Dynamic Data Assimilation
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Lecture - 39
Deterministic predictability

Student: Let us start sir.

Good, if you recall; in a first lecture, we said data mining data assimilation and prediction are parts of a continuum data mining relates to development of models. Almost all the models that will come to be originally arose from considerations that can be related one way or the other to the concept of data mining as we understand it today.

Once the notion of a model was well established the notion of being able to use the model to create prediction came about. Once we are able to predict we would the interest became improving the quality of prediction became fundamental became a problem of fundamental importance; how do you improve the quality of prediction you need to use the data.

So, data assimilation is the process by which you can fit the models of the data so that the model can be calibrated to make reasonably good prediction. So, I have created the model. I have observed the data, I have created the assimilated model; what is next to create prediction. So, in this course, we are going to see the various aspects of the quality of prediction who controls the quality how; what is predictability what are the in why is the interest in predictability studies and that is the theme of this last set of modules.

So, first we would like to be able to split our discussion of predictability into deterministic predictability where deterministic refers to prediction generated by a deterministic model stochastic predictability relates to predictability of analysis of predictions arising from stochastic models.

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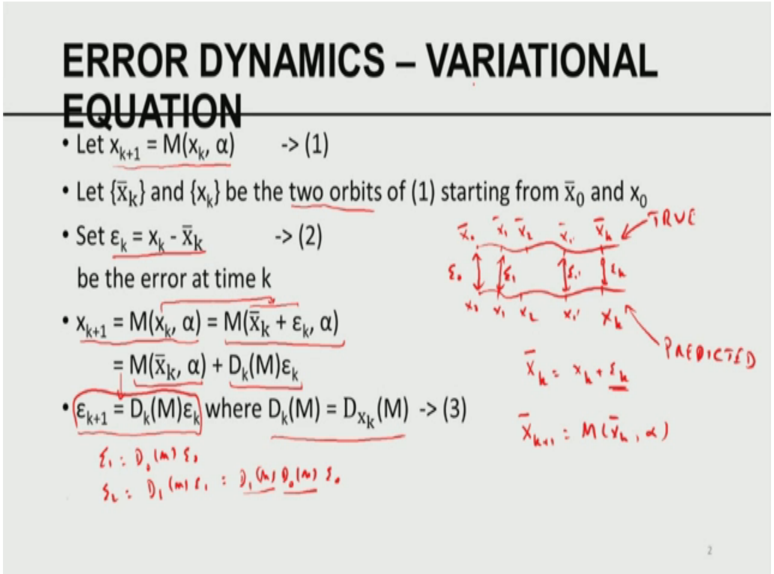

So, as a starting point we are going to talk about what is called the error dynamics or variational equation which we have already alluded to; when we talked about for your methods. So, we will quickly start developing some of the basic tools that we would we would need in in in in the analysis of deterministic predictability let x k plus 1 is equal to M; M of x k alpha be a deterministic model x naught is the initial condition alpha are the parameters.

Now, let us consider the model running from one state called x naught bar another state called x naught. So, that is one prediction there is another prediction the trajectories in; is also called orbits. So, let x k bar let the sequence x k bar the sequence x k be. So, what is the sequence x 1 bar, x 2 bar x i bar here is x 1 x 2 x up. So, let us consider 2 orbits or 2 trajectories generated by the models starting from 2 different initial conditions.

Let epsilon k is equal to x k minus x k bar therefore, the initial difference is epsilon naught the initial differences epsilon naught then this difference is epsilon one this difference is epsilon i, this difference is epsilon k. So, what is epsilon k epsilon k is the difference in the forecast of the model a time k induced by the difference epsilon naught in the difference in the initial condition that is used to generate the 2 different forecast.

So, epsilon k is called the; so, if the epsilon naught is the error in the initial condition epsilon k is the induced the other in this state of the system at time k, why is this important; in many of the 4 D VAR based methods as well as forward sensitive methods within the context of deterministic dynamic data assimilation scheme our goal has been to be able to estimate the initial condition the estimate of the initial condition based on finite samples will always have errors.

Therefore, if you used one set of observation, I will get one estimate for the initial condition, if you use another set of observations I will use another set of initial conditions. So, no matter which method you use if you are trying to estimate the initial condition from a finite set of observation the estimate will not be equal to the true value that I would need to be able to make better forecast in the another words estimates have always errors embedded in them. So, you can think of epsilon naught as the error in the initial condition estimate. So, if I want to give some life to this trajectory let x bar a, let x naught bar be the true unknown initial state.

So, this could be the true trajectory; this could be the trajectory this this is the predicted trajectory, this could be the predicted trajectory the predicted trajectory is different that truth trajectory because the initial state that is used to generate the prediction was different from the true state the initial state estimate. So, you can think of x naught as the estimate of the initial condition the estimate epsilon naught is the error in the initial condition.

So, the error in the initial condition; however, small it is it is going to be reflected in the forecast I am interested in the error dynamics. So, the evolution of epsilons induced by the model is called the error dynamics that is the title of the slide. So, I would like to be able to now derive an equation for the evolution of epsilon based on the evolution of the true state as well as the predicted state.

So, what does that we one would expect if the error in initial condition is small if the prediction at time k the error the is closed to the truth state; that means, the error in the prediction at time k is also small in other words small initial errors leads to small errors in the forecastle future time then you would say the model forecast is more reliable if the error in the initial state explodes in time then the quality of prediction given by the model deteriorates in time.

So, what does this relate to it relates to the sensitivity of the model to the variations are errors in the estimate of the initial condition. So, if the initial errors are magnified by the model; that means, the model is very sensitive if the initial errors do not grow, but grow goes down to 0; that means, the models are more stable. So, stable models can be used to create better prediction, but not all models are stable. So, the quality of prediction depends on essentially to the sensitivity of the model to errors in initial condition this is the fundamental theme of predictability analysis this is the fundamental theme of predictability analysis.

So, that is the path way we are going to be taking in this set of slides that describes analysis of deterministic predictability. So, let x k plus 1 actual model using it is the actual model starting from the state x naught. Now to a first order approximation I can say the unknown true state is equal to the known prediction plus epsilon k. So, it says gain a first order theory. So, x k is equal to this x k can be replaced by the true state plus epsilon k alpha.

So, here what is that I am assuming alpha is the parameter you know, if I change the alpha the solution changes. So, I am going to pretend for a time being I know alpha precisely and I have used alpha the precise alpha if there is going to be a forecast error, it is all due to only errors in the initial condition and not the errors in the parameters. So, I want to separate, I do not want to assume errors in too many things again, I want to emphasize the forecast errors can come from errors due to 3 sources, one errors in the initial condition errors in the parameter are errors in the model, I want to be able to enjoy analyze each one of them separately.

So, in this case what is that we are interested in assume the model is perfect deterministic assume, I know all the parameters in the model absolutely precisely. So, if there is any forecast error that is directly attributable to only errors in the initial condition and the errors initial tension where do they come from they come from estimates while data assimilation schemes gives you a reasonably good estimates optimal estimates in some sense because the estimates are derived out of finite number of samples the estimate from the finite set of samples may only be closer to the truest, a true value still there could be a non-zero error our aim is to be able to see how the model treats this more often inevitable error the error coming from estimation of the unknown initial condition based on finer samples. So, that is the kick that is the real key.

So, if I now express; this map in the form of a Taylor series, it can be seen this is the base value this is the perturbation. So, from here if I; from here if I. So, please also remember x k plus 1 bar is equal to M of x k bar plus alpha therefore, this equation this equation becomes this because this is equal to x k plus 1 bar x k plus 1 minus x k plus and bar is epsilon k the right hand side is becoming the Jacobian of M at time k a time k means what; at evaluated at x k times epsilon k. So, this becomes the dynamics for the evolution of the error where D k is essentially D of x k of M, I did not want to complicate the notation.

So, D of x k of M is simply decay. So, what does this tell you epsilon one is equal to D 0 M epsilon 0 epsilon; epsilon 2 is equal to D 1 M epsilon one which is equal to D 1 y M times D 0 M times epsilon naught, what is this? This is the Jacobian evaluated the first state this is the evaluation at the next state. So, it is the product of Jacobians this equation 3 in mathematics is called variational equation in meteorology literature. It is also called the tangent linear system so tangent. So, in the in the context of in the context of a 4 D watt we considered tangent linear system essentially with respect to understanding the propagation of perturbation here there we induced, but we induced we thought of inducing a perturbation initially to be able to correct the forecast errors.

Here, there is no correction there is no data there is no I am we have already done the data simulation. So, I have a model x naught be the estimated initial state I am going to run the model from this estimate, I am going to compare the performance of an assimilated model with respect to the unknown true state and that is the analysis we are trying to do. So, this is the first data assimilation analysis of model forecast to be able to see how the initial errors are treated by the model.

