Real Analysis

Prof. S.H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

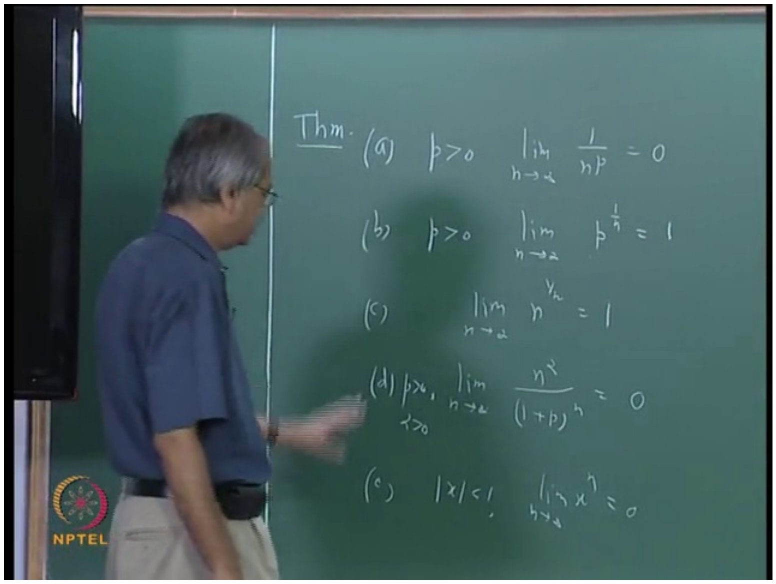
Lecture - 09
Sequences of Real Numbers (Continued)

Well in the last class, we have seen what is meant by saying that a sequence is convergent or not convergent. And seen some methods of showing that sequence either converges or diverges directly by using definitions or by some other methods. And especially showing the convergence we have seen that, it is better to make use of several theorems about the sequences to show that some sequences are convergent. And in that we listed those theorems for example, if you take 2 convergent sequences then, there is some convergent, sequence product etcetera.

And one of the important aspects in that is if so called the sandwich theorem namely that if you take three sequences, let us say x n, y n and z n. And if x n is less than equal to y n less than equal to z n for each n, which we describe by saying that y n is sandwich between 2 x n and z n. And if these 2 sequences x n and z n converges the same limit then, y n also converges and it converges to the same common limit. And this is something which is used very often in proving that many well known sequences are convergent. Let us see some examples of this. Basically, what we will do is that, we shall make a list of some sequences whose, converges we shall be using very often.

Let me just write it here. So, let us say p bigger than 0 then, limit n tends to infinity 1 by n to the power of p is equal to 0. Then, again for p bigger than 0 limits n tends to infinity p to the power 1 by n is 1. And let us say similarly limit n tends to infinity n to the power of 1 by n that is also 1 by n. By the way this power 1 by n we have already defined, we have seen that given any positive number then, any natural number n then, exists this is a unit power of the number x to the power n is p. So, p put to the power of 1 by n means that number. Then similarly, we can say that limit as n tends to infinity.

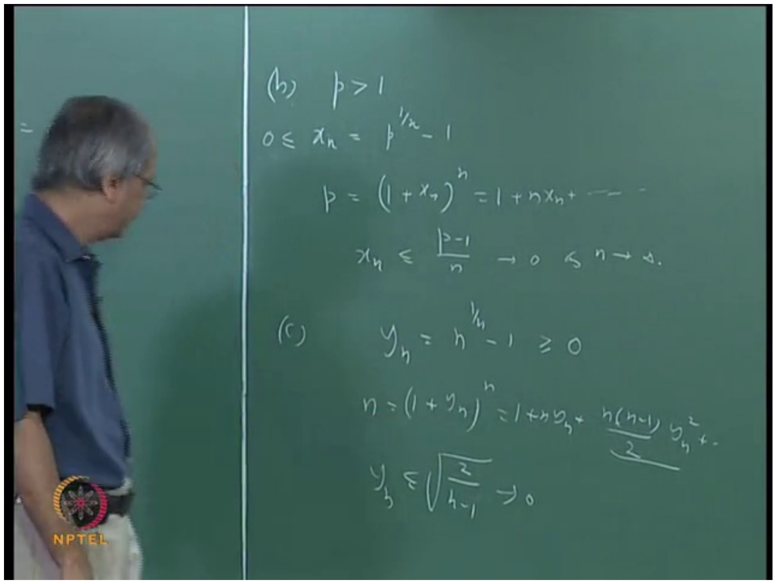
(Refer Slide Time: 01:35)



Suppose here I take something like n to the power alpha divided by 1 plus p to the power n. Here also I take p bigger than 0 then this limit is 0, alpha is also bigger than 0. And finally suppose you have mod x less than 1 then limit n tends to infinity x to the power n, that is also 0. We shall see proofs of some of these things and other session view as an exercise. Anyway this we did not discuss, but we have shown already that 1 by n limit and how this Archimedean property or n is not bond above that comes into picture and showing this.

And this 1 by a to the power of p, you just multiply it sufficient number of times if p is an indices, other cases also you can take care of it. Let us just look at this now, p is bigger than 0 we will need to make, if p is equal to 1 there is nothing to prove. This becomes a constant sequence, so we need to surround the cases p bigger than 1 and 0 less than p less than 1. So, let us just see the proof of this.

(Refer Slide Time: 03:54)



Let us see the proof of the case when p is bigger than 1, p is equal to 1. The case is trivial and other possibilities 0 less than p less than 1, but you will see that if we prove this case and if you take 0 less than p less than 1, 1by p will satisfy this. And if p to the power 1 by n, if the limit for that is 1 limit of you can look at limit of 1 by p to the power x, that will also given. So, the other case also can be taken care. So, it is enough to prove this case. Let, say I will take this x n as p to the power 1 by n minus 1.

