Dynamic Data Assimilation
Prof. S Lakshmivarahan
School Of computer Science
Indian Institute of Technology, Madras

Lecture - 04
Finite Dimensional Vector Space

In module 1 the introduced various aspects of data mining, data assimilation, and prediction and their relations to other branches of science and engineering. This course is a mathematically oriented course on data assimilation. So, we are going to provide all the mathematical tools and techniques that would be needed, to be able to pursue research and education in data assimilation areas. With that in mind in module 2 we have several sub modules in 2.1 we are going to quickly review the concepts of finite dimensional vector spaces. The notion of finite dimension vector spaces is fundamental to performing any computational process. In 2.2 we are going to be talking about all the results that one would need from matrix theory, and then we will review concepts from multivariate calculus, then as the last part in this module, we will also review some of the basic principles from optimization theory.

So, a strong grounding in basic understanding and (Refer Time: 01:40) vector space matrix theory, multivariate calculus, and optimization tools and techniques are fundamental to any serious pursuit of data simulation. So, we will start with a quick review of fundamental principles from finite dimension vector spaces.

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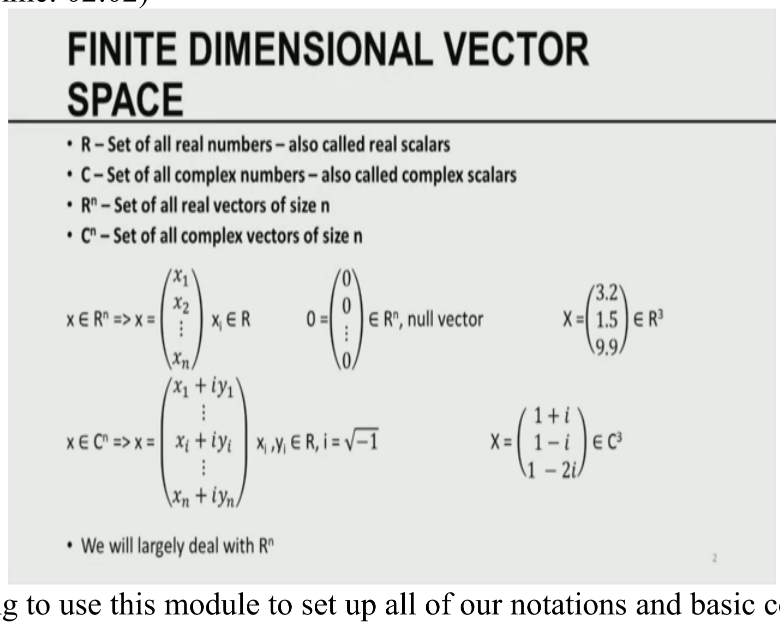

I am also going to use this module to set up all of our notations and basic concepts. So, R is a set of real numbers, they are also called real scalars, C is a set of complex numbers is called complex scalars R n R to the power n refers to set of all real vectors of size n, C of n set of all complex vectors of size n, we are giving some examples now. X belong to R n implies x is a vector with n components, the components are written column wise each of the component x i is a real number, 0 is a vector, 0 vector consists of all zeros is called a null vector. Then n is 3 here is an example of a vector 3.2 1.5 9.9. 3.2 is the x 1 to 1.5 is x 2, 9.9 is x 3. Here is an example of a complex vector in a complex vector again there are n components each component is a complex number. So, first one is x 1 plus i y 1 the ith 1 is x i plus i y i and x n plus i y y n is the n th element to the complex vector. Here x i and y i are real numbers, i is the unit imaginary number square root of minus 1, here is an example of a complex vector 1 plus i 1 minus i 1 minus 2 i this is a complex vector of size 3.

Even though we talked about complex as well as real spaces of vectors, largely in this course we will deal with real spaces especially r to the n.

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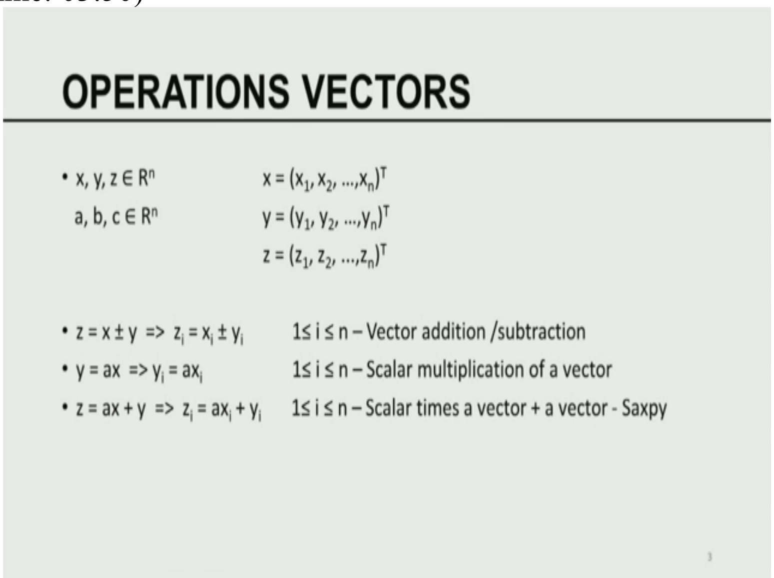

I am going to quickly review some of the concepts from operations and vectors, x y z be vectors let a b c be vectors, I am sorry there is an error a b c belongs to R not R to the n. So, there is an error we will we will correct that. X is a vector, y is a vector, z is a vector. Z is a sum of x and y are difference of x and y, Z i is the i th component. So, z i is either sum of the 2 components or the difference of the 2 components, this is called vector addition vector subtraction y is equal to a times x a is a scalar. So, this is called scalar multiplication of a vector, y i is equal to a times x i scalar multiplication of a vector x by a scalar a z is equal to a x plus y in this case i th component z is equal to a times x i plus y i this is called Saxpy scalar times the vector plus a vector. So, these are the basic operations on vectors that we will be dealing with.

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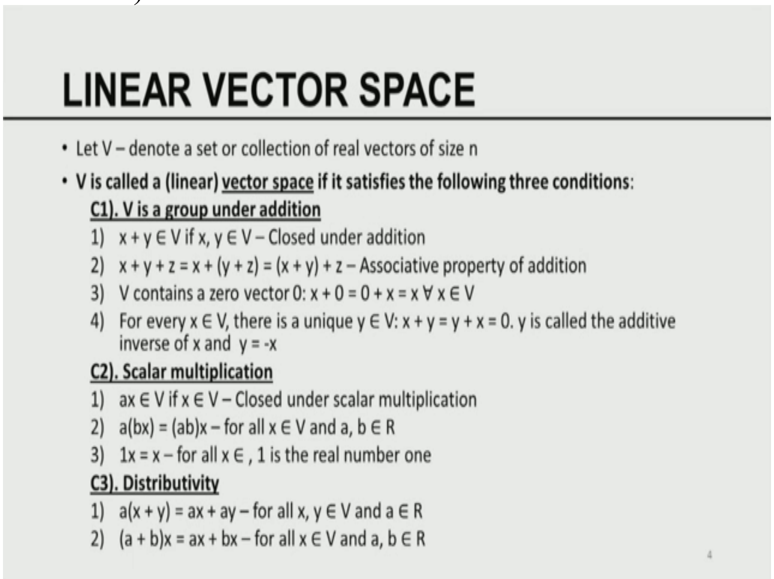

Now, I would like to introduce the notion of what is called a vector space, let v denote a collection of real vectors of size n. So, this v to be called a linear space or a vector space or a linear vector space, there are several names associated with it, if it satisfies the following 3 condition. The first condition is v 1 is a group under addition what does it mean? If |I took any vectors in V that sum is also in V it is closed under addition, this operation of vector addition is also associative. So, if I am given 3 vectors x plus y plus z, in order to find please remember addition is a binary operation I can only add 2 numbers at a given time. So, if I given 3 vectors I have to make 2 additions, you do 1 at a time either you add y plus z and to the sum you add x or you add x plus y to the sum you add z, such a property is called associative property of addition.