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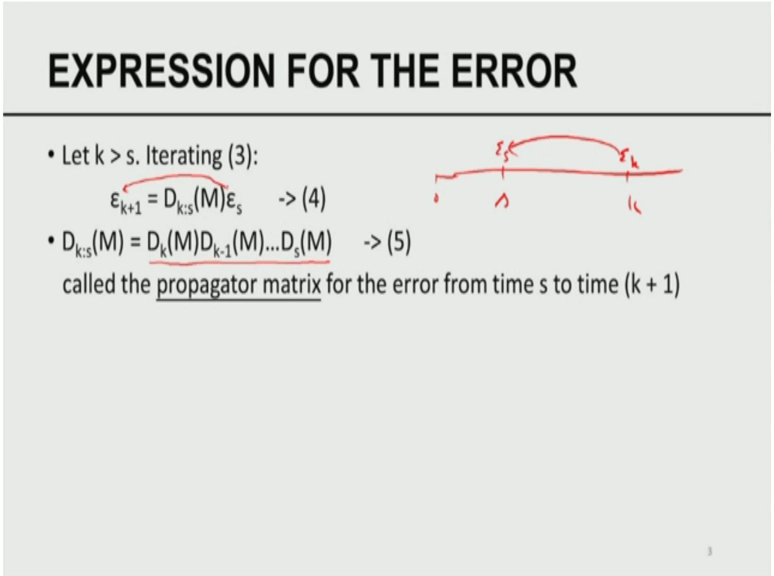

So, I am naturally interested in 3; 3 efforts to the dynamics of evolution of the model errors. So, if I iterate 3, I get this expression what is this expression. If I picked 2 instances in time yeah, I am sorry, this is 0, this is time s and this is time k, I would like to be able to express how the error at time s is related to the error at time k and that is the relation. So, s could be 0 or anything else. So, this is general expression does this tells you how the errors at 2 instances and times separated in time are related by the dynamics D; D of k column s of M is simply the product of the Jacobian evaluated along the trajectory from time s to time k.

This matrix has a special name it is called a propagator matrix what does it do it tries to relate the errors between 2 different instances on time the propagator matrix is simply a product of the Jacobians.

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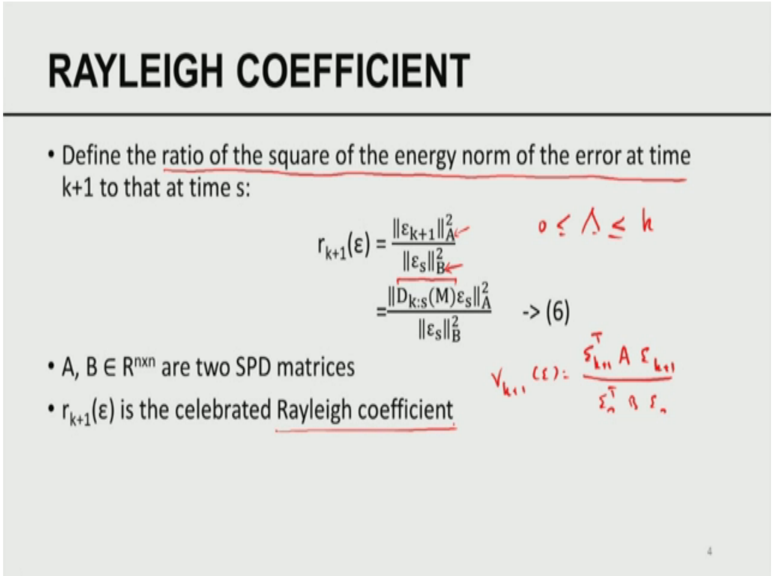

Now, I am going to introduce a numerical measure please understand yet the error epsilon k. So, I have now understood how the errors propagate to a first order accuracy through the model.

Now, but error is the vectors we would we understand certain scalar measured better than vectors. So, I am now going to introduce a scalar measure that tries to that tries to quantify the properties of errors and that is done by a quantity called Rayleigh coefficient the Rayleigh coefficient is simply the ratio of the square of the energy norm of the error at time k plus 1 to that at time 0.

So, in the previous slide we related the error at time k plus 1 2 error at time s error time k plus on epsilon k plus 1 is a vector the error time s is a vector s in general could be 0 or anything any number less than greater than 0. So, s is greater than 0 less than or equal to k. So, that is the essentially the range for us. So, epsilon k plus 1 is given by the product of Jacobian from kts of M times epsilon s the energy norm I am trying to evaluate a numerator with energy norm based on in SPD A, the bottom line SPD B.

So, A and B, let A and B be 2 given symmetric positive of a matrices evaluate the numerator using the energy norm based on the matrix A evaluate the denominator using the energy norm based on the matrix B in principle A and B could be equal, but the treatment is some; is a simple when you do not have to consider A and B, for example, what could be the thing in meteorology I would like to be able to say hey initially I am interested in pressure, but at time at a later time; I may be interested in certain water city.

Initially I may be interested in temperature difference at a later time, I may be interested in rain. So, what do A and B bring to the bring to bear on the problem I would like to be able to consider the sensitivity of the rain in the future time and it depends on a temperature distribution to the previous time, I could be; I am; I may be interested in some quantity at time at a later time and its sensitivity based on another quantity.

Therefore at this stage, when I am trying to talk about quantities sensitivities, I do not have to make them to be the same, therefore, Rayleigh coefficient what is it? It is simply the ratio of quadratic forms what is the quadratic form. So, the numerator, r k plus 1 epsilon is essentially equal to epsilon k plus 1 transpose a epsilon k plus 1 divided by epsilon s transpose B epsilon s i that is exactly that is exactly the relation that is involved in here. Now please understand epsilon k plus 1 and epsilon s are related through the model; the relation through the model is given by the product of the Jacobian along the lines. So, how does the initial. So, what is that the you can think of it now suppose I have an grid, I have an initial condition, I have a model, I am going to introduce a thermal bubble in a small locale in the at the initial time in involving certain small number of grid points.

So, there is a perturbation of temperature distribution initially how does the model reacts to the thermal bubble at a time in the feature how does this thermal bubble introduces other changes in the model. So, that could be one way of thinking about thinking about this ratio. So, A and B in general are 2 symmetric positive matrices this ratio is called it is the celebrated Rayleigh coefficient in matrix theory and that comes to our that is a very useful measure for us to be able to consider predictability analysis. So, I would like to be able to understand the behavior of this ratio.

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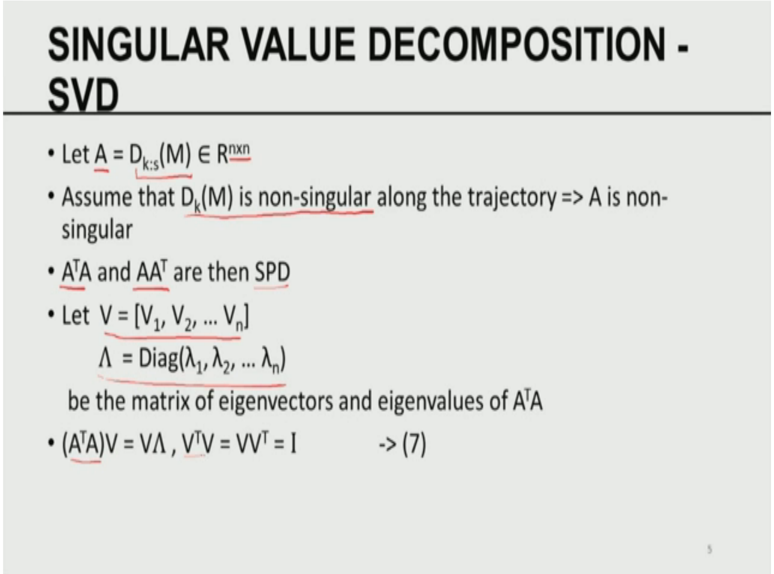

Now, let us see, what does ratio represents this ratio in general represents a measure of the sensitivity you can you can see if the energy in some quantity at a later time related to the energy of some other quantity at the initial time. So, this ratio captures the spirit of the sensitivity that is involved in the predictability analysis suppose this ratio remains less than one if this ratio is a scalar right, if there is ratio remains small for all time beyond s; what does it mean?

The error does not grow if the error does not grow what does it mean the predictions are pretty done accurate, on the other hand, if this ratio grows in time; what does it mean, the error the initial errors magnify therefore, analysis of the properties of this Rayleigh coefficient for appropriate choices of energy norms of the numerator the denominator essentially provides a very good clue to the quality of forecasts generated by the model that is the that is the that is the secret of using Raleigh coefficient.

So, in order to further analyze the Rayleigh coefficient, let us concentrate on the propagator matrix what is the propagator matrix it is a product of the Jacobian from time s to time k. So, let us call that matrix a for the sake of simplicity in notation I am also going to assume the model Jacobian is nonsingular if the model Jacobian nonsingular means what the models well formed.

Non singularity the Jacobian at every point along the trajectory goes to a test to the well-formed nature of the model itself the model.

Sometimes the model could be screwed up how do you measure what is the measure of the screw up in the model look at the Jacobian the model actually gives you a solution you try to evaluate the values Jacobian along the trajectory is the valid the Jacobian remains full rank the model is well formed if the model if the if the rank of the Jacobian along the model varies then the model is not well formed model.

So, I am assuming the model is well formed in the measure in the sense that Jacobian along the trajectory are nonsingular. Now a square matrix; please realize that A is a square matrix; even though A is a square matrix, in general A is not symmetric, A depends on the starting time s and the starting time k. So, if I vary any one of the timings way A changes, but for the analysis I am keeping s fixed k fixed. So, A is fixed. So, A transpose A and AA transpose are the 2 Grammians coming out of this model both of them are SPD; why both of them are SPD, if each of the components of a component matrices in this product are nonsingular the product of nonsingular if the if; if a matrix is nonsingular, it is Grammian is full rank the Grammian is not only full rank is also symmetric and positive definite these are all immediate conclusions we have alluded to several times earlier especially within the context of development of SVD severe weather decomposition.

So, let V 1 to V n be the eigenvalues this is the eigenvectors of the matrix A transpose A. So, A transpose A v is equal to V lambda V; V transpose V transpose these i.