Is it clear that this is always bigger not equal to 0 and suppose we show that x n tends to 0 as n goes to infinity then, it will win this. Because the difference between this is x n, so if we show that x n similarly the meet of x n is 0 as n goes to infinity, that will prove the required thing. Now I say that we are going to use Sandwich theorem, so what I will do is that I shall construct a sequence, which is bigger than it is, let us say y n and show that 0 less than equal to x n less than equal to y n for each and y n goes to 0.

See how this can be done. Now, this is same as that p is equal to 1 plus x n less to n. Now, this is same as 1 plus n x and plus, there will few more thing that I do not bother about. Is it clear that because x n is all negative in fact positive. All the terms here are positive so any term that is less than equal to p. So, it follows from here so I will say that that is whatever this part is is any term or any 2 or three or any number of terms. So, I can say from here that x n is less than or equal to p minus 1 by n for all n. You can call this 5 n if you want and this tends to 0 as n. Let us clear this tends to 0 as tends to infinity.

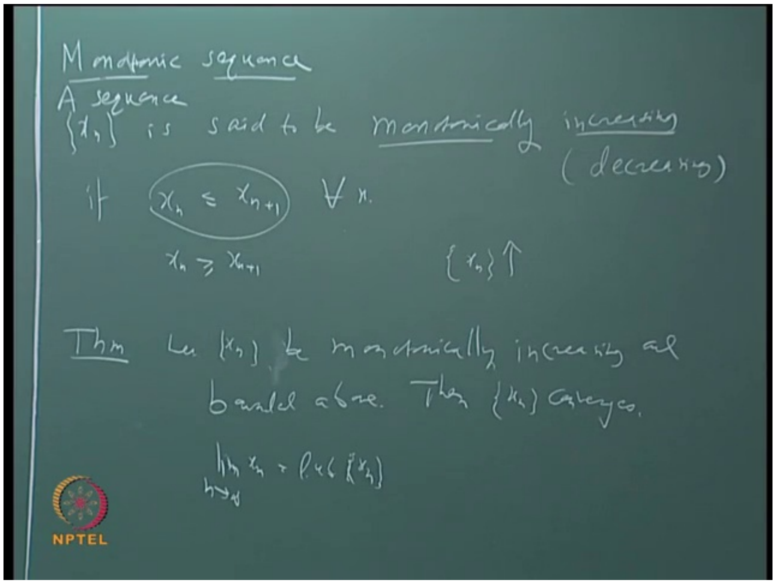
And so, x n is bigger or equal to 0 and x n is less than equal to y n and y n tends to 0. So, you take the constant sequence 0 and this sequence this sequence given sequence x and n is sandwich in between this 2 so its limit must be 0. Let us look at this again, there is also its similar I will again consider this x n as or let us say y n as n to the power of 1 by n minus 1. Now, this is same as that n is equal, it is clear that this is always bigger or equal to 0. And again f of is a similar way, I can say from here that this is same as saying that n is equal to 1 plus y n to the power of n.

Now, suppose I do the same thing here also, I can say that this is same as 1 plus n by n, it should be etcetera, but you can see that if I earlier here every term is positive. So, I can take any number of terms or group of terms and use this, but you can see that if I just stop here and proceeding this way. What I will get is y n is less or equal to n minus 1 by n and limit of that is not 0. Instead of p minus 1 by n, you will get n minus 1 by n, n minus 1 by then limit of that is 1, so you cannot use sandwich theorem. So, what will do is we will see whether we take one more term. So, what is the next term here 1 plus n, the next term will be n into n minus 1 by 2 into y n square.

This is also less than or equal to this, so what you get from this? n into n minus 1 by 2 by square that is less than or equal to n. Now, that is same as same that y n is less than or equal to square root of 2 by n minus 1 because this term is less than or equal to n. So, from that you will get it by n is less than for equal to square root of 2 by n minus 1. And now this tends to 0, so y n also is equal to 0 and that is same as same at the limit of n to the power of 1 by n is 1. I shall give the proof for this last two.

Again next one is again similar; there you will need more than this. Here we have stopped with the term y n square, there we will need some more terms, but again the idea is very similar. As soon as that you tried to do this to your own take it an exercise; and you can see that once you do this 3 e is straight for as you can just take alpha is equal to 0 here then, that will give you e. Then there is another technique in showing that the sequence certain sequences are convergent.

(Refer Slide Time: 10:31)



Again you may finer with this, but let us again repeat that to do this we need the concept of this what is called a monotonic sequence. Let us just recall definitions first, when suppose you are sequence x n is said to be monotonically increases. Well it is already we know when is facing, if each succeeding term is bigger than or equal to previous term, but monotonically increasing if you can say x n is less than or equal to x n plus 1 for all. And similarly, you can define what is meant by monotonically decreasing.

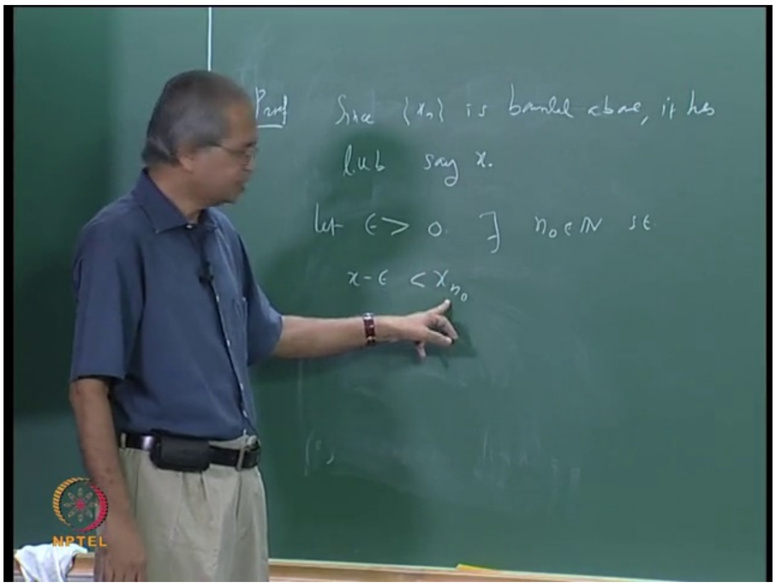
You can say it is decreasing instead of this, this will be replaced by x n less than or equal to x n plus 1. And what is meant by a monotonic sequence? Monotonic sequence is increasing or decreasing, monotony increasing or monotony will be decreasing, it is monotony sequence. Then, we come to a very well known theorem in this, if a sequence is monotonically increasing and bounded above then, that converges. And that is a very one of the well know techniques in showing that is several sequences are convergent sequences. Let us just recall that x n by the way this is a standard notation to say like that x n.