So, what does the associative property essentially tells you? The order in which you add does not affect the results of the computation. V contains a 0 vector, what is the profit is 0 vector? If you add 0 to any vector it remains the vector does not change, x plus 0 is 0 plus x is equal to x for all x. For every vector x there is a unique y such that x plus y is equal to y plus x is equal to 0 that is called additive inverse of x and y is called minus x any collection of vectors that satisfies these properties closed under addition, it is an associative property it contains a 0 vector, and its it also an additive inverse such a set is called a group. So, v first be a group, second one there are properties (Refer Time: 07:05) scalar multiplication a times x, if you if x belongs to v a x also belongs to V so; that means, any vector if you multiply by a constant it is in the same set. If a and b are 2 scalars you can multiply the vector x by b and then by a, that is equal to multiplying a and b and then multiplying with x that is again a kind of an associated with respect to scalar multiplication. One times x is the x. So, one is the real number 1 if you multiply any vector by the number 1 it does not change is itself.

The third property is called distributive property, a times x plus y is equal to a times x plus a times y for all x and y. So, first one is scalar multiplication distributes itself with respect to addition, the second one is scalar addition distributed itself with respect to vector multiplication by a vector. So, a plus b times x is a x plus b x for all x. So, any collection of vectors that satisfy these 3 properties c 1, c 2, c 3 is said to constitute what is called a linear space a vector space or a linear vector space. For all computations there must underlie always a finite dimensional vector space, has the bay as a base on which it is all the computations are done.

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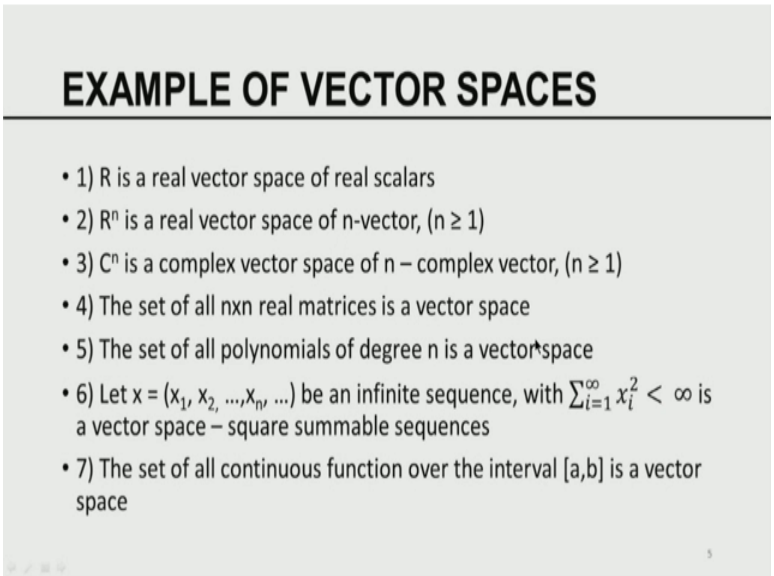

So, this is the general definition of what a vector space, is vector space comes in various shapes and forms. The set of all real numbers is the vector space, it satisfies all the properties. The set of all real (Refer Time: 08:42) satisfy these properties the vector space set of all complex vectors satisfy all the things. The set of all n by n real matrices is a vector space; set of all polynomials of degree n is a vector space. If we have sequence infinite sequence such that the sum of the squares is finite, its called square summable sequence the set of all square summable infinite sequences they also form a vector space. The set of all continuous functions or interval a b is also in vector space. So, you can see vector space of functions, vector space of sequences, vector space of polynomials, vector space of matrices, vector space of complex numbers, real numbers and real vectors. So, vector spaces are abundant every one of these vector spaces constitute the basis for computational processes, and data assimilation is largely a computational problem because I need to be able to estimate fit the model is a solving an inverse problem, being a computational problem I must always be concerned with what is the vector space in which I am performing all these computations.

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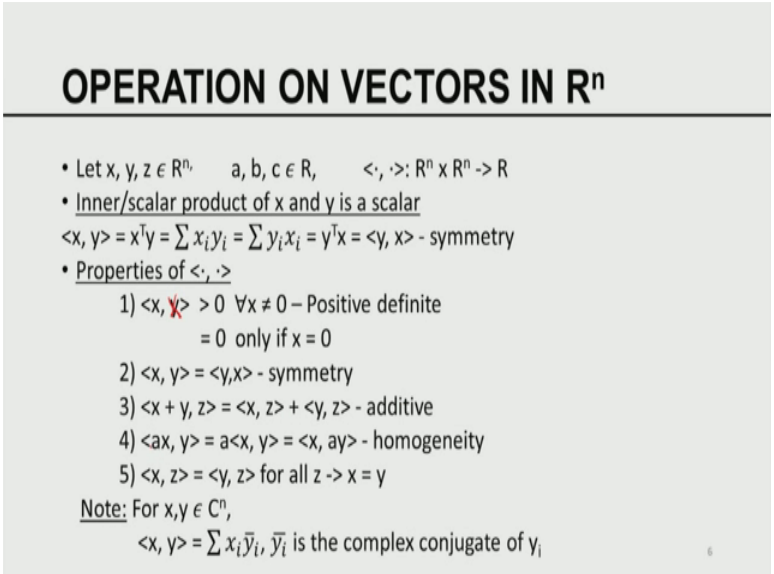

I am now going to quickly review operations and vectors, some of you who have taken a course in linear algebra may already know this I am assuming that all the people who are going to be reading this may not have the same background and so to bring uniformity in the reader, I am going to quickly review many of these concepts. So, let x y z be 3 vectors a b c be 3 real numbers, I am going to introduce this bracket notation opening bracket dot comma dot closing bracket that is going to be a binary operation on vectors. So, that binary operation in here is called an inner product. So, parentheses inner product y x and y defines an inner product which is defined at x transpose y, which is defined as some of x i, y i, x i y i is also equal to y i x i, because multiplication of real numbers is commutative. So, that is equal to y transpose x is equal to y comma x. So, this means the inner product, I am not only defining the inner product, but I am also showing the inner product as an intrinsic property it is symmetric so, the symmetry property.