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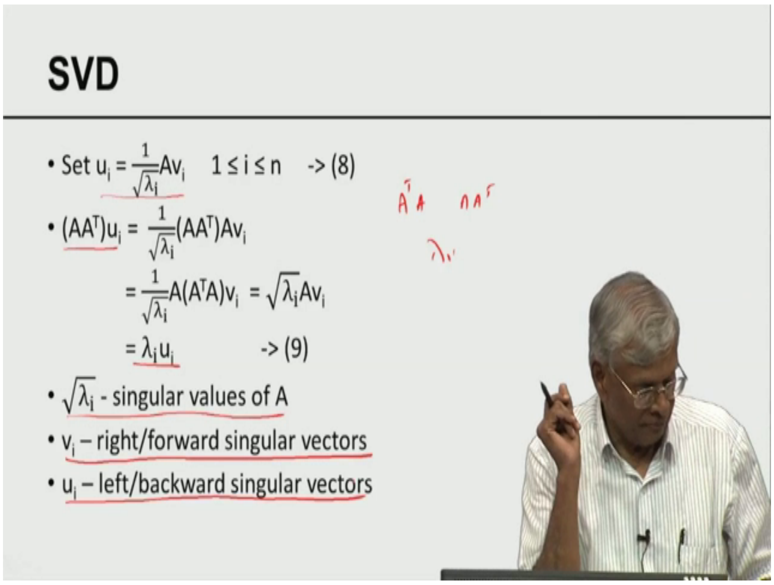

Now, I am going to define a new vector u i which is equal to 1 over lambda I am using the similar formalism as we did in the SVD the only differences earlier we defined SVD for the context in the context of matrix h which is rectangular now I am talking SVD within the context of the matrix a which is square h is a single matrix the forward operator a is a complex matrix is a product of matrices along the trajectory if I assume each of the matrices each of the Jacobian matrices are non-single a is not silver. So, I am just trying to reinforce the idea where a comes from.

So, if vi's and lambda I's are the eigenvector eigenvalue pair for A, I can now define a new make a vector u i; u i also is an M vector which is one over square root of lambda i times A v i from this derivation. So, I am now going to compute AA transpose u i, if you substitute the expression for u i and simplify it, it turns out, i get 9. So, what does 9 say AA transpose u is equal to lambda i u i. So, what does it mean if vi's are the eigenvectors of A transpose A u i's are the eigenvectors of AA transpose both the matrices share the same set of eigenvalues in this context lamp at the the lambda i's are the eigenvalues of A transpose A as well as AA transpose.

Both of them are matrices of the same size. So, square root of lambda i is called the singular value that is the definition. So, singular value of a matrix is the square root of the eigenvalue of the Grammian that is the definition and the square root of the eigenvalues the Grammian are all going to be positive if the Grammian is symmetric positive definite a Grammian is symmetric positive definite, if the component matrices are full rank.

So, full rank condition essentially tells you the problems are well formed and that is the condition we have been looking at all through full length matrices all good things in life and that is exactly the kind of theory we have developed. So, lamb square root of lambda are called the singular values vi's are called the right or forward singular vectors ui's are called left or backward singular vectors these are the general nomenclature that is well understood within the applied mathematics linear algebraic context now from 8; sorry from 8; sorry.

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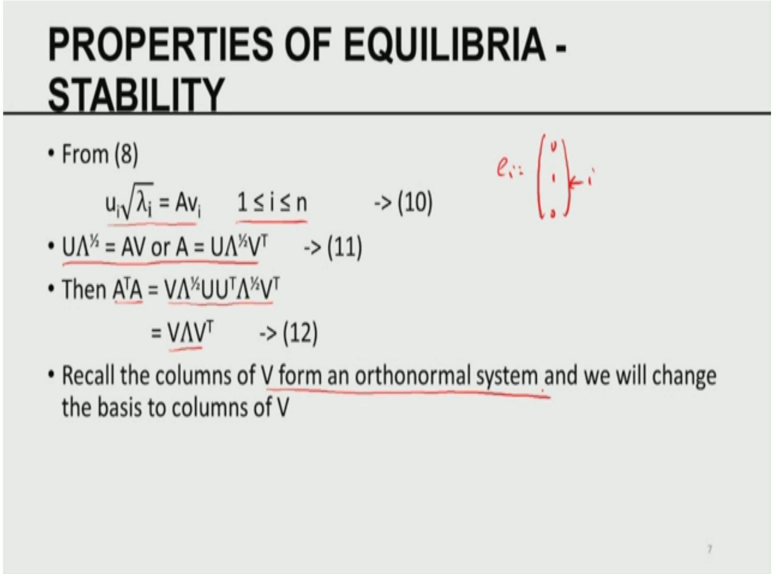

I am going backward, I would like to talk about some of the properties of equilibria from 8, I can rewrite my equation.

For the definition of u i, this way, there are n, if I collected all the relations the matrix form it becomes the matrix version. So, 11 is the matrix version of 10. Therefore, A transpose A is given by this decomposition because UU transpose U; transpose U they are all identity U is orthogonal V is orthogonal; therefore, I get this expression. So, the columns of V are orthonormal system. So, what is that we are we are now going to do we have been doing analysis in the general coordinate system given by the standard basis which is e i what is a what is a standard basis e i is equal to 0 1 0 and this is the ith location i for one to that the standard basis if I have any other basis I can do the analysis on that basis why would you change the basis if the analysis the new basis can be made simply it can be made simple nothing is lost.

In fact, lot may be gained by changing the coordinate system from the standard to a given coordinate systems now we are going to gain by changing the analysis from the given simple coordinate system to an orthogonal system defined by V the columns of V what are the columns of V they are the eigenvectors of the matrix A transpose A given a space that we should always have in n a n basis vectors if the basis vectors are orthogonal it becomes orthogonal basis if the basis vectors are not orthogonal it is basis, but doing arithmetic in non-orthogonal basis is a little bit more involved.

So, we are simply changing from one orthogonal basis to another orthogonal basis and that is going to throw a lot of light on the behavior of the behavior of the Rayleigh coefficient.

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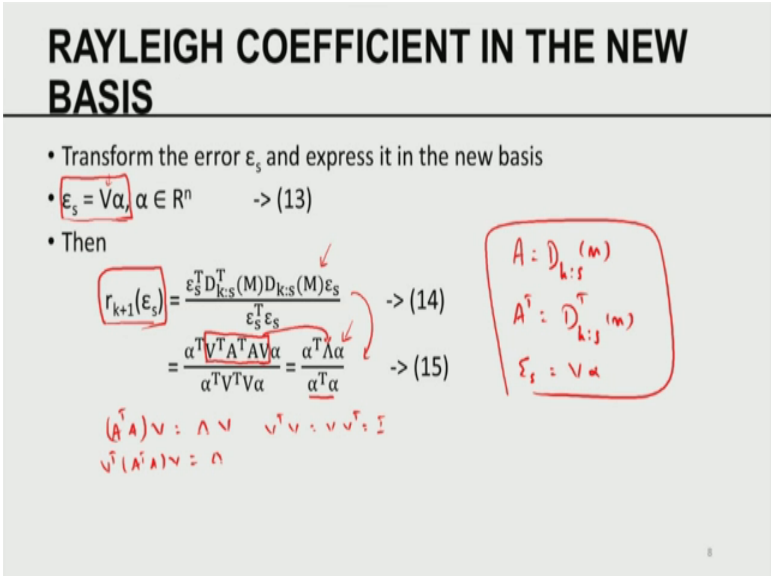

So, what is that what is the transformation we are going to be talking about now please understand epsilon s is the initial error at time s, V is the new basis. So, I am going to transfer epsilon s 2 alpha. So, what is alpha? Alpha is the new vector alpha and epsilon are referred to the same point in this space epsilon s is the coordinate of the same point in the original basis alpha is the coordinate of the point in the new basis whose columns whose basis vectors are columns of V and these 2 values of coordinates of the same point the to coordinate system are related by the expression in 13.

Therefore now I am going to re express my value of the Rayleigh coefficient of time k plus 1 with respect to the Rayleigh coefficient of time s this is the definition of the Rayleigh coefficient in the standard basis which is fourteen in the transform the domain my Rayleigh coefficient takes this form. So, what is that I am going to do please remember my a is equal to D colon s of m. So, a transpose is equal to D ks s transpose of M also remember my epsilon s is equal to is equal to V transpose alpha. So, if you substitute all this fourteen becomes 15 fourteen becomes 15. What is the real kicker?

Here the real kicker the numerator and the denominator get simplified what is that A transpose A v is equal to lambda V because V the columns of V are the eigenvectors of that therefore, if I multiply this by V transpose A transpose A v that is equal to lambda and that is exactly what is occurring in the numerator in the numerator therefore, this complex matrix in the numerator becomes simplified as a diagonal matrix in the numerator we also know V transpose V is equal to VV transpose that again come from the orthogonal the eigenvector that is equal to i the denominator gets simplified by this.

Therefore I have decoupled both the numerator and the denominator from the dependence and. So, in in in here what are the one of the various values of am assuming I am assuming a is identity I am sorry I am computer I am concerning the energy norm, I do not think; I do not want to go back right now sorry. So, I hope this is clear. Now the expression for the Rayleigh coefficient is given by 15.