You write this upward arrow that is the notation to say that x n is monotonic and inclusive and similarly, the arrow pointing downwards for monotony decreasing. So, I will just use this let x n be the monotony increasing sequence, let x n be monotonic increasing and bounded above then, x n converges. In fact we can say something more; we can also say exactly what it converges to.

It converges to the least upper bound of this set, it takes the range and the sequence takes its least upper bound and there it converges.

We can say that limit of x n as n tends to infinity is the thing, but this LUB, least upper bound of x as now on word also called supremum of x n. And by the way if a properly increasing sequence converges just to a limit x then, we simply denote it by this symbol; that x n is monotonically increasing and converging to x. And in view of this theorem it also means that x is link LUB of that, this is something, which should be using very often.

(Refer Slide Time: 14:53)



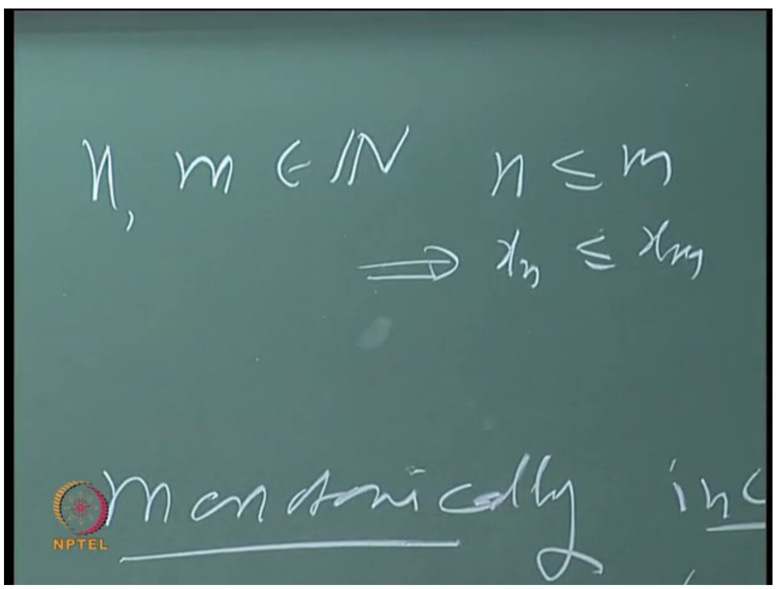
What is important to realize is that, this theorem follows again because of the n u b x because it is a non empty set it is bounded above, so it has a least upper bound. And so the n u b x plays a very important role in this. So, let us say that since x n is bounded above, it has the least upper bound. Suppose, I call it least upper bound x then, how does one show the sequence converges? Then, we have to take a charge for x bigger than 0 where as n 0 etcetera. So, let us take epsilon bigger than 0 then, you have seen already how does this epsilon and l u b comes into picture that is if x is on least upper bound.

Student: ((Refer Time: 16:09))

See it is a least upper bound, it is a least upper bound that means x plus epsilon is not an upper bound. If x plus epsilon is called an upper bound what should happened, x minus epsilon is not correct, so what should happen? Then, there should exist some element x n, which is bigger than x minus epsilon. So, we can say there exists n 0 in n such that, x n 0 this must be bigger than x minus epsilon. Now this is something we could have said about any bounded set for any sequence, but now in case of this sequence we know that it is monotonically increasing.

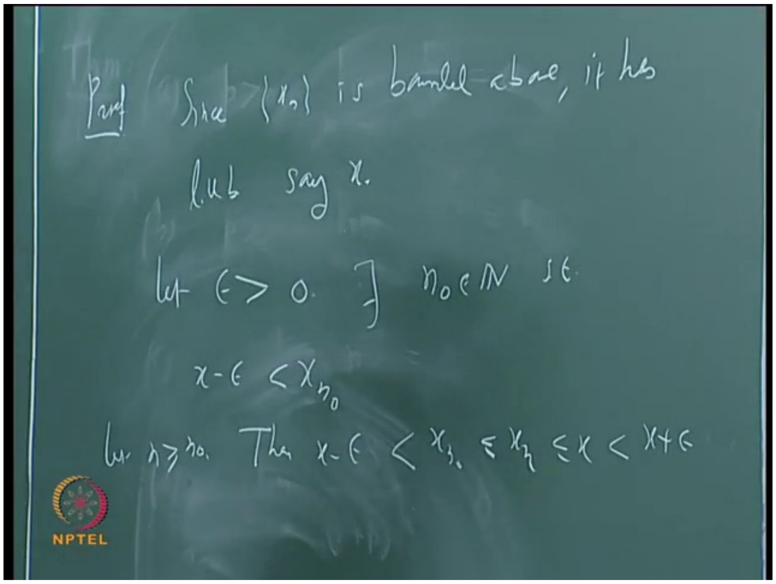
So, if it is monotonically increasing, for example, if you take n 0 plus 1, it should be bigger than equal to x and n 0 0 plus 2, it should bigger next n 0 plus 1. In other words for any n bigger than or equal to n 0 x n 0 should be less than or equal to x. In fact that is something we can say about any monotonically exist x n is less than or equal to x n plus 1, x n plus 1 will be less than or equal to x n plus 2, that will be less than or equal 2 x n plus 3 etcetera.