So, the properties of inner product x dot y is greater than 0, if x is not equal to 0 it is 0 only if x is equal to 0. So, this y must be this y must be x 1 second I will tell that now this y must be x, then the definition is correct. So, inner product of x is itself is greater than 0 when x is not zero inner product of x is with the x is 0 only when x is 0 that is called a positive definite property. We have already seen the symmetric property, inner product is also additive inner product of x plus y with z is inner product of x with z inner product of y with z the sum of the 2 it will product is said to be homogeneous what does it mean. If I multiply 1 of the components by a constant a; inner product of a x comma y is a times inner product of x and y, it is also same as x times inner product of a y that is called the homogeneity property.

If the inner product of x and z and y and z are equal for all z that x and y must be equal that is again another fundamental property of inner product, we will use all these properties in developing a joint technique or joint methods when we do four d var methods. When x and y belong to C n the inner product is defined by x i y i bar y i bar is the complex conjugate of y. So, the inner product definition has to be appropriately modified when you go from real domain to the complex domain. Since we are going to be dealing only with the real domain, these five properties of inner product are sufficient for our purposes.

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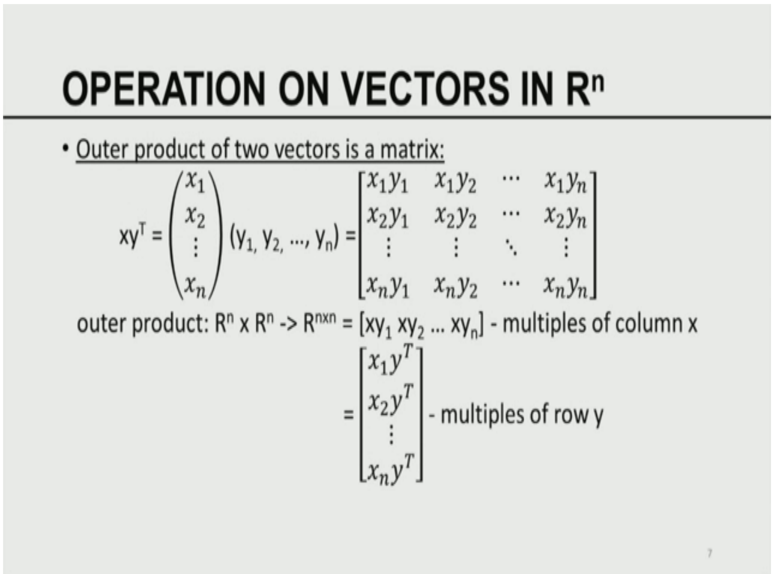

Now, I am going to define other operations and vectors x is a vector, y is the vector by vector i always mean a column vector. So, y transpose is a row vector. So, x y transpose is the product between a column vector and a row vector, the product of column vector row vector is called an outer product of 2 vectors, the result is a matrix x 1 y 1, x 1 y 2, x 1 y n and so on you can see the elements of the matrix coming in here. So, the outer product can be written in many ways, the first column is a multiple of the vector x by y 1, the second column is the multiple of x by y 2, the last column is a multiple of x by y n likewise I can also consider as a multiples of rows. The first row is the multiple of the row y with x 1 the second row is the multiple of the row y with x 2, the last row is the multiple of the row y with the x n. So, I can think of it as a matrix or multiples of column x or multiples of row y. All these are properties of the outer product of matrices outer product of matrices is a fundamental operation.

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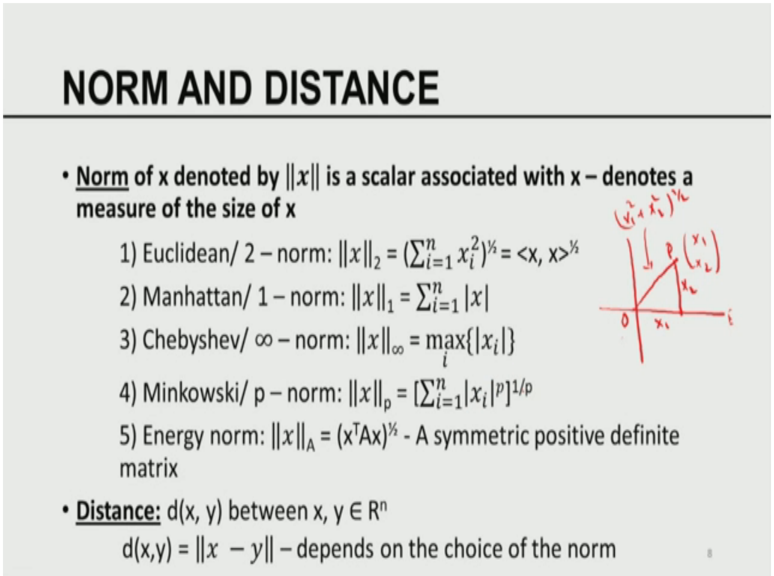

The next one is called the norm of x and the notion of a distance. The norm of x is denoted by x within that sign 2 vertical to the left 2 vertical to the right, a vector is 1 object the norm of a vector is another object, the norm is a scalar associated with every vector there is a norm. Norm is a measure of the size of the vector, the size of the vector is denoted as a scalar. The norm of a vector arises in many ways, one is called the equality norm, another is called the Manhattan norm, another is called Chebyshev norm, another called Minkowski's norm, another is called the energy norm. The Euclidean norm is a standard 1 that comes from the Pythagorean theorem. The norm of x is equal to square root of the sum of the squares of x, that can also be expressed as square root of the inner product of x with the x the Manhattan norm or one norm is essentially sum of the absolute values of x i would like to be able to bring the distinction between Manhattan norm and the and the Euclidean norm. So, if I have a 2 dimensional plane, if I have a point here, if this is x 1 this is x 2, the Euclidean norm refers to this distance and what is the value of this distance this distance is equal to x 1 square plus x 2 square to the power half.

That comes from the length of the hypotenuse is this is x 1, this is x 2, it is a right angle triangle with 2 sides x 1 and x 2 that is the length of the hypotenuse. So, that is the 2 norms. The one norm on the other hand is if I want to go from 0 to this point, I have to go by x 1 then I had to go by x 2, it is sum of the distances from 0 to there let us talk about this now. Suppose you have to go. So, let us assume this is the point o this is the point p, if I want to go buy a taxi from point o to point p i will first go along the street to the east and then I will go along the street to the north. So, the total distance travelled b y a taxi cab is x 1 plus x 2, but if i had a helicopter I could fly directly from o to p, and that is the Euclidean norm. So, that is the 2 ways of differentiating the 2 norms Chebyshev norm is called the infinitely norm and that is Chebyshev is a famous Russian mathematician.

And he defined the norm to be the maximum of the observed values of I that is under useful definition of a norm. Minkovski another mathematician from Russia he defined what is called a p norm, the norm of p is given by what is that you need to do you take the absolute value at each component raise it to the power p, it is the p th. So, one or p th root of the sum of the p th powers of the absolute values of x, I hope that is clear from the expression. We will often talk about another useful in meteorology when we talk about error growth and other things is called energy norm. Energy norm of a vector x with respect A matrix A is defined to be x transpose a x to the power of half, A in this case its a symmetric positive definite matrix.

So, norm refers to the size; size can be measured in many ways, there are at least five different ways I have illustrated 1 can measure this size of a vector with. Once I have a size, I have the notion of a distance. So, if x and y are 2 points, the distance between x and y is simply the norm of the difference of the 2 vectors x is the vector y is the vector difference of a vector is a vector I can transfer the norm of the vector. So, the distance between 2 vectors is simply the norm of the vector associated with the difference with the difference.