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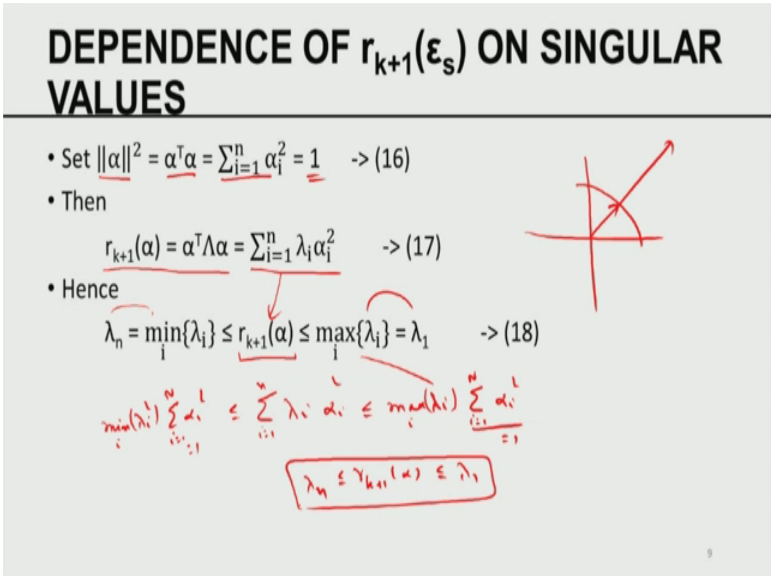

Now, what is the, denominated 15 what is that is the inner product of inner product of alpha with itself the numerator is simply a quadratic form of alpha with that of a diagonal matrices. So, what is that? So, alpha is a vector. So, alpha is any vector that is going to represent my epsilon s. So, I am going to now consider a normalized vector. So, the norm of alpha defined by this I am going to set to be one what does this mean if I have a vector here I can consider the normalized part of it. So, that is essentially a normalization the direction is more important than the magnitude itself that is the idea here. So, I am going to consider the ratio in 15, then I confine my attention to alpha unit vectors that is a that is a whole that is a whole point of the game.

So, in this case, the numerator becomes one; therefore, r k plus 1 alpha is essentially is essentially the numerator which is alpha transpose lambda alpha which is again given by lambda i square root of square of alpha i square from here the following inequality becomes follows directly this is the actual value since the sums since the sums of alpha i square is 1 since the sums of alpha i square is 1.

So, you can readily see lambda i alpha i square i is equal to 1 to n is less than or equal to is less than or equal to maximum over i of lambda i times alpha i square i is equal to 1 to n and that is equal to one. So, that gives raise to this this is also right minimum over i of lambda i square times summation alpha i square i is equal to 1 to n; that is equal to 1 therefore, lambda 1 is the maximum eigenvalue lambda n is the minimum eigenvalue it readily follows r k plus 1 alpha is less than or equal to lambda one and lambda n where lambda one and lambda n are the maximum and minimum value of the eigenvalue.

So, what is that we have done we have narrowed the values that the the ratio of the energy norms of the errors can take by moving simply from the standard coordinate to the new coordinate formed by the eigenvectors of the Grammian A transpose A assuming a is full rank. So, this is a very nice treatment of the analysis of the analysis of the Rayleigh coefficient.

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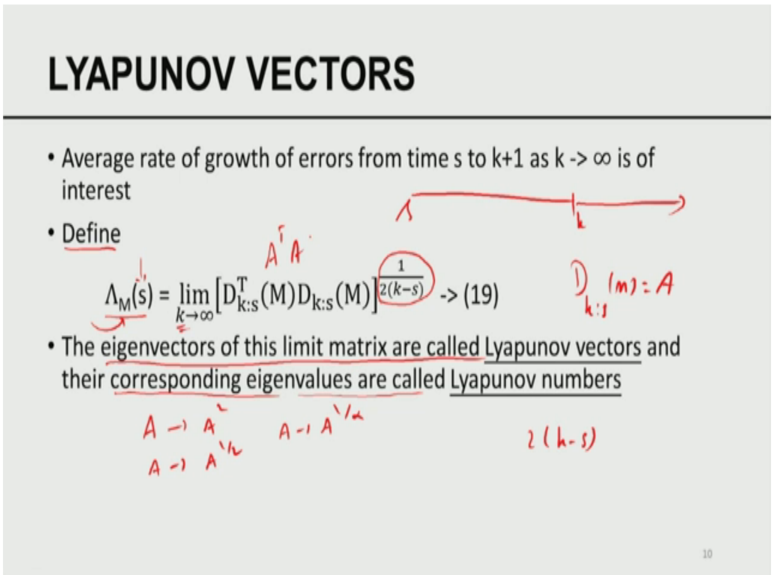

Now, what is that we are interested in we are interested in determining what is called the average rate of growth of errors from time s to k plus 1 then k goes to infinity that is of interest let us look at this why are we interested in the average growth of errors, you if I commit an initial error if the error grows how much error, I can afford to crap my forecast such there forecast till meaning is still meaningful is the question. For example, in weather forecasting we have make tremendous this improvement over the past 50 years, but we still do not know how to make monthly forecast or models are some of the best known models in the last hundred years and we have the best computers we have all kinds of observation systems we still cannot make predictions over.

So, the predictions that we make in today with all the things we know with all the technologies available computer sensors satellite models and everything else we can probably believe 3 to maximum 4 day forecast. So, what is 4 day; 4 day is the predictability limit. So, what does it mean today I am going to make one day forecast 2 day forecast 3 day forecast side a forecast the error one day forecast is there, but it is smaller they error in need to do today forecast slightly larger error with the five day forecast is more.

Why do you do not want to do the ten day forecast is not that I do not know how to make a ten day forecast I can make the ten day forecast to, but the error by the time ten days into the future comes into being you. So, much that it dwarfs the signal. So, there is a signal on which the forecasts are superimposed if the magnitude of the superimposing error is comparable to the signal then signal plus noise overwhelms the information. So, it is a classical study in the engineering literature called signal to noise ratio.

So, in any estimation problem in any detection problem we will always like to be able to analyze the signal to noise ratio and we would like to be able to maximize the signal to noise ratio. So, in some sense, the Rayleigh coefficient tries to capture the spirit of the computation of signal to noise ratio. So, so one measure one question is to how far does it take to be able to double the initial error.

So, what is the norm of the initial error what is the number of the error at a future time; how long does it take for the initial error the norm of the initial error to double that is called error doubling time; if the error doubles in 10 days of a ten days the error is even more than double. So, can I make a forecast that the error is double than the initial one no. So, what is the amount of error you may be able to tolerate; how long how far the error grows how fast the error grows ok.

Now, error rate of growth varies from time to time to time to time. So, if the error relative growth varies from time to time to time what is one measure average rate of growth of growth of errors? So, to understand the quality of prediction we need to understand the predictability limit what is the predictability limit predictability limit is the limit beyond which errors overwhelm the signal if the errors overwhelm the signal the prediction is useless.

Now, here comes some of the basic thing some processes are predictable perfectly the lunar solar eclipse IBM price I can predict for tomorrow, but I do not think I can predict the price of a maybe M stock day after tomorrow is much more volatile that lots of factors that depend on how do you predict the price of a barrel of crude oil the barrel of crude oil depends not only at the cost of production, but also the cost of transportation from place A to place B, either you transport the unpurified base liquid or purified form.

But then the cost of crude oil is controlled by. So, many events in the world if there is a war in the Middle East or if there is a problem in the Middle East we believe that the supply maybe affected the price immediately goes up. So, there are a number of factors that affect the price of a barrel of crude. So, the barrel of crude predictability effect is extremely difficult problem. So, a likewise the weather prediction we have gained enough knowledge about the weather the behavior of it. So, that we can make good short term prediction long term prediction we try to do, but we are not able to do long prediction which are reliable for example, I cannot predict seasonal weather precisely i do not let alone climate why it is not because they are not intelligent because the model we use exhibit extreme sensitivity to the initial condition.

So, the initial condition you use to generate the forecast have even smaller errors if the model is very sensitive to small errors the errors blow up into up. So, that is the fundamental team behind predictability analysis. So, the interest now goes to analyzing what is the average rate of growth of errors that is important. So, what is? So, given s is a starting point fixed this is k; I would like to get to go to infinity.

So, from a fixed starting point if I want to be able to make asymptotically large values of time if I want to be able to make forecasts for such large values of time over the time what is the average rate of growth of errors as into the model to that end I am going to define a new matrix. So, what is this matrix you remember D k s M which we will have also called a I am going between a and this whenever we need to. So, this is simply A transpose A there is a Grammian; s is fixed s is fixed it depends on M I am going to let k go to infinity. So, I am interested in the long term value of the Grammian A transpose A; I am also interested in the in the 2 k minus one 2 times k minus one the root of this matrix.

Please understand please understand given a matrix A, I can consider a square given matrix A, I can consider a to the power of one half given a matrix A, I can consider A to 1; 1 over alpha for some alpha. So, this is matrix I am considering one over 2 times k minus s; what is k minus s k minus s is the time difference between the starting point and the final time k goes to infinity. So, k minus s goes to infinity. So, this this matrix plays a fundamental role the eigenvalues of this limiting matrix are called Lyapunov vectors and the corresponding eigenvalues of this limiting matrix are called Lyapunov of numbers we are going to talk about these two, but before we go there I would like you to be able to appreciate the definition of this matrix.

So, what is this it is a product of the I am sorry if the product of A transpose AA transpose a is the Grammian A transpose A depends on s and k and keeping s and M the model fixed I am letting k goes to infinity I am interested in the in the in the 2 times k minus s the root of this product matrix and that is the matrix that limit matrix that limit matrix has an eigenvalue eigenvectors the eigenvalues are called Lyapunov vectors eigenvalues are called Lyapunov numbers Lyapunov is the famous Russian mathematician who propounded the theory of stability in the early decades of 20th century and so in his honor, it is called Lyapunov vectors Lyapunov numbers.