(Refer Slide Time: 17:52)



So, in general we can say that if you take two natural numbers n and m, which n less than or equal to m. This will be x n is less than or equal to x for monotonically increasing sequence and in equality will be reverse for monotonically decreasing sequence.

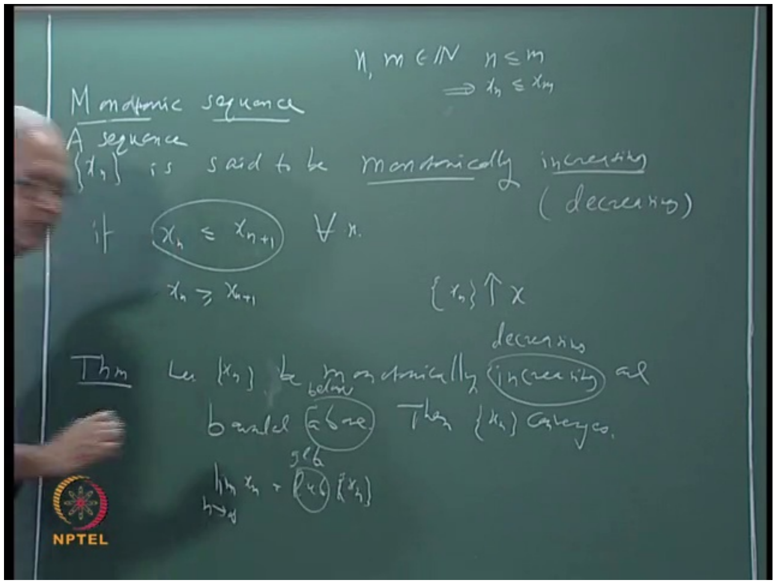
(Refer Slide Time: 18:13)



So, what you can say is that then for all n, so let us say that let n be bigger than or equal to n 0. Then x minus epsilon this is strictly less than x n 0, this is less than or equal to x n. And we can always say that x n must be less than or equal to x because it is a upper bound. And in fact x is always less than x plus epsilon so, x is less than x plus epsilon. So, what did we show that, for every n bigger than or equal to n 0, x n lies between x minus epsilon and x plus epsilon. Thus, anything that x n converges to x.

So, we have proved that every sequence, which is monotonically increasing and are with above converges not only that, we also shown that it converges to its least upper bound. In a similar way we can show that every sequence, which is monotonically decreasing and bounded below will also converge and it will converge to its let us learn about.

(Refer Slide Time: 19:30)



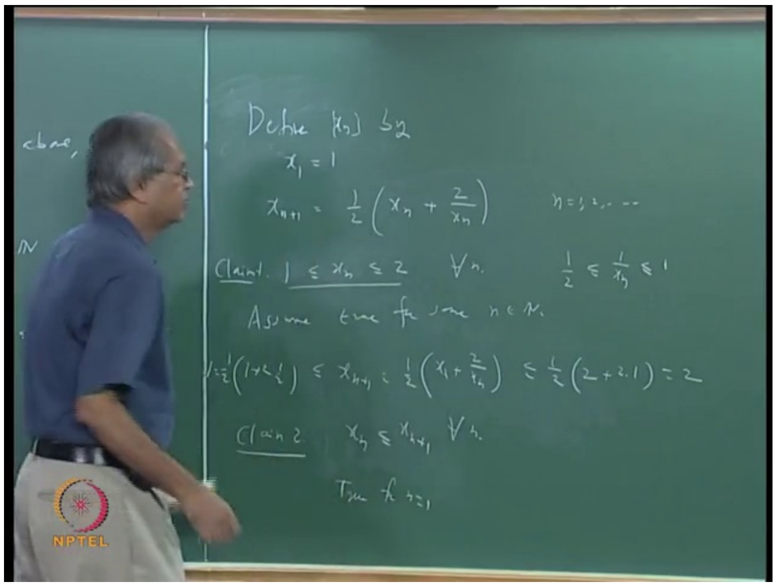
So, instead of increasing suppose you change this to decreasing and in fact it should be bonded below. Suppose, we take it is bonded below then that sequence also converges and it will not converge to l u b, but it will converge to g l b. And proving that is simple, once you prove something for an increasing. You can imitate proof in the same way, other one possibility is that if x n is monotonically increasing or x n is monotonically decreasing, minus x n is increasing. If x n is monotonically then, minus x n is increasing.

So, use the theorem for increasing sequences and then, let require it. Anyway that proof I shall discuss here that you can prove on your own. Now this is theorem, which is quiet useful in showing that the several well known sequences, which we come across are convergent and it will also be quiet useful subsequently. And also to say also to define something that we want to talk about little later, but before that let me just take 1 or 2 well known examples.

You also know that one of the ways is defining a sequence because the sequence is the function of set of all natural numbers and the set of all natural numbers that is what we have called principle of induction. So, for example I can give the value of x n and then I can tell how to compute x 2 from x 1, x 3 from x 2 etcetera, this is what is called defining the sequence in depth to you, that is give the value of what is x 1. And then say how x n plus 1 is competent from x n then, that defines all the numbers in the sequence.

Again this is just one possibility I can also say, I can also give the value of x 1 and x 2 and then say how to compute x n plus 2 from x n and x n plus 1 that will also define the whole sequence. So, this is called defining the sequence in derivatively, but once the sequence is defined in derivatively then, every property of the sequence also has to be checked by mathematical induction.

(Refer Slide Time: 22:09)



And one of the ways of showing the sets sequence are convergent or divergent is that, you show the sequences monotonically increasing or bounded above or the other way decreasing or bonded below. Let us just take one example of that form, so define x n as follows. Let us say x 1 is 1 and x n plus 1 is equal to let me say half into x n plus 2 by x for all and equal to. So, when a sequence is defined in this form, the only way to discuss its convergence or divergence is that you have to see whether it is increasing on it.