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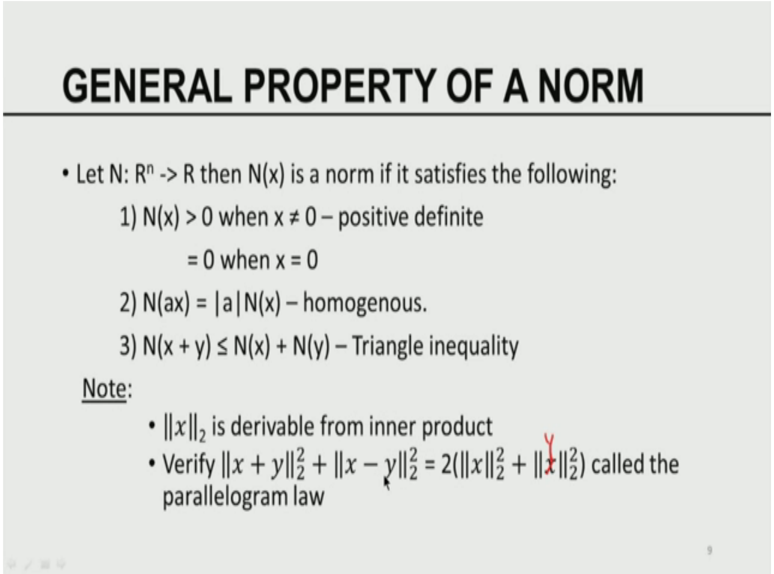

What are the general properties of norm? You can define norm any way you want no one is going to be able to come, and dictate that this should be the only way to be able to design norm. So, if you want to define your own norm I am going to tell you what are the basic properties a norm must possess. So, given a vector x N of x is a norm, if it satisfies the following 3 condition. N of x must be positive definite N of x must be go homogeneous in other words the norm of a scalar multiple of a x is simply a times the norm of x that is called the homogeneous. The third property the norm is that the sum of the norm of the sum of the 2 vectors is less than or equal to norm of x plus norm of y that is called the triangle inequality. So, the norm should be positive definite, and norm must be homogeneous, a norm satisfy the triangle inequality, I would like to point out that every norm that we define the 5 norms, we define all of them satisfy these properties in addition to this 5 you can define your own norm, you can any norm that you want to use must satisfy these 3 conditions.

Now, a special note Euclidean norm is very special, because Euclidean norm is the only norm, that can be derived from inner product of the 5 norms only Euclidean norm is associated the inner product and nothing else then I am I have a home work here verify that the norm square thus of the sum plus norm square of the difference is 2 times the square of the norm of x plus square the norm y. So, this must be norm of y 1 second this must be norm of y. So, that is that rule is a very basic rule, that is called the parallelogram law, any the norm based on the Euclidean definition always satisfies this Parallelogram law.

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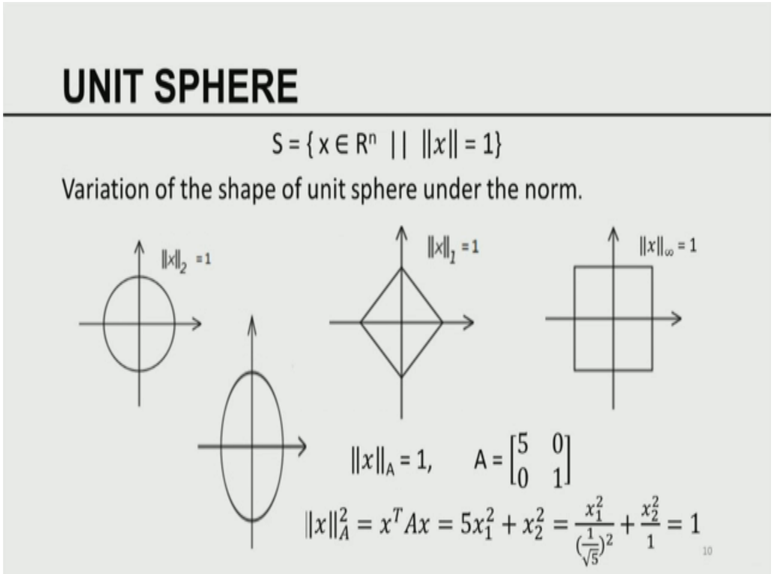

Then the notion of what is called a unit sphere comes into play, the unisphere in 2 norm is given here. Please remember 2 norm is called the Euclidean norm. The unit sphere is in 1 norm takes this shape one norm is the is the Manhattan norm, this is the convention of geometric norm this is the infinity norm the unit sphere in the infinity.

So, what is the unit sphere? Unit sphere is a set is a locus of points which are at unit distance from the origin. So, if you take a circle of radius 1 centered at the origin, if the circle is defined as a locus of all points at constant distance of one from the origin. So, if we pick the norm to be Euclidean norm that is the circle. The circle becomes this trapezoid, when you change the norm. The trapezoid becomes a square if we change the norm. The unit circle becomes an ellipsoid if I change the norm. So, when you pick matrix A to be 5 0 0 1 that is the symmetric positive definite matrix, if you consider the square of this norm you get the an equation to an ellipse which is given by here, x 1 square by a square plus x 2 square by b square is equal to 1 is an equation to an ellipse.

So, you can readily see the equation 2 then ellipse is depicted here. So, what is why am I doing this? I want you to understand that the geometrical figures naturally marks the shapes changes if you change the definition of a norm. Again I want to insist here mathematics is a man made science, you have total freedom to do whatever you want the only condition is you must be consistent. So, for a norm to be consistent you have to satisfy those 3 rules. So, consistent with those 3 rules, we have seen several different norms and this is one way to geometrically explain the intrinsic differences between the properties of these norms.

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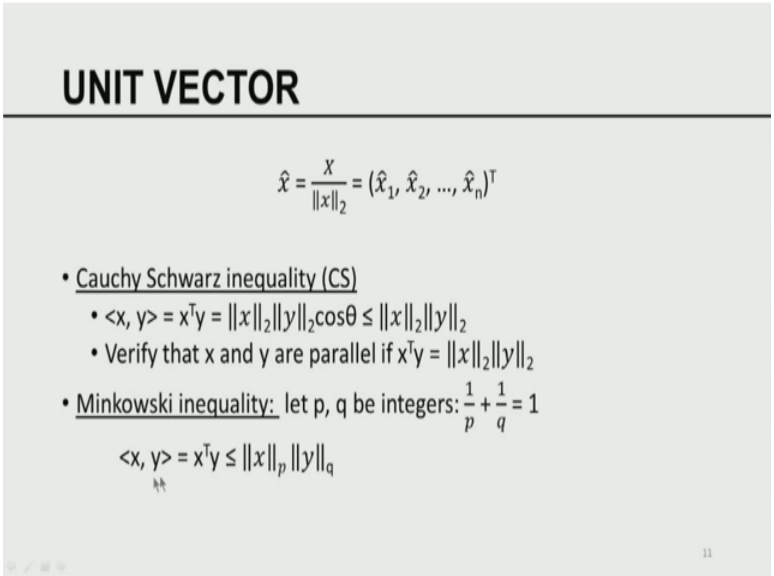

Then you have the notion of what is called the unit vector. Unit vector in the direction x is simply x divided by the norm of a x we all know that very well, then there are a couple of fundamental inequalities what is called Schwarz inequality what does it say. If I have an inner product between x and y, the value of the inner product by definition is x transpose y and that is equal to the norm of x norm of y times the cosine of the angle between the 2, that is a cosine of theta. Cosine of theta is always less than less than or equal to 1 therefore, this product is always less than or equal to product of the norm of x and norm of y. So, this inequality namely inner product of x and y is less than or equal to the product of the norms of x and y that inequality is called Cauchy Schwarz inequality.