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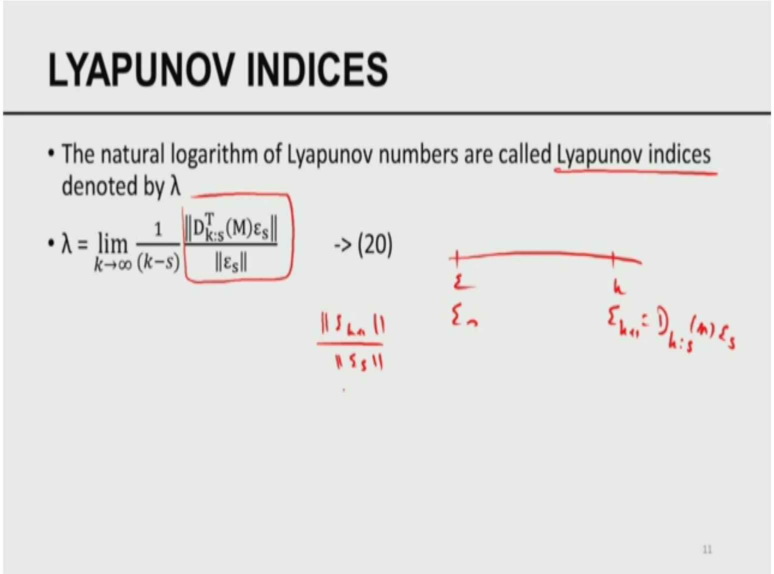

The natural logarithms of the Lyapunov numbers are called Lyapunov of indices. So, let us look at this number. So, this is called Lyapunov of indices Lyapunov of indices. So, what is a Lyapunov of index? it is the let us look at this. Now let us look at this ratio what is this ratio? It is a norm of the error at time. So, if I have time s if I have time k I have epsilon s I have epsilon k plus 1 the norm of the norm of the ratio of the. So, this is equal to D k times s of M of epsilon s.

So, the norm of the error the epsilon k plus 1 divided by epsilon s. So, what does it tell you by running epsilon as through the model I am able to change the initial error? So, then the norm of the raise ratio is a number and this number depends on the time interval k minus s I am going to normalize this by one over k minus s; s is fixed k goes to infinity. So, this ratio is the; you can you can think of it as the average. So, this average value lambda is called the Lyapunov index and this Lyapunov index essentially tells you the average rate of growth of the errors that makes sense; I want to think about it now think about it now. So, epsilon k plus on by epsilon s what does it mean that is the rate of growth of error during the time from s to k, I am trying to divided it by 1 over k minus s.

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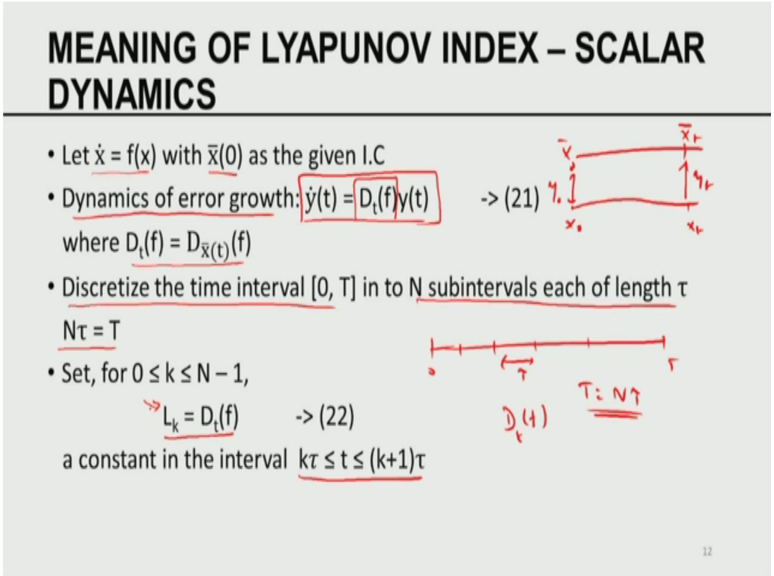

So, that is the average rate of growth of errors. So, what is the meaning of this Lyapunov index I am going to introduce it by illustrating it on a scalar dynamics? In fact, the entire theory of dynamical chaos in deterministic system is centered around this notion of Lyapunov indices average rate of growth of errors sensitivity to initial conditions. So, what is the idea if a system exhibits a set of the extreme sensitive to the initial condition that system cannot be used to generate prediction for long intervals of time?

If the system exhibits less sensitivity to initial condition such systems can be used to create long term forecasts it turns out most of the models of interest in climate studies most of the models of interest in oceanography atmospheric sciences exhibit do exhibit extreme sensitivity to initial condition that is the reason why we still are not being able to make ten day forecast a monthly forecast seasonal forecast we still have not captured all aspects of the model sensitivity. So, the current models exhibit extreme sensitivity that is the reason why we are able to do what we are able to do.

So, to understand this let us work a simple example, let x dot is equal to f of x be a dynamical system a x if x naught bar is a given initial condition that is the unknown that the true value of the state. So, this is x naught bar. So, from here I can compute the solution x of t I have x naught this is x naught bar I can compute x of t. So, barred quantities are true unbarred quantities are coming from estimated values, I have an initial difference which is epsilon naught I am doing the entire theory, but some place in a simple one dimensional models. So, the dynamics of the error growth the variational equation for this case becomes where our t is equal to Jacobian of f times y f t this y plays the role.

So, instead of instead of epsilon I am sorry instead of epsilon I am going to consider this as y naught. So, let y naught be the initial difference that y of t be the difference at time t epsilon we use in the contacts with a discrete time y of t I am going to use it in the case of continuous time. So, y of t is related to a the rate of growth of y of t is given by this equation this equation is a linear dynamical system the linear dynamics varies along the trajectory again. So, this is the Jacobian if you discretize this in the interval 0 to t there are young subintervals let the sub intervals of time tau. So, n tau b k, this is 0 this is t I am going to divided into intervals where the subintervals are time r of length tau and t is equal to n tau.

So, now I am now going to assume that this equation which is given by 21 is such that during a given interval of time my D t of f remains constant; that means, the interval of time tau is. So, small that my Jacobian does not change too much within that small interval of time therefore, I am now going to assume the dtf in a small interval going from k tau to k plus 1 tau a remains constant and that constant value is L k. So, what is the L k; L k is the constant value of the Jacobian of the system during the time of evolution from k to k plus 1 or k tau 2 to k plus 1 tau if tau is small, this is a reasonably very good assumption to make therefore, y dot.

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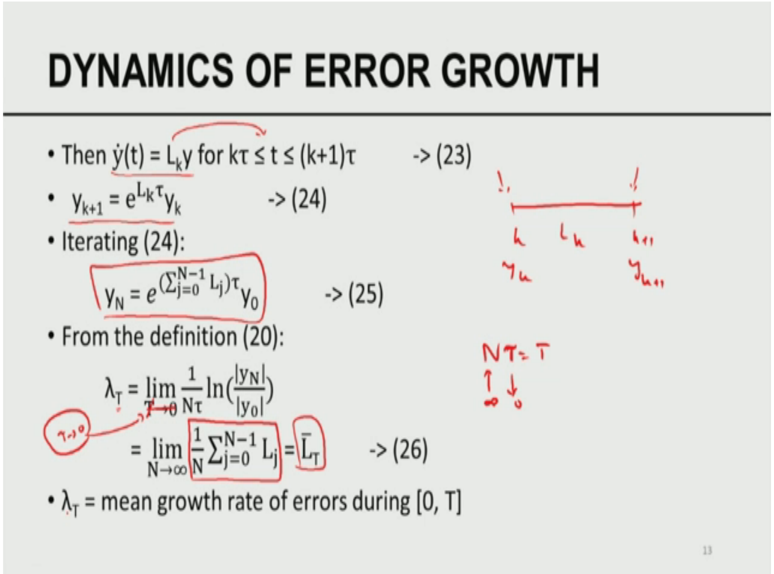

Therefore y dot dt of fyt by combining 22. So, 22 when substituted and 21 becomes an equation like this; sorry, becomes an equation of this time that is a linear equation. This equation depends on k you can see L k depends on this interval, I can solve this equation very readily there is a y k plus 1 is equal to e to the power of L k tau y k. So, if a time k this is time k plus 1 if y k is the initial condition, I would like to be able to compute the solution at y k plus 1 the matrix L k remains the same.

So, the solutions are 2 intervals of are 2 endpoints are related by this equation 24, if I trade this from y naught to Y N I get this relation you can readily see by iterating 24 and using the definition of L k being constantly that in the domain Y N is equal to e to the power of summation j is equal to 0 to n minus 1 the L j times tau y naught.

Now, from the definition of the Lyapunov index; the Lyapunov of index are time t, I am assuming might for catalyzing is capital T. So, when my forecast horizon is capital T from the definition of 20. Now it should be tau pointing to 0. So, this would be sorry as tau goes to 0 not t. So, t is fixed tau goes to 0 as tau goes to 0 n goes to infinity n tau represent the total time capital T. So, n tau is always capital T please understand n tau is capital n tau is capital T. So, as tau goes to 0 n goes to infinity, but the product is fixed t; the product is fixed t.

Now, I am taking the logarithm of Y N divided by y naught what is Y N absolute value of Y N is the error at time capital n y naught is the error at time 0 I am taking the ratio it is a logarithm of the ratio of the magnitude of the error at time n to time zero. So, y is there the errors now I am interested in the errors that affect the prediction now if I substitute 25 the expression for Y N in here and simplify what does it become it essentially becomes the average of Lj that is L bar t a that is a beautiful expression that comes from a simple. So, lambda t which is the function of t; t is finite is becomes l bar t what is l bar t is the average value of the Jacobian along the trajectory that is beautiful that is that is beautiful.