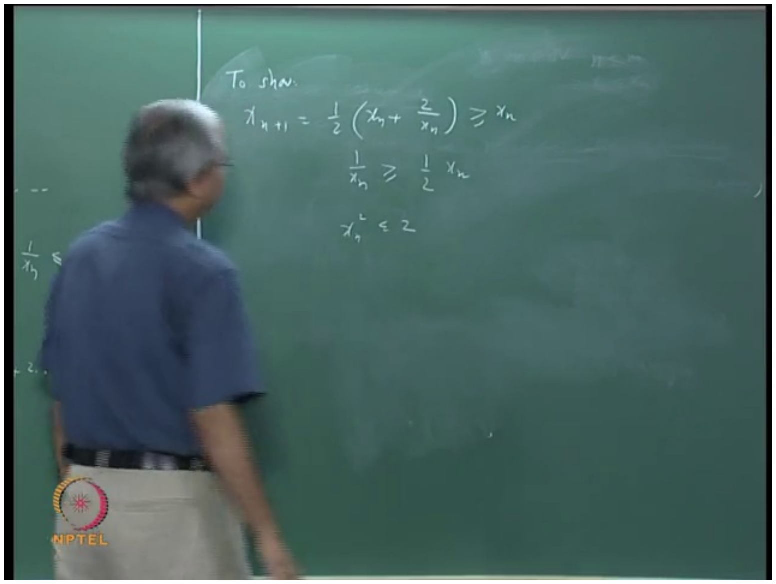
Let us first see whether it is increasing or decreasing that you can see, but we look at x 2 it will be half into 1 plus 2, so it is bigger than. So, if at all we go to show anything it has to monotonically increasing. What about the boundedness? We can try to show that x 1 see for example, you can see that x 1 and x 2 both is less than or equal to 2. We can say that, we can attempt to show that it is bonded above by 2. We did face we can choose some other number, but let us try with this. So, first of all x 1 is 1 obviously we can say that 1 less than or equal to x 1.

In fact let us say that we want to show this, 1 less than or equal to x n less than or equal to 2 for all n. Let us say when we want show something we will write that as a claim. This is true for n equal to 1, so assume that it is true for some value of n. So, assume true for some n in n. And then we should try it is true for x true for x n plus 1. Now, since x n plus 1 also involve the reciprocals, let us write this equality in terms of reciprocal also. See, 1 less than or equal to x n less than or equal to this also means 1 by 2 less than or equal to 1 by x n less than or equal to 1 because this also may be needed in improving.

So, now look at x n plus 1 that is suppose, this is true, we want to prove similar thing is true for x n plus 1. Now x n plus 1 is equal to half into x n plus 2 by x n. So, this is less than or equal to half into x n is less than or equal to 2, and then plus 2 into 1 by x n less than or equal to 1, 1 plus 2 into 1 and so, half into 4, so this is 2, so this part is fine. So, x n plus 1 less than or equal to is fine, can we also x n plus 1 bigger than or equal to 1. Let us see how that follows, this is bigger than equal to half into x n is bigger than or equal to 1 plus 2 into 1 by x so, this half into 1 plus 1, so that is ok.

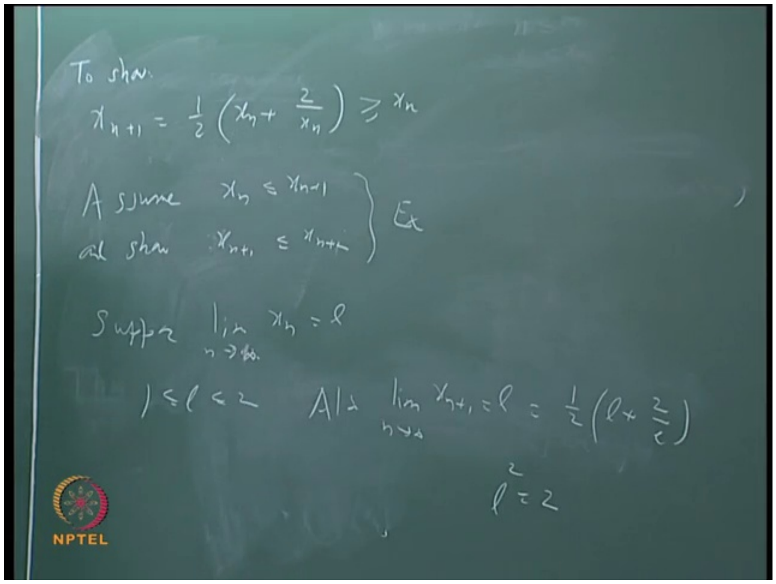
So, we have shown that if 1 less than or equal to x n is true for some value of n, that is also holds for x n plus 1. So, by induction it holds for all n. Now, what is the next thing we need to show? We want to show that x n is monotonically increasing. So, you can call this claim 1. So, claim 2 x n is less than or equal to x n plus 1 for all n. And as you know now in most of the induction proofs the case for n equal to 1 is trivial. So, what about n equal to 1, x 1 is 1 and you can calculate whatever is x 2 and just think that it is bigger than or equal to 1 so, true for n equal to 1. Let us see now next to show that x n is less than or equal to x n plus 1, what is it that we need to show that x n plus 1.

(Refer Slide Time: 27:59)



Let me just say what is that we need to show? We show that x n plus 1 is half into x n plus 2 by x n. We want to show this is bigger than or equal to x n. So, what does it mean? Suppose I bring this 1 by 2 and this, so that is 1 by x n is bigger than or equal to half x, that is x n minus this half x. So, that is same as saying that we must show that x n square is less than or equal to 2. So, suppose we might show that here instead of 2, suppose I taken root 2 and try to show that x 1 is do not root 2. Then, itself it would have been over here, but obviously you that cannot work, so again we have to use the induction.