Its one of the most fundamental inequalities, again I would like you to work as an exercise, verify that this Cauchy Schwarz inequality becomes an equality only when the vectors x and y are parallel to each other is a very simple exercise and I would like you to ah prove it yourself to be able to understand the power of the Cauchy Schwarz inequality a def a an extension of the Cauchy Schwarz inequality is called Minkovarsky inequality if p and q are 2 integers with the property, 1 over p plus 1 over q is 1, then Cauchy Schwarz inequality can be extended to the inner product of x and y is equal to x transpose y is less than or equal to the p norm of x and a q norm of y. When p is equal to q is equal to half, 1 over half plus 1 over half is 1 p is 2 q is 2 i get the 2 norm. So, the Minkowski inequality reduces to Cauchy Schwarz inequality when i pick the 2 norm. So, you can see the generalization between 2 norm, p norm, q norm Cauchy Schwarz Minkowski all these related properties of vectors.

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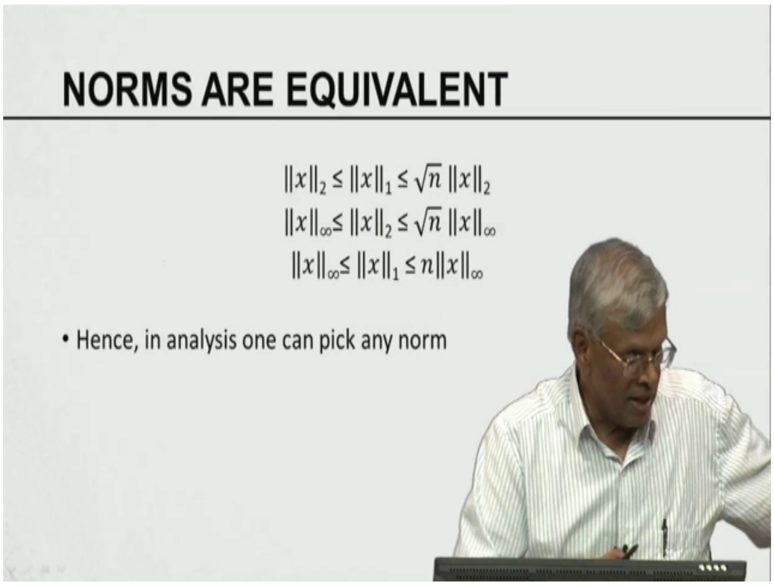

So, now that we have known that there is one norm, there is 2 norms, there is infinity norm all these norm they are related, I am not going to prove them, but you can readily see what does it mean. If I have a vector x, if its if the norm of a particular vector x is finite in 1 norm, it has to find it in every norm that is what it essentially says. The length of the vector in 2 norm is less than or equal to the length of the vector in 1 norm, which is eq the less than or equal to square root of n times with the length of vector in 2 norm likewise all other inequalities, I do not want to repeat it, you can read it for yourself. This essentially tells you that all these norms are intrinsically interrelated. So, what does this mean? This means that you as an analyst has total freedom, you do not have to confine your analysis either to 1 norm or 2 norm or infinity norm of the energy norm, you can do the analysis by picking any norm that is convenient to you. If you can prove one result in

1 norm, you can extend it to any other norm using these inequalities. So, that is the fundamental aspect of this uh relation between various norms.

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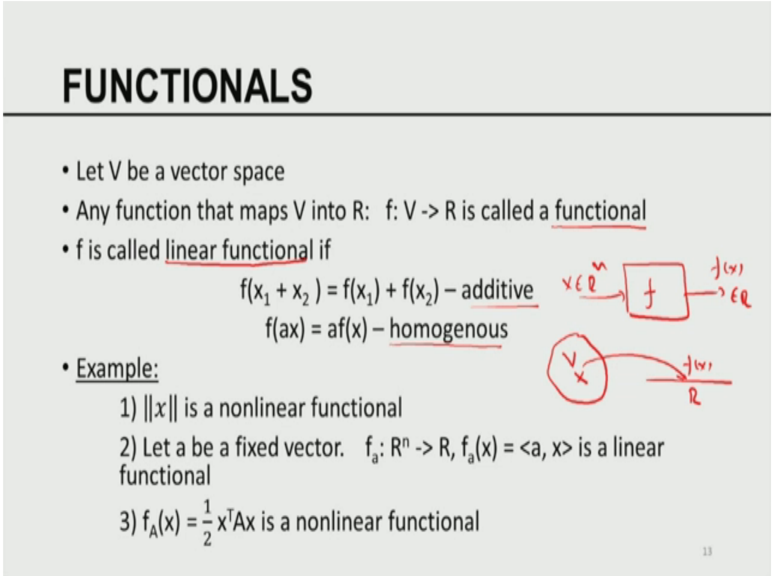

Now, I am going to introduce the other concept which is called a functional, let V be a vector space. Any function; that means, that maps V into R, R is a set of real number, f is a function that takes a vector x as input. So, let me give you a little picture here. So, I have a box which is f, I give an x the x belongs to R of n, it spits out a value f of x and f of x is a real number. So, what does it mean? It takes vectors and converts them to real numbers. Any function that converts a vector into real number that is called a functional, function is different from functional it is a very technical term. So, I would like you to be aware of the intrinsic differences between. A functional is a function, but not all functions are functionals. So, functionals are special cases of functions; f is called a linear function. So, once I have a functional, a functional can be a linear functional or a non-linear functional a functional is said to be a linear functional the emphases a linear functional, if f of x 1 plus x 2 is f of x 1 plus f of x 2; that means, it satisfied additive property, it also satisfy what is called the homogeneity property f of a x is equal to a times f of x. So, any functional that satisfies these 2 properties is called a linear functional.

Now, I am going to give you examples of linear and non-linear functional and norm is a non-linear functional. Given a vector x and norm is a number. So, norm converts a vector into numbers is a functional is a non-linear functional. For any fixed vector a f of a mapping R n to R; that means, f of a of x is a times x for a f of x that is a linear functional, another example of a non-linear functional given a matrix a, I can talk now about 1 half of x transpose a x that is an example of a non-linear functional. So, functions functionals leave functional non-linear functional functionals defined over vector space. So, vector space is the basis. So, you can think of a functional to be as follows a functional is here is a vector space V, here the real liner R, a functional takes a vector and maps it on real number. So, that is how you can look at your functional mapping a vector to a real number.