So, lambda t refers the main growth of errors during the interval 0 to t now what do I want to get the Lyapunov index, let t go to infinity that is the definition of Lyapunov index.

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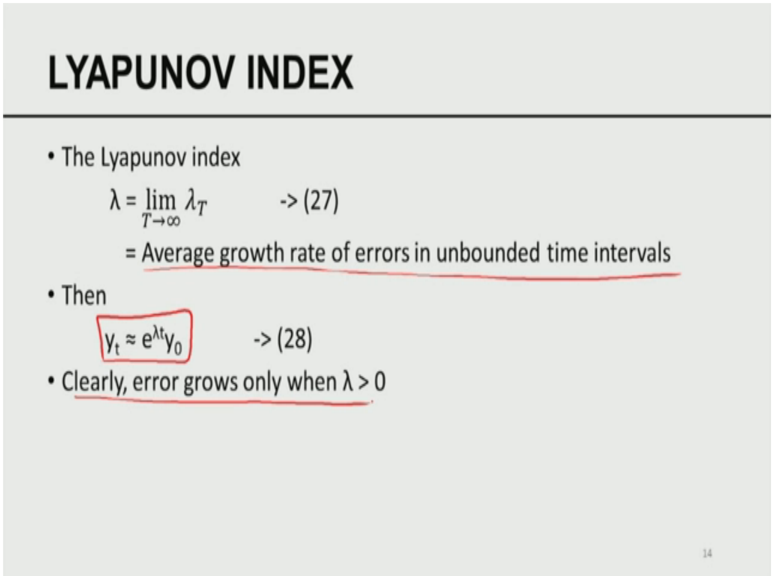

So, what is the Lyapunov of index is the limit of capital is the limit of lambda t with a capital T going to infinity and that gives raise to that gives raise to the definition of average rate of growth of errors in unbounded intervals. So, what does that tell you if lambda is defined that way you can now see that y t at any time is equal to e to the power of lambda t times y naught clearly the error grows when lambda is greater than 0. So, what does it mean one can analyze the quality of prediction by analyzing lambda if lambda is greater than 0; what does it mean average rate of growth of greater than 0 the average rate of growth is greater than 0 error grows.

So, in systems where lambda is greater than 0 that is initially predict predictability limit in systems where lambda is less than 0 there is no predictability limit I can predict for the whole feature. So, that is import of this analysis and. So, I want you to go back and understand the theory reasonably well. So, we presented a general theory; we are all trying to illustrate the general theory based on a very simple scalar continuous time dynamics, we have introduced the notion of average growth of errors. So, it is the average rate of growth of errors over asymptotically long intervals of time that helps to indicate the quality of prediction generated by the model. So, lambda depends on the model.

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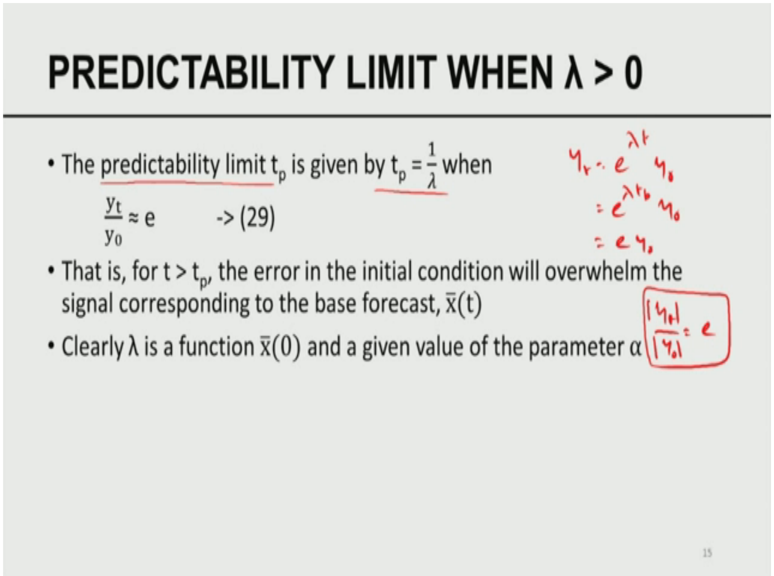

So, lambda is going to be able to give us the guideline how to believe and how long I can believe the forecast generate by the model.

So, how do we define the predictability limit the predictability limit tp is given by tp is equal to 1 over lambda. So, if lambda is positive my prediction can hold water only up to the forecast horizon tp which is equal one over lambda. So, what is one over lambda if y of t is equal to e to the power of lambda t y naught if t is equal to t p y naught if tp is equal to now lambda y of t is equal to e times lambda is y y naught; that means, the ratio of the magnitude of the initially a ratio the magnitude of the error a at time t to the initial error this is the order of e a that is a particular unit.

This limit for atmospheric models used to be one or days about 3 4 decades ago now has gone to of the order of five to seven days; it is this improvement from one to 2 days to five to seven days is the achievement by the meteorological community by bringing in the signs of data of assimilation and analysis of analysis of predictions created into the model. So, you can see the role of data assimilation the role of errors in the initial condition the role of trying to measure the ratio of the errors at 2 different times and understanding the rate at which the errors grow these are some of the beautiful mechanisms used in which one can create estimations of how good a forecasters that is the that is the ultimate key.

So, creating forecast is one thing trying to attest the goodness of the forecast or something else we have learnt how to create forecast by. So, many different methods of data assimilation the analysis of data assimilation is not complete until we understand the predictability limit how long the predictability how long the prediction can hold water that is the idea. So, that is for time greater than tp; tp th particular for time greater than tp the error in the initial condition will overwhelm the signal corresponding to the base forecast clearly lambda is a function of x naught. So, and the given value of the parameter lambda varies as the function of the initial condition also the parameters.

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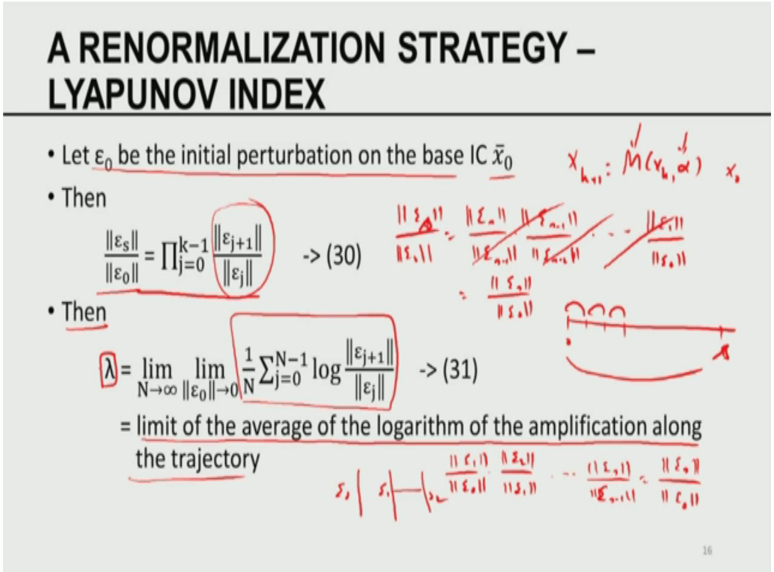

Now, I am going to talk about how to compute lambda for real systems. So, we have seen the importance of Lyapunov index. Now I am going to talk about the renormalization strategy by which one can implement an algorithm on any model to be able to compute the Lyapunov index. Once a quantity is of great interest we need to be able to compute it. So, let us let us look at an algorithm now let epsilon naught be the initial value of a perturbation based on the initial condition x naught bar. So, the initial condition is fixed given model is fixed alpha is fixed.

So, please understand my model is x k plus 1 is equal to M of x k alpha. So, M is fixed alpha is fixed I have initial condition that is also fixed. So, if I am interested in the ratio epsilon s to epsilon naught sorry epsilon naught epsilon s to epsilon naught, I can essentially express this a product like this. So, what is that epsilon s divided by epsilon s minus one times epsilon s minus one divided by epsilon s minus 2 likewise the epsilon one divided epsilon naught all the cross terms will leaving behind this equal to epsilon s divided by epsilon naught therefore, this ratio is equal to the product of the ratios between 2 successive times. So, what is that we are interested in we are interested in 0 e s I am interested in trying to multi the product, I am sorry, now I am interested in the ratios 2 successive times the product there off the product there off gives you the product from epsilon s 2 epsilon 0.

So, you can readily see this product is given by this ratio. So, then from our definition what is lambda the Lyapunov index; Lyapunov index is the average of the log of the average of the log of; so, I have to take the log of both sides if the log of both sides log of the product is the sum of the logs. So, log of an a b is equal to log of a plus log of b. So, if I took the logarithm on both sides of 30; I get an expression. So, the ratio of the log of both sides as epsilon naught goes to 0.

So, I would like my epsilon not to be as small as possible I want my end to go to infinity what is the n the number of discrete intervals of time and that is the limit of the average of the logarithm of the amplification along the trajectory that that is the basic idea. So, I would like to be able to compute this if I can compute that you want is the cover algorithm I am that and to be able to express 31 the key is in expressing the ratio of the errors a ratio of the norms of errors a time s to 0 as the product of ratios at consecutive times. So, this is the product of ratios in the consecutive times the product of the ratio is consecutive times the numerator denominator cancels leaving behind epsilon s epsilon naught. So, that is a very clever way of writing the ratio of the norms a time s 2 time 0.

So, this is this is one of the ways of computing there. So, how do we how do we do that. So, here is an algorithm now, let epsilon; epsilon naught.