(Refer Slide Time: 29:21)

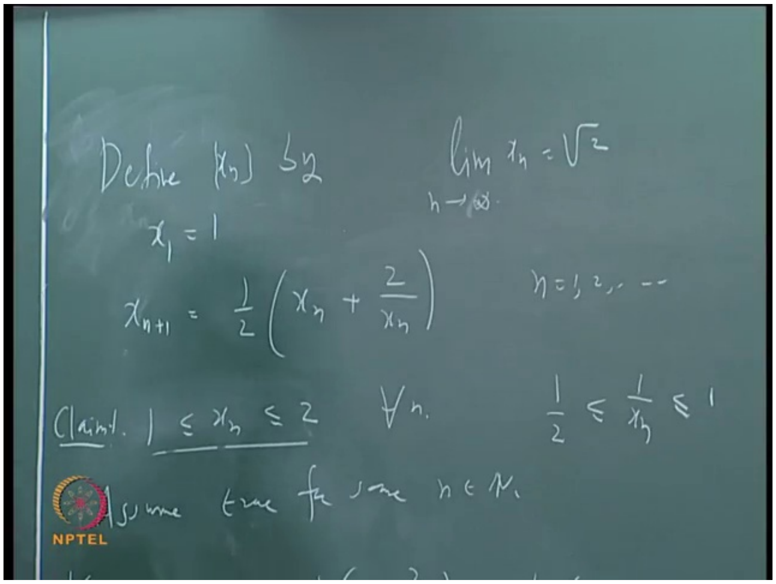


Since, this proof is fairly rooting I will not go into this. Let us assume that x n is less than or equal to x n plus 1 and show x n plus 1 is less than or equal to x n plus 2. Take this as an exercise. So, I am not discussing it here. The next question is suppose we prove this to this, it will be in this sequence converges. The next question is if it converges what is its limit? Again you will know the well known argument for this if x n converges suppose limit of x n is equal to n, suppose limit of x n has entered infinity is equal to l.

Then, first of all it will in that 1 less than or equal to 1 less than or equal to 2 because of what we have proved here. First of all 1 less than or equal to l less than or equal to 2, but there is one more important thing limit of x n plus 1 as n goes to fee that is also l. Write that is limit of also limit of x n plus 1 as n tends to infinity, that is also l, but x n plus 1 is half into x n plus 2 by x. So, limit of this we can also convert limit of this in different way. So, limit of this is also nothing but half into 1 plus 2 by 1.

And so, what we will get is that this is l is same as half into l plus 2 by l. And if we simplify this, you will get half l is same as 1 by l and you will get this as l square is equal to 2. So, l is equal to that will give l is equal to plus or minus root 2, but you have this.

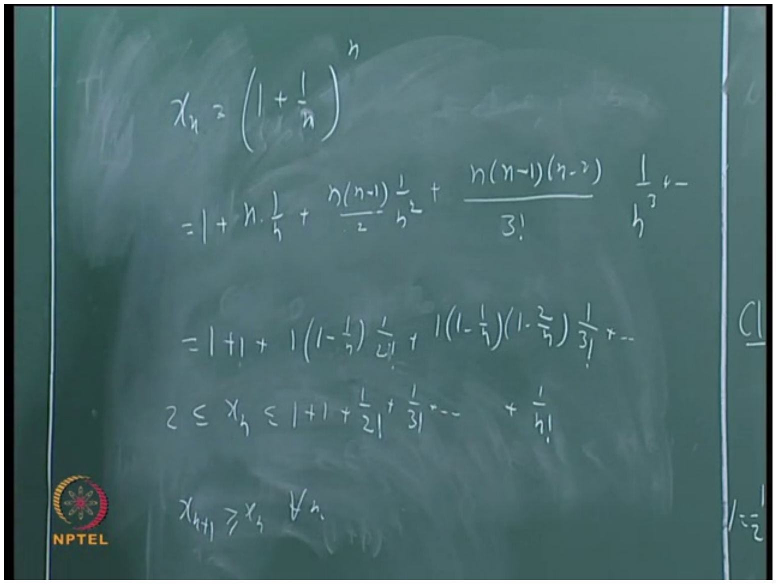
(Refer Slide Time: 31:50)



So, it can't be minus root 2, so limit of this limit of this sequence limit of x n minus n tends to x that is the thing. This is one of the well known techniques in finding the limits of many sequences, which I had defined in this fashion. Let me again tell you that here I have given the value of x n and we have defined x n plus 1 in terms of x n. It is also possible to give x 1 and x 2 and define x n plus 1 in terms of x n and x n minus 1. Let us take the two previous things and similarly three terms etc.

These are nothing but small modifications on this same id. Now, before going to the next concept, let us also complete the discussion of this well known sequence x n is 1 plus 1 by n raise 2.

(Refer Slide Time: 32:43)



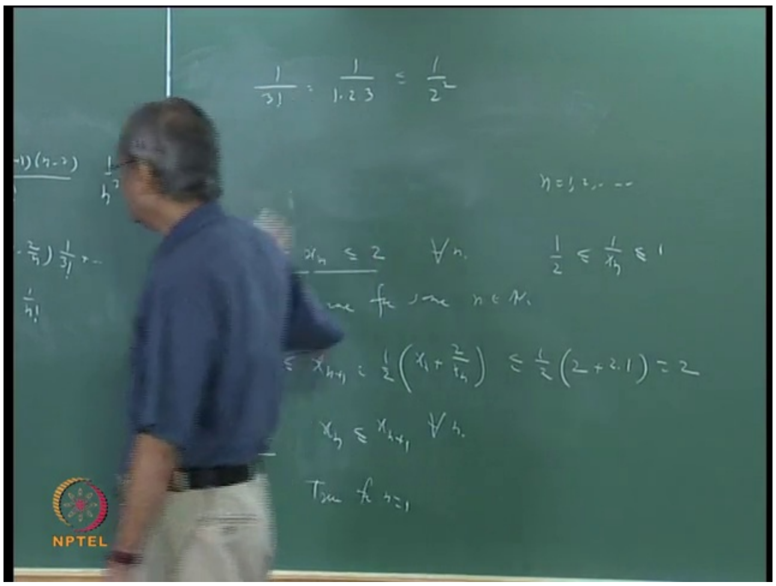
Again here also we can show that, this sequence is monotonically increasing and bonded above. Let us just quickly look at what are the steps in showing that. So, suppose you expand this by using binomial theorem then, this will be 1 plus it is next term is n into 1 by x, so that is also 1 n into n minus 1 into n minus 1 by 2 into 1 by n square. The term will be n into n minus 1 into n minus 2 divided by factorial 3 into 1 by n cube etcetera. Now, what we can see from here is that this should be 1 plus let us see who will write this term in a slightly different manner.