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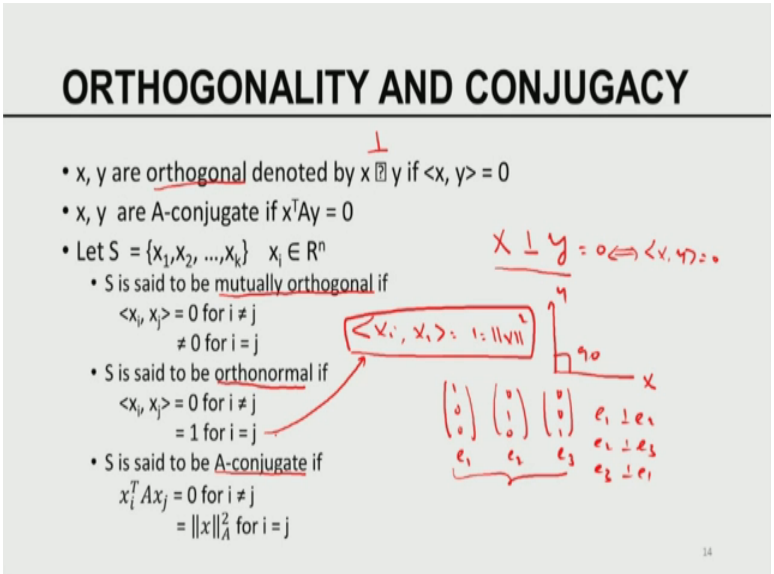

Now, I am going to quickly talk about the notion of orthogonality and conjugacy of vectors, why do I need conjugacy? Later when we are going to do optimization we are going to be talking about conjugate gradient method. So, I would like to be able to introduce the notion of conjugacy pretty early enough. So, let x and y be 2 vectors, we denote their vector this one I am sorry yeah good this symbol has to be perpendicular like this, I think my computer did not have that is. So, we say x perpendicular y is equal to 0 to imply the inner product implies and implied by the inner product of x and y is 0 the inner product to x and y is 0, I say the vector is orthogonal, orthogonal vectors are denoted by this symbolism x perpendicular sign and y. So, 2 vectors are said to be orthogonal if the angle between them is 90. So, if this is x, if this is y, x this is y angle is 90 degrees. So, we say x and y orthogonal. Now in extension of the notion of orthogonality is called a conjugate c 2 vectors are said to be a conjugate if x transpose a y is 0, now I can extend the notion of a conjugate c to a set of vectors, let x be a set of k vectors each of them in R, n s said to be mutually orthogonal if I pick any 2 vectors x i x j it is 0 if i is equal to j it is not zero. So, we call it mutually orthogonal; that means, if I took any pair of vectors there are orthogonal.

So, what is an example of any pair of vectors there is orthogonal you already know this example, 1 0 0, 0 1 0 and 0 0 1 we generally call this vector e 1 we call this vector e 2, we call this vector e 3. You already know e 1 is perpendicular to e 2, e 2 is perpendicular to e 3 and e 3 is also perpendicular to e 1. So, e 1, e 2, e 3 are unit vectors, they are mutually perpendicular to each other that is the notion of mutually orthogonality. Then s said to be orthonormal if eric if I took if I| pick to 2 distinct vectors, the product is 0 the inner product is 0 if I pick the same vector and compute the inner product with itself, then the value is 1 in which case it is called orthonormal, orthonormal means the vectors are normalized they are also orthogonal. So, what do I mean by saying the norm of x i come on now uh in the product of x i is equal to 1, that simply is equal to the square of the norm of x is 1 say that is what that what this means that is what this means that is what; that means, that means vectors have unit length every 2 vectors are orthogonal.

Now, if I look at my vector e 1, a this is of unit length, this is of unit length, this is a unit length. So, I have examples of 3 unit vectors which are mutually orthogonal. So, this these 3 vectors they are not. So, they are not only mutually orthogonal, but also orthonormal. I hoped you see the difference between normality and simple orthogonality. The same set of vectors are said to be A conjugate if x i A trans x i transpose a x j, equal to 0 if i is not equal to 0 the energy norm of x with respect to the matrix A square of it if i is equal to j. So, this is an extension of the notion of mutual orthogonality. So, these 3 concepts are orthogonal orthonormal a conjugacy of a collection of vectors is one of the fundamental properties of vectors that we would be very much interested in our analysis.

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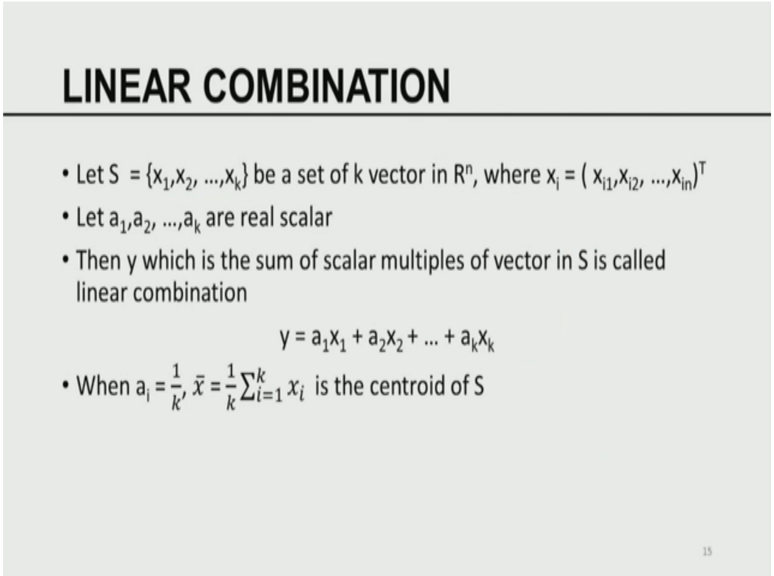

Now, I am going to introduce a very simple notion what is called whats the linear combinations of vectors, we will also have a lot of occasions to talk about this concept. Let x be a set of k vectors, each of the vectors are going to be in R n. So, each of the vector. So, I have k number of vectors each of them in R n. So, I wanted to distinguish 2 things the size of the vector is the n, but k of them. x i the i th vector has the n components the n components of i th vectors is i 1, i 2, i n. the first index refers to the index i of the vector x the second indices refer to the components of the vector. let a 1, a 2, a k or the be the real scalars let us define y to be the sum a scalar times the vector plus a scalar times the vector plus scalar times a vector; y is simply sum of the multiples of each of the vectors. So, y is called a linear combination of the vectors x in I have.

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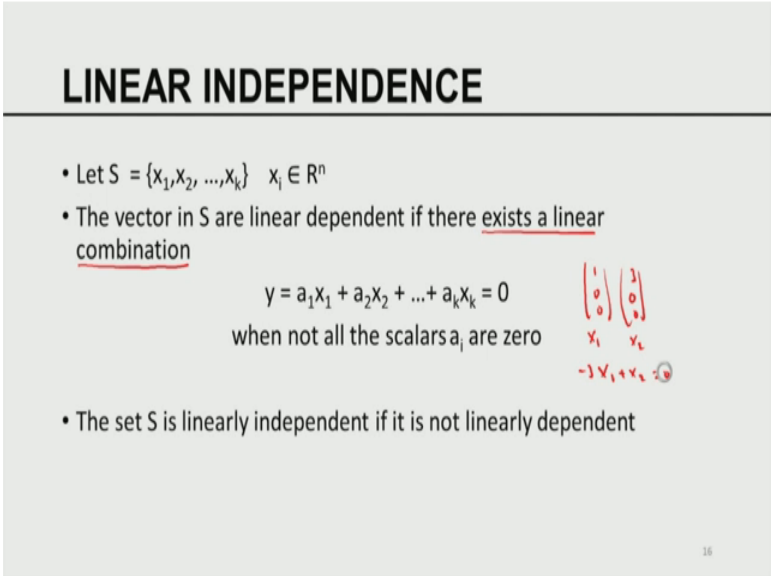

So, this is this is called the linear combination thats very fundamentally linear combination y is the a 1 x 1 plus a 2 x 2 plus a k x k.