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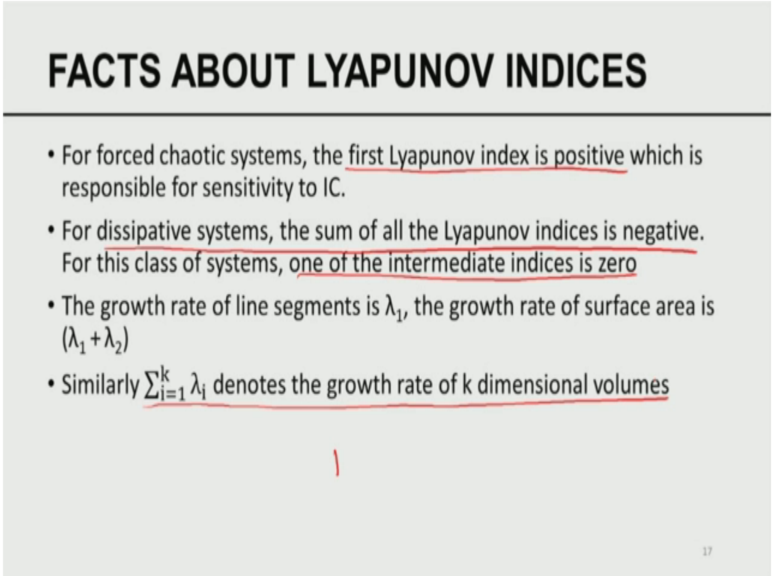

Let epsilon; I am sorry; that is right, let epsilon naught be the initial error then run the model one sip interval try and compute the error epsilon 1 compute the ratio the norm of epsilon one by norm of epsilon 0.

Then run the model from epsilon one to epsilon 2 then multiply this by norm of epsilon 2 times epsilon one continue if I continued; I get epsilon s divided by epsilon s minus one that is equal to norm of epsilon s divided by norm of epsilon 0 . So, by running the model once I am at a time by running the by computing the ratios by taking the product of those ratios taking of the logarithm of the ratio I can in a way estimate lambda. So, this provides an easy algorithm to be able to be able to evaluate lambda. So, this is what is called renormalization strategy.

Now, I am going to give you well known information about certain facts about Lyapunov indices for forced I have not defined what a chaotic system. Now let me talk about the definition of a chaotic system when do I say a dynamic system is chaotic if given a dynamical system if you compute the Lyapunov index if the Lyapunov index is positive in some part of the domain then it is called chaotic why it is chaotic in a chaotic system error grows on an average right.

If there the error grows an average rate then the predictability becomes very very very very difficult. So, what is the measure of the; that predictability is Lyapunov index. So, Lyapunov index is being positive is one of the signatures of the chaotic system and also lambda greater than 0 is an indication of the fact that the system is extremely sensitive initial condition because the initial condition errors grow at a rate lambda. So, Lyapunov index sensitive is initial condition being chaotic all these things are related concepts.

So, given a force chaotic system the first Lyapunov index is positive which is responsible for sensitivity the initial sensitivity a responsible for the sensitive initial condition. Now I only talked about a i Lyapunov index, you may ask what is the what is the first Lyapunov index again that requires a little bit of an explanation you remember we talked about the matrix let us go back in the definition of Lyapunov indices we talked about this matrix in nineteen lambda M is a matrix a matrix have M different eigenvalues. So, I am now if I talked about the analysis of the first eigenvalue that leads to first Lyapunov index second eigenvalues leads to the second Lyapunov of index.

So, if I system; if the order of the system is 3 in general it should have 3 Lyapunov indices I only talked about the maximum value of the Lyapunov index or the Lyapunov index corresponding to the first eigenvalue. So, in n dimensional system has the n eigenvalues n dimensions can have n in Lyapunov indices. So, if there are n Lyapunov indices you can talk about the first second third and so on; I am now going to talk about some of the well known facts about Lyapunov indices for different types of systems.

So, for a dissipative system the sum of all the Lyapunov index must be negative that is a fact this can be proven it is proven on many good books on introduction to chaos theory for this for this class of system one of the intermediary Lyapunov indices is also 0, I am got a illustrated further these are some of the well known facts I am trying to summarize explaining each of this will take at least one or 2 lectures, but here I am trying to collect many of the many the results the growth rate of the la.

So, what is lambda 1 tells you. So, if I have a 3 dimensional system I have 3 Lyapunov indices lambda 1 tells you the great growth rate of arose along one line lambda one plus lambda 2 refers to the growth rate of surface areas lambda one plus lambda 2 plus lambda 3 refers to growth rate of specific volumes; that is where different Lyapunov indices come into play.

So, likewise the sum of the first k Lyapunov of indices tells you the great of growth of k dimensional volumes. So, these are all very many simple facts.

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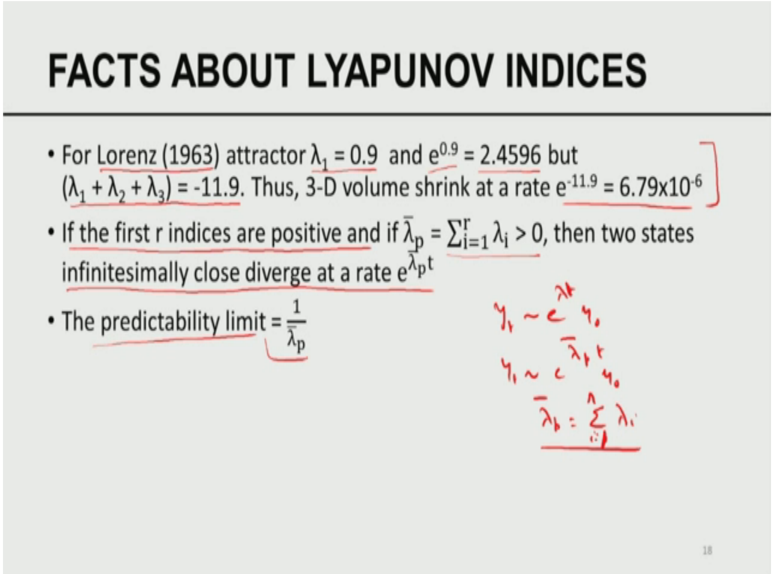

So, now I am going to talk about some of the specific values for specific systems many of us are introduced the notion of Lyapunov 1963 model; it is Lyapunov in 1963 for the first time introduced r accidentally found the presence of chaotic system. So, for it is there is this 1963 model of Lyapunov is one of the most thoroughly analyzed and understood systems of differential equation.

It consists of 3 ordinary differential equation coupled with non-linearity it is also a dissipative system because it is dissipative it is in general does not go to infinity it is the other orbits remains bounded the first Lyapunov index for this is 0.09; therefore, e to the power of 0.9; 0.9 is equal to 2.4596. So, what does it mean any line segment any error along the line or any; yeah any error along the line will magnify at the rate 2.4596 that is the average rate of growth of errors, but the total Lyapunov indices for this is minus 11 0.9; that means, the 3 dimensional volumes decrease at this rate decrease at this rate the life.

So, what is the idea here the volumes decrease the attractor wave the where the where the where the orbits of the system exists; you cannot predict where the orbits will be at what time because it is intrinsically chaotic because lambda one is positive. So, that is the overall characteristics of what is called the Lorenz system and examples of values of the first Lyapunov index if the first r Lyapunov indices are positive, then lambda prime p is equal to sum of all the Lyapunov indices which are positive 2 states, there are incident definitely close diverges the rate lambda bar e to the power of lambda bar p times t.

So, this is again the average rate of growth this is an extension of y t being equal to e to the power of lambda t times y of t. So, in this case it is y t times e to the power of lambda p times t times y naught where lambda p is I am sorry lambda bar p sorry where the lambda bar p is equal to the sum of the first r some of the first r Lyapunov of indices which are positive. So, the sum of the positive Lyapunov indices refers to the average and great of growth in this case the predictability limit is given by one over lambda at to the one over lambda bar p.

Now, you can see if there are more than one Lyapunov index that is positive lambda bar p is larger if lambda p lambda bar p is larger one over lambda p bar is smaller therefore, if a system exhibits larger number of positive Lyapunov indices in such system the predictability limit is much smaller if a system does not have any positive Lyapunov index then then I can make predictions for a long periods of time predictions for the long periods of time. So, that is the ultimate essence of the theory of Lyapunov index.

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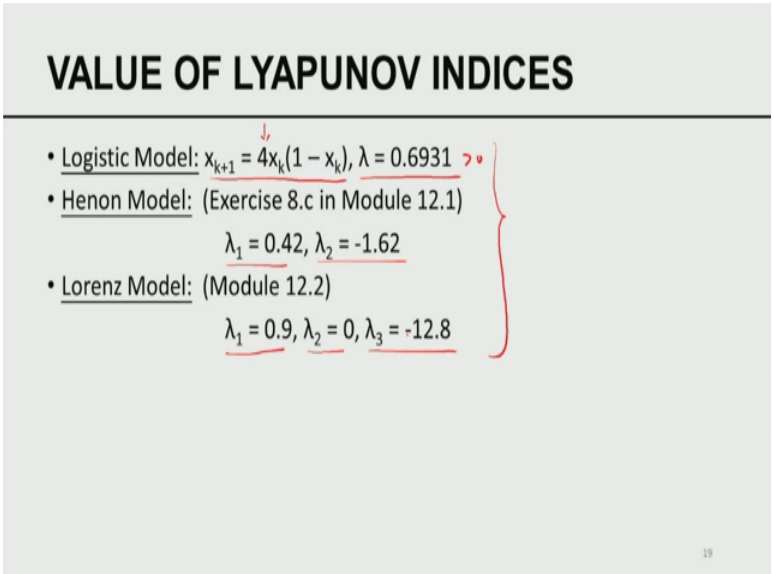

Now, I am going to give you these are exercised problems, but I am going to cote you some of the values of the Lyapunov indices or some of the very well-known mathematical models for the logistic model that the parameterization 4 the Lyapunov index is 0.96931 that is positive. So, logistic models are chaotic because they are extremely sensitive initial condition; there is a model called Henon model; the Henon model has the first Lyapunov index is a 2 dimesnional; I am sorry, yeah, for the Henon model, I have the lambda one is point for 2; lambda 2 is minus 0.162 for the Lorenz is already talked about first one is 0.9; second one is 0, last one is this. So, these are some of the examples of simple systems there is a difference between the properties of the same logical logistic model is the discrete time model that tries to capture some of the principles of population dynamics.