I will divide by N Square to this terms n into n minus 1. So, this term can be written as minus 1 into 1 minus 1 by n into this 1 by factorial 2. Then, the next co-efficient will be 1 into 1 minus 1 by n into 1 minus 2 by n into 1 by factorial 3 etcetera. This is just basically rearranging and to see that it is convenient to see. Now, if I take x n plus 1, what is going to happen? Here you are going to get 1 by n plus 1 into n plus 1. So, this is going to be changed to 1 by n plus 1, 2 by n plus 1 etcetera.

Is it clear that this is bigger than or equal to this, 1 by n plus 1 will be smaller than 1 by n, but you have to subtract it. So, the corresponding term in this will be bigger than or equal to corresponding term here. And plus you will have one extra term because the power is n plus 1. So, x n plus 1 bigger than or equal to x n for all n that, is it clear. In fact that is the reason writing in this fashion.

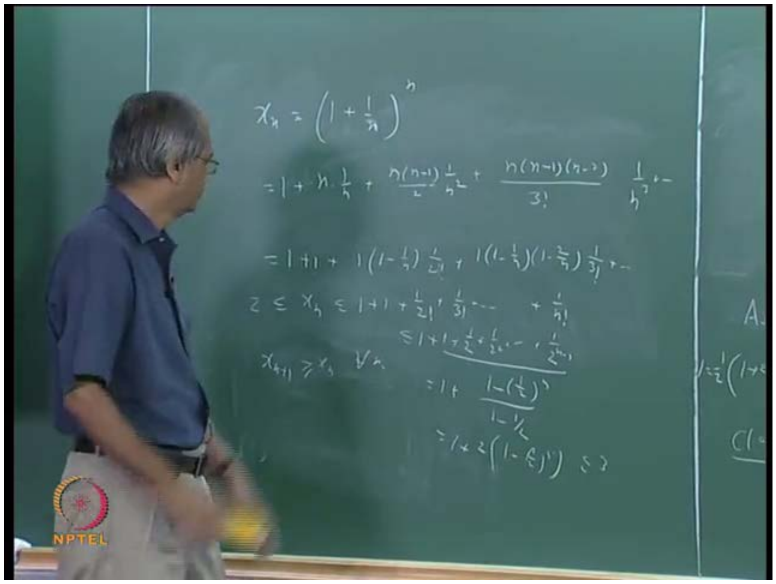
See remember each of this terms are non negative, so everything is here is this 1 plus 1 is 2. So, we can always say the 2 is less than or equals to x n always. And also x n is less than or equal to each of this term is less than or equal to 1, 1 minus etcetera. Whatever is occurring here is less than or equal to 1. So, x n is less than or equal to we can say that 1 plus 1 plus 1 by factorial 2 plus 1 by factorial 3 etcetera plus 1 by factorial n. And you can say one more thing here that is for example, if you look at this 1 by factorial 3, see 1 by factorial 3 it is nothing but 1 by 1 into 2 into 3, so this is less than or equal to 1 by 2 square because this 3, I am replacing by 2.

(Refer Slide Time: 36:17)



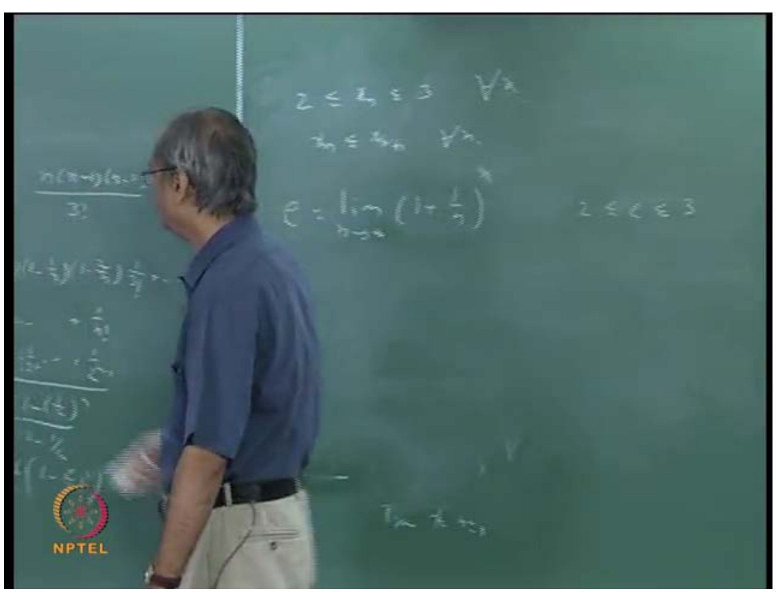
Similarly, 1 by factorial 4 this 3 factorial all of them I can replace by 2 and I can say that is less than or equal to 1 by 2 q etcetera.

(Refer Slide Time: 36;34)



So, in general I can say that, this is less than or equal to 1 plus 1 plus 1 by 2 plus 1 by 2 square etcetera plus 1 by 2 to the power n minus 1. And then you know that further there any one this is a geometric progression. So, some of this part will be less than or equal to 2 because some of this part will nothing but 1 minus half. Let us write it this is same as 1 plus 1 minus half to the power n divide by 1 minus half right. And which is same as saying this 1 plus 2 into 1 minus half to the power n. And obviously this 1 minus is anything less than or equal to 1.

