision decides what b is the computer precision decide what f is epsilon is decided by the smallest value of the computer can store.

So, all these factors are decided by the computer architecture who depends. So, what else. So, your relative error is bounded by can be magnified by the product kappa a times the some other relative errors. So, if kappa is large your solution could be much more erroneous your know kappa by a small your solution could be much more precise therefore, this is the reason why we call kappa the condition number it is a conditioning the matrix that relates to the quality of the solution obtained by any method that you use to solve A x is equal to b now what is any method I saw what are the methods we know how to solve A x being a ton of methods no matter whatever than a third you use this you are you are bound by the inequality.

So, if kappa is. So, if kappa is 10 to the power of twenty means why you a relative error can go up to 10 to the power of twenty if the relative error can go up by twenty power of twenty what does it mean you have spent the money, but the result is not what the paper written it. So, that is the importance of the notion of condition number why is this important people in meteorology will say I am using a 3 D-VAR I am using a 4 D-VAR, I am using this I am using that yes all those algorithms are very well understood very well known, but you need to be cognizant to the fact that the solution that these algorithm output the quality of it is decided by the nature and properties of the matrix that go into the computational process. So, since kappa is greater than one errors in a and b are amplified.

So, this is the keyword amplified the larger kappa more sensitive the system to the round of errors round of errors comes because of finite precision arithmetic. So, how. So, here is a beautiful idea. Now I have a problem to solve, I have an algorithm to solve the problem I have a computer architecture on which the algorithm is implemented here we talk about the effect of computer architecture the finite precision arithmetic could have on the quality of the solution that you are going to put. So, it is a beautiful combination of algorithms and architecture how they are melded together to give a solution whose quality can be quantified like this.

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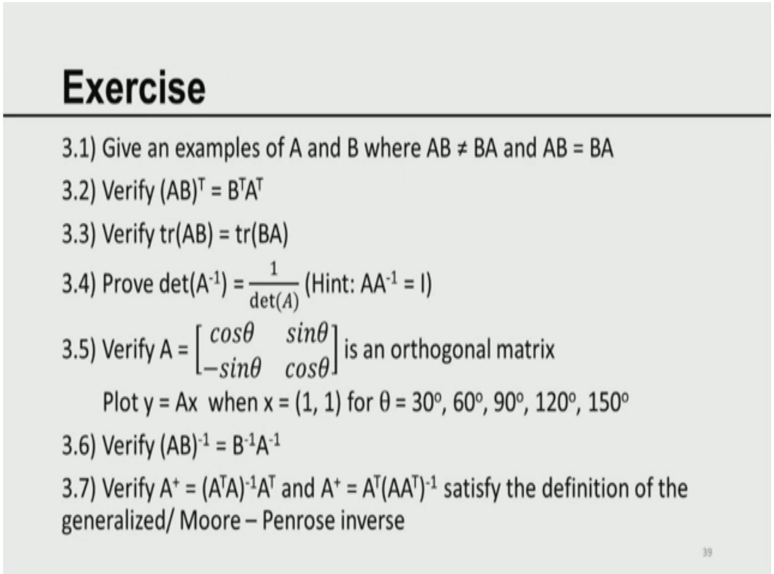

With that we end the coverage of review of matrices I am going to suggest several exercises and they are given in these problems.

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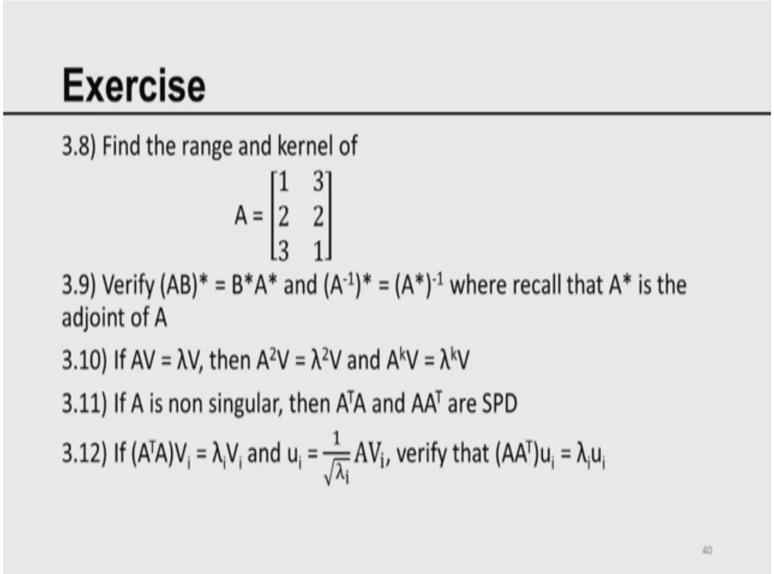

There are about twelve problems in here and I will strongly encourage students to use pencil paper work do not go; do not write a program you should know it first to be able to do with hand before you do with computers. So, all these problems are very simple and fundamental to understanding many of the concert will be cover.

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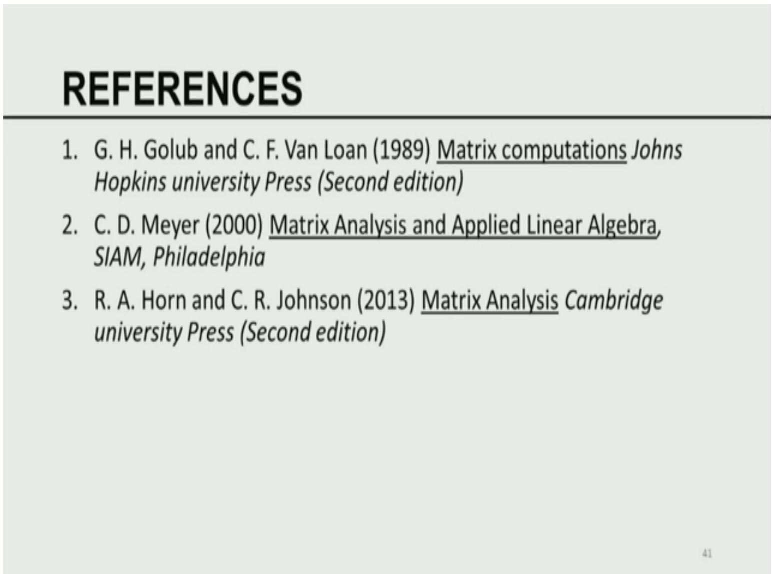

If you want proves of many of the things that we have done in this lecture you can refer to the 3 times standard textbooks these are my favourite one Golub and Van Loan Meyer Horn and Johnson. So, with this we conclude our coverage of overview of many results from matrices you can see we have reviewed a ton of results you may wonder do we need all of them you will soon see, we will use almost all of them in our analysis of algorithms expressions.

Thank you.