Henon model is a mathematically oriented model it may not correspond to any particular physical system Lorenz is model how did he obtained the 3 differential equation he obtained it from starting with some vorticity equation oh i and in this case not a vorticity equation I am sorry it is it is the it is the it is the heat transfer problem and from that he applied the spectral methods and he from that derived the 3 sets of equation whose ligand wall whose Lyapunov indices are positive 0 and negative. So, Lorenz model theoretically for the first one to be able to exhibit the notion of Lyapunov index.

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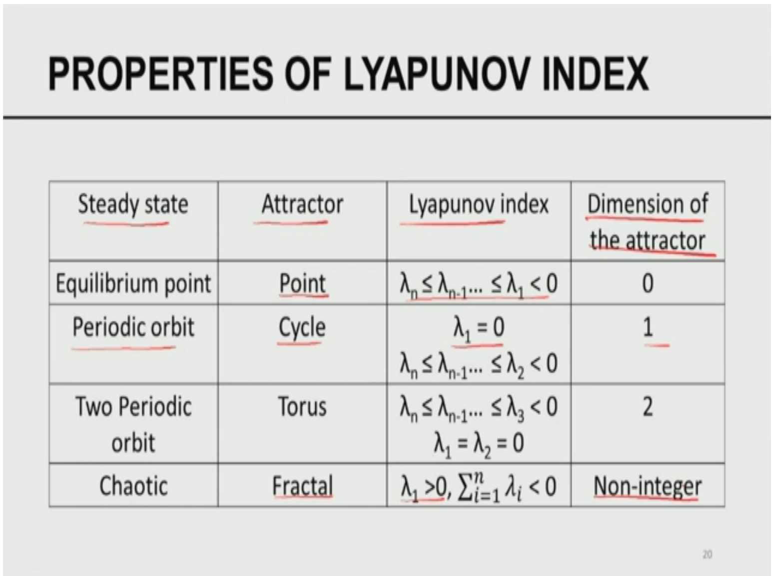

PROPERTIES OF LYAPUNOV INDEX

Steady state	Attractor	Lyapunov index	Dimension of the attractor
Equilibrium point	Point	$\lambda_n \leq \lambda_{n-1} \dots \leq \lambda_1 < 0$	0
Periodic orbit	Cycle	$\lambda_1 = 0$ $\lambda_n \leq \lambda_{n-1} \dots \leq \lambda_2 < 0$	1
Two Periodic orbit	Torus	$\lambda_n \leq \lambda_{n-1} \dots \leq \lambda_3 < 0$ $\lambda_1 = \lambda_2 = 0$	2
Chaotic	Fractal	$\lambda_1 > 0, \sum_{i=1}^{n} \lambda_i < 0$	Non-integer

And the role of Lyapunov index and predictability studies here are some other properties of Lyapunov indices, I am going to now create a comparisitive a comparison steady state behavior attractor Lyapunov index dimension of the attractor these are summaries are very well known result a steady state for a dimensions can be in equilibrium point what is an equilibrium point there is an attractor to which all the solutions come and settle down those steady state are called equilibrium point equilibrium point; the attractor is a point the Lyapunov indices are all negative.

So, therefore, the dimension of the attractor 0 for a periodic orbit it is a cycle; there is one Lyapunov index is 0 rest of them are negative the dimension of the attractor is one; I know, I have not talked about the dimension attractor much more elaborately, but I believe; I wanted to provide you a quick summary of the notion of predictability and some of the models for which the predictability can be answered.

So, what does it mean if a system exhibits an equilibrium predictability is complete if a system exhibits periodic orbit their predictability is complete; I do not have to worry about predictability I can predict for all for all times if the system exhibits that steady state with respect 2 periodic orbits the attractor is set to form yeah the yeah terrace in which case 2 eigenvalues are 0.

There is the eigenvalues are negative the dimension of the attractor is 2 for a chaotic system the attractor is called fractal. I have not introduce even the notion of fractal object fractal object is a complicated object in the case of a fractal object in the case of a chaotic attractor such as the Lorenz attractor the at least one Lyapunov index is positive the rest of them could be 0 or negative and the sum of all the overall Lyapunov indices totally together could be less than 0 that; that indicates the system is totally dissipate dissipative and the dimension of the attractor is non-integer; that means, if they have what is called fractal dimensions.

So, the notion of fractal attractive fractal dimension chaotic behavior they are all interrelated with each other.

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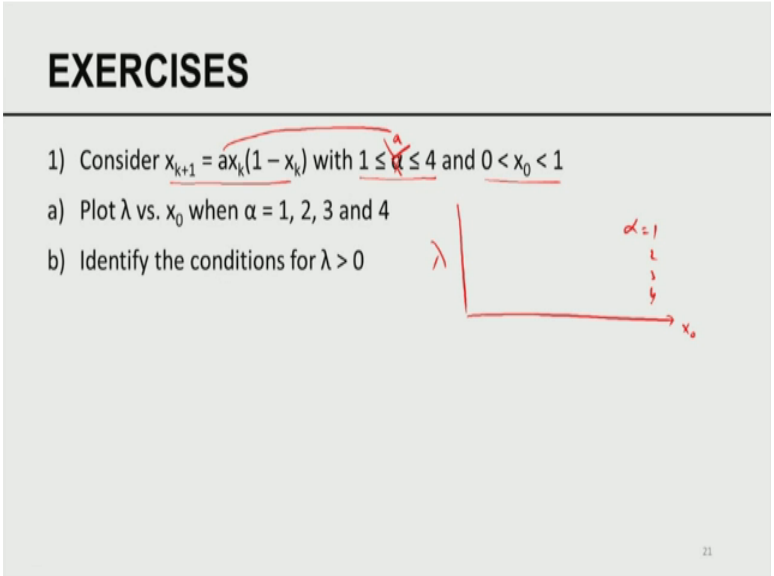

With that we conclude our discussion of deterministic predictability I would like you to pursue these this exercise what is the exercise consider the simple logistic model this is a very good exercise consider alpha consider this is not alpha a the same parameter a as in here consider the range of parameter from one to 4 x naught is now is equal to 0 to 1.

So, I would like you to pick an x naught pick an alpha compute lambda. So, plot this is a very good computing exercise Lyapunov index versus x naught for a given value of alpha sorry for a given value of for a given value of alpha. So, alpha is equal to 1 draw this curve alpha is equal to 2; draw this curve alpha is equal to 3 4 draw this curve you will see you will verify when alpha is 1 and 2 and 3 the system does not exhibit any sensitivity initial condition, but when alpha is equal to 4 the system tends to exhibit extreme sensitivity initial condition and that is the model we talked about with this I have; I conclude our discussion on predictability of deterministic system as a quick summary predictability analysis is of great interest after we assimilated data into the model once you assimilate data in to the model I am going; I have the capability to make prediction.

The question is how far for how long the prediction will hold water; the answer to the question of how long the prediction will hold water depends on the intrinsic property of the model itself it relates to the sensitivity of the model solution with respect to the initial condition this sensitivity is essentially relate to the forward sensitivity we have already defined. So, forward sensitivity analysis and analysis the Lyapunov index are intimately associated in each other even though I am not exploring that discussion in this in this in this in this set of lectures one way to be able to summarize the initial condition sensitivity is through a parameter called Lyapunov of index.

Lyapunov of index could be positive 0 or negative for dissipative system the sum of all the Lyapunov index is must be less than 0; at least one Lyapunov index is 0, if one of the Lyapunov indices the greatest Lyapunov index is positive such systems are supposed to exhibit extreme sensitivity it turns out many many of the model that are currently being used in geophysical sciences to predict different types of geophysical phenomena have exhibited extreme sensitivity to initial condition that is the reason why we are not able to make long term predictions despite 50 plus years of progress in model building as well as data collection and data analysis and with all the glances we are able to increase the predictable limit to about five six no more than seven days these days.

So, what does it mean the prediction problem continues it continues to dominate it continues to be a problem of great challenge and 1 of the goals of this study models data; data assimilation all relates to improving the predictability limit. So, what is the ultimate goal of doing all these things I would like to be able to make long range prediction very reliable long range prediction until such time we achieve the ability to make long range prediction our job is not done and there is no telling how long it may take though it all depends on it all depends on very many different aspects of very many different aspects of models data; data assimilation sensitive to the model initial conditions so on and so forth.

So, I have provided a simply a rudimentary ideas; I simply scratched the surface unpredictability is a very deep discipline I would encourage you hope this will encourage you to be able to look at some of the interesting books related to productivity analysis as well as chaos theory, but as a starting point this exercise analyzing logistic model by choosing various alphas and choosing various x naughts; I am trying to compute will be a very good opening game in your understanding the theory of predictability.

Thank you.