What is the standard example of any linear combination if I have a set of vectors x 1 x 2 x k if i compute the average x bar 1 over k times summation i equal to 1 to k x i a what is that thats called the centroid in geometry we consider center of gravity the center of gravity is the centroid centroid is simply a linear combination of vectors. So, i this is an example of the notion of the linear combination that often occurs in in statistics in many computations. So, the notion of linear combination is fundamental once I have the notion of a linear combination I am now going to talk about the notion of whats called linear independence and linear dependence again this is another fundamental property from vectors a of spaces that 1 needs to be very toughly let x be k vectors the set of vectors the set of vectors in s are linearly dependent if there exists a linear combination y defined by a 1 x 1 a 2 x 2 a 3 x 3 in a k x k whose sum is zero, but the condition is that not all a are zeros.

When not all a of zeroes means even when I can i can annihilate them by I can annihilate them by picking by picking some of them to be not zero as an example if I have a vector 1 0 zero if I have a vector 3 0 zero you can readily see this is the vector let us say x 1 this is the vector x 2 x 1 can I can say minus 3 times x 1 plus x 2 is equal to 0 do you see that place. So, these 2 vectors are not linearly independent they are linearly dependent. So, the notion of a linear dependence is very clear.

So, when do i say something is linearly independent the opposite of dependencies independence your set of vectors are s said to be linearly independent if it is not linearly dependent. So, you define what dependent say s and then say independence is something that is not dependent. So, the notion of linear dependence is is fundamental if an observer absolutely uh you have very replace a very basic role when we deal with rank of matrices when we talk about solutions of linear systems and. So, on

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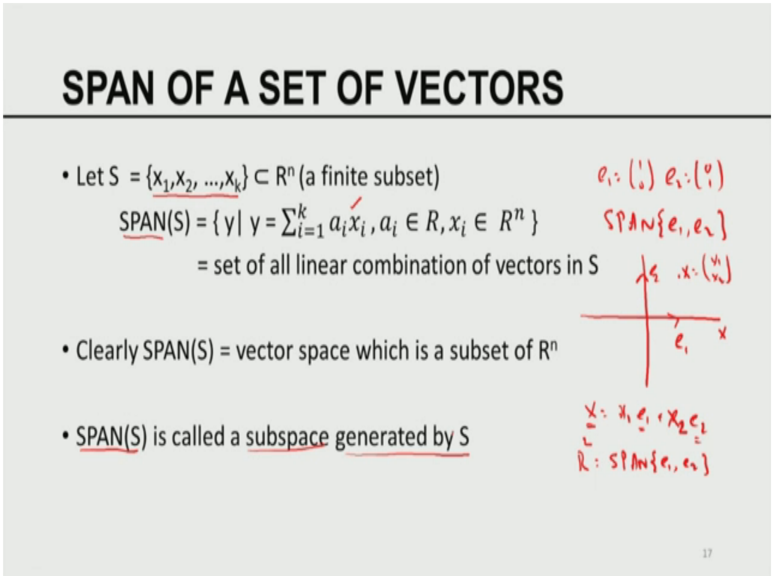

the next concept is the notion of whats called span of a set of vectors. So, let us assume I am given a set x of k vectors in r n n is the dimension of the space k is the number of vectors i had picked I am going to define a concept called span span of a vector what is that it is a set of all that vectors y it is a set of all vectors y such that this y is the linear combination of the set of all vectors in s. So, x i is are all inverse a i's are constants. So, y is the linear combination of vectors in s and each of the a i's are real numbers x i is are in r n. So, think of it now i I have i have been given a fixed set of I have been given a fixed set of numbers I have been given a fixed set of vectors which are x 1 to x k. So, these xs are fixed I have a choice in a is each of the a is are real. So, for each coefficient there are infinitely many choices there are k such coefficients. So, there are k way infinity of combinations that what, but that is possible the set of all linear combination if you put them all together we call the span effect. So, thats called the set of all linear combinations of direction

I will give you in a quick example now let e 1 be the vector 1 0 let e 2 be the vector 0 1 span of e 1 e 2 let us consider the span of e 1 e two. So, this is the x axis x 1 axis this is the y axis. So, e 1 goes like this e 1 goes like this e 2 goes like this every vector in this space the x can be replaced as x 1 times e 1 plus x 2 times e 2 we all know that right hey any vector x here is equal to x 1 x two. So, what does this mean x 1 times e 1 plus x 2 times e two. So, any vector x is the linear combination of e 1 and e 2 therefore, the 2 dimensional space r 2 is the span of e 1 and e 2 i hope thats very clear to you now. So, the 2 unit vectors span the whole space. So, thats the power of the notion of span. So, clearly a span is a vector space and its a subset of r n we say the span of s is a subspace generated by the set of vectors s. So, in summary what is the concept here using the concept of linear combination and by picking a set of k vectors I am able to define a subspace generated by a subset of vectors a sub subset of k vectors a subset of k vec k vectors in here. So, thats the notion of a span of a set of vectors

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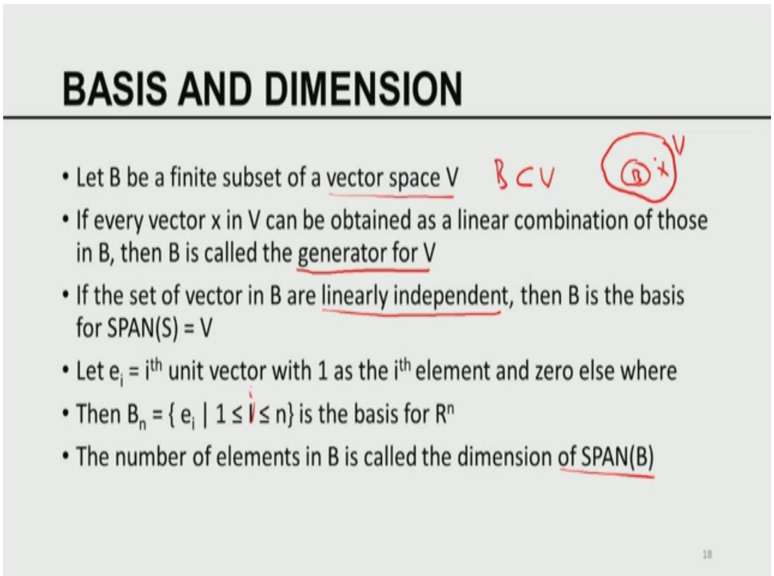

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The next concept is called the notion of a basis and dimension you can really see I am not proving any of these concepts I am trying to introduce all these concepts because you must be aware of these concepts. So, you should have a good access to a good book on linear algebra to be able to further explore these concepts, but i want to bring all these concepts to the forefront to emphasize this these concepts play an intrinsic role in the development of algorithms for data assimulation data assimulation as a discipline belongs to computational science it is a branch of applied mathematics it has very deep roots in many of the different sub disciplines in mathematics I am trying to expose such ah ah basis 1 would need to be able to do that assimilation thoroughly.

So, I am now going to do the next concept called basis and dimension let let us consider the vector space v let b be a subset of the vector space. So, b is a subset of the vector space I am not going to be talking about a particular property of vector sub subset if every vector. So, what is the basic idea here this is the vector space v b is a small subset of it if every vector x in v can be obtained as a linear combination of those in b; that means, every vector in v can be expressed as a linear combination of vectors in b that b plays a very basic role b is very important because everybody in v depends on b such a subset is called a generator for v the notion of a generator.

For example the 2 unit vectors e 1 and e 2 generate the whole 2 dimensional space because every vector in a 2 dimensional space is a linear combination of the 2 vectors e 1 and e 2 if the set of vectors in b are linearly independent then b is said to be the basis we already know the notion of linear independence. So, b is the basis or span of s e i is the unit vector with 1 as 1 the i th element and 0 else where. So, thats called the i th unit element b n the set of all unit vectors i 1 to n this must be i i is equal to 1 to n this is the capital i must be little i the set of all unit vectors is the basis for b of n therefore, you can readily see the n dimensional space is essentially created by a linear combinations of of of of vectors in the basis the number of elements in b is called the dimension or the span of b. So, the the dimension relates to the number of generators. So, what does it mean what is the minimal number of element that you need to be able to create the whole space if i had n unit vectors I can define the hole in dimensional space if i had 2 unit vectors I can define the whole 2 dimensional space. So, the notion of big notion of dimension and you can certainly see all these things are intimately related to the notion of linear combination linear dependence linear independence and these are fundamental concepts relating to vector spaces

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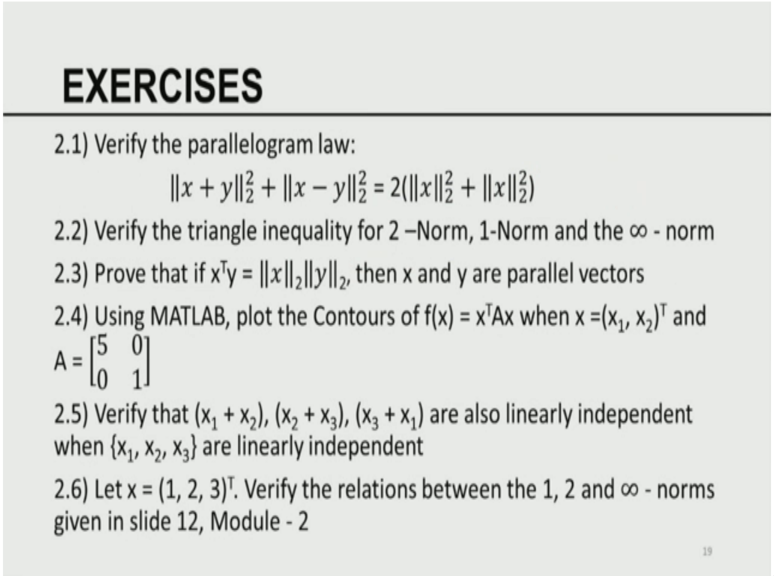

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Now, I am going to conclude with a set of problems which i would like you to work and and and and extend your understanding to be able to work some of these problems you may need to consult some of the other books that uh describe all these methodology lot more uh in in in clear detail, but i would like to I have hit on major concepts 1 must be aware of to be able to pursue things to follow. So, verify the parallelogram law verify the triangle inequality for the 2 norm 1 norm and infinity norm they are very good mathematical exercises prove that the inner product is equal to the product of the norms you for if x and y are parallel vectors this essentially comes from the cauchy schwarzs inequality using matlab plot the contours of f of x when a when x transpose a x in other words f of x is x transpose a x. So, this is the quadratic function i would like you to plot the contours of this using a matlab

Matlab is 1 such example you dont have to use matlab you can use mathematica or any other software system that you are comfortable with, but matlab has very powerful graphics and that makes the job of ploting all almost trivial verified that if the x 1 x 2 x 3 are 3 linearly in independent vectors that x 1 plus x 2 x 2 plus x 3 x 3 plus x 1 are also linearly independent let x be a vector 1 2 three; that means, I am now giving a very specific vector with comp components 1 2 and 3 i would like you to verify the relations between the 1 norm 2 norm infinity norm given in slide twelve of module 2 point 1 the module is essentially 2 point 1 in this in this particular model i dont have to even say in this particular module, but the model number is 2 point 1 thats thats thats what i would like to able to emphasize in here. So, this is 2 point 1

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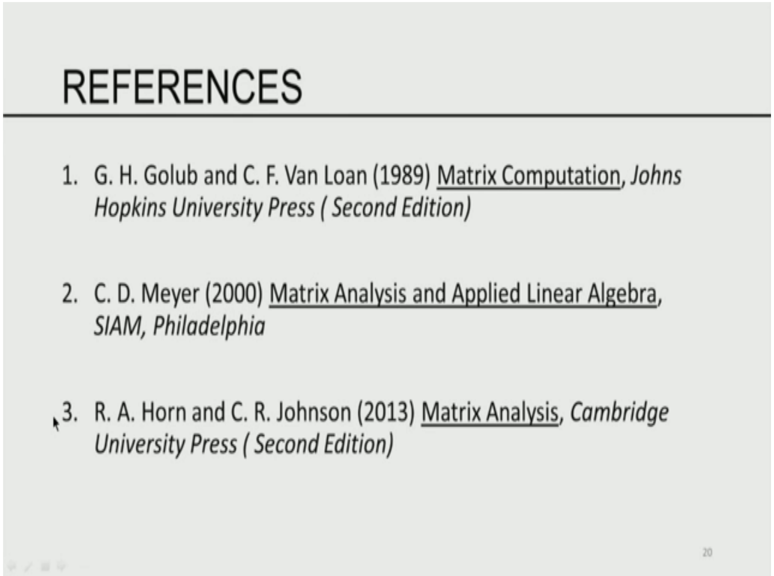

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With that i think we come to the end of the coverage i told you you have to go into other books for further reading I am giving you 3 references 1 is a book by golub and van loan thats 1 of my favourite I have a copy of that I have the second 1 is matrix analysis and applied linear algebra the third 1 is horn and johnson matrix analysis the third 1 is little bit more advanced second 1 is quite elementary third as first 1 is rather intermediary I have all the 3 copies of these books these are extremely useful anybody who wants to do fundamental work in data assimulation must have at least 1 of these 3 my preference is 2 the book by meyer published by siam is an excellent book with primary emphasis on not only on matrix theory, but also on computational aspects of matrix theory.

With this we conclude our overview of the basic principles of vector spaces. So, what are the basic things we covered vectors norms distances concept of linear dependence concept of linear independence orthogonality conjugacy basis dimension these are the nuts and bolts of linear algebra that you would need to master to be able to proceed further.

Thank you.