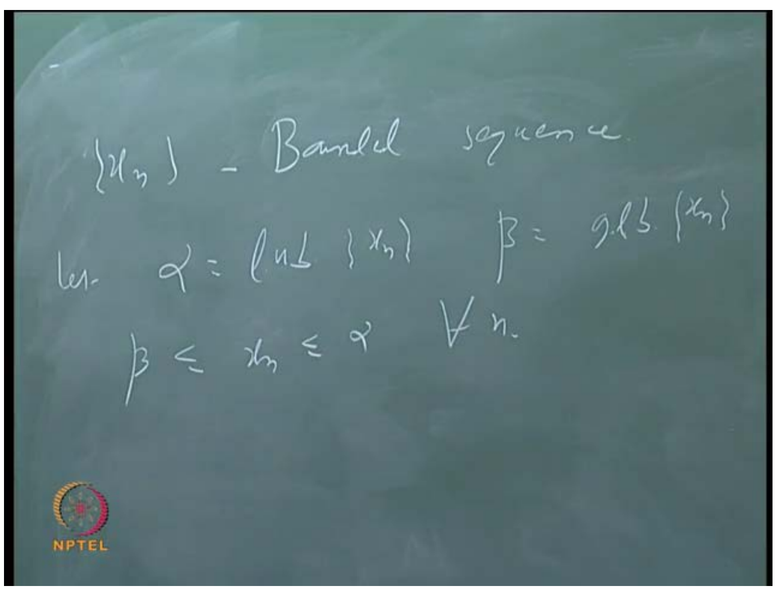
(Refer Slide Time: 37:40)



So, this is less than or 1 plus 2, so this is less than or equal to 3 anyway. So, what will be prove that for each x n 2 less than or equal to x n less than or equal to 3 for all n. And x n is less than or equal to x n plus 1 for all n. So, we have proved its sequences practically increasing and bounded above. Of course we also put it is bounded below, but that is not relevant. So, this sequence x n converges this sigma x n converges and you all know that its limit is called the number e, its limit is denoted by number e. So, e is limit n times to infinity, in fact there are various ways of differing this number e, this is one of those type 1 plus 1 by n raise to n. And because of this we can say that this number e lies between 2 and 3, so 2 less than or equal to e less than or equal to 3.

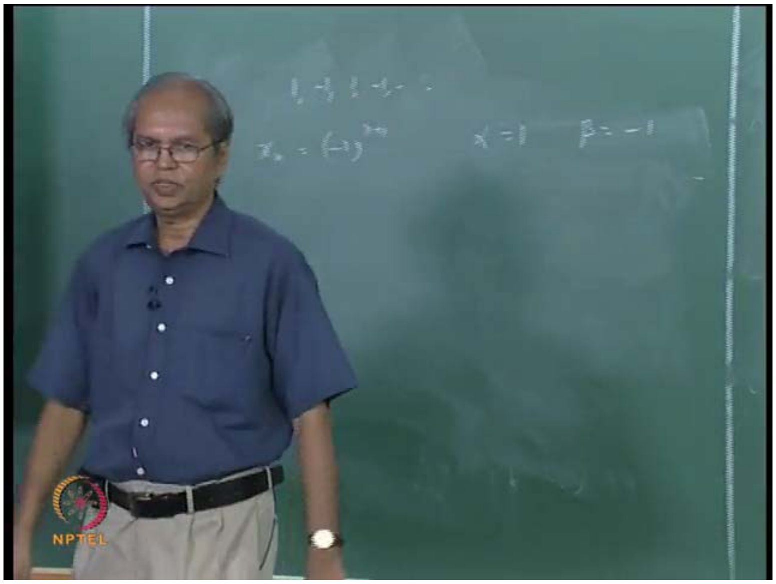
And if you want to get an approximate value of e, you take some large value of n and conclude the corresponding fraction here. And you will get the approximate value of e. Now, let us proceed to the concepts. Is it clear so far whatever we all have done? We are showed that every sequence, which is practically increasing and bounded above, converges similarly, for decreasing sequences. And we are seen how to make use of this theorems to show the converge of some of the well known sequences.

(Refer Slide Time: 39:31)



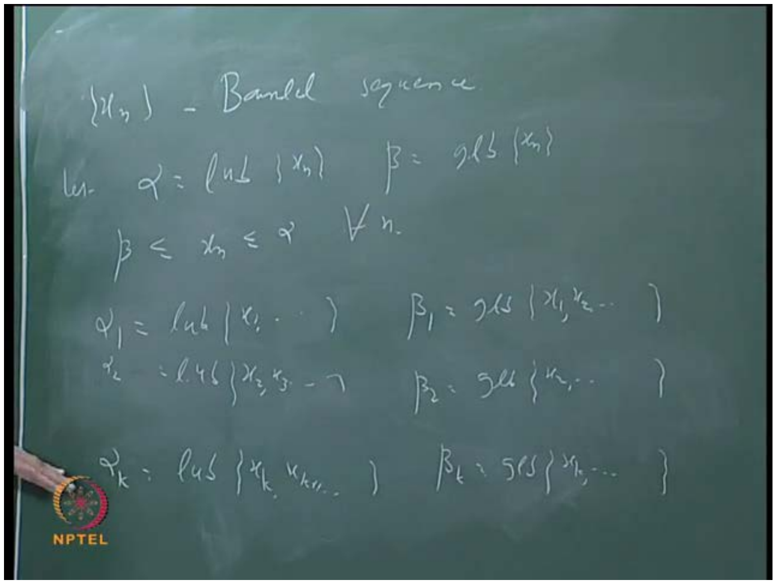
Now, let us just take a sequence, which is just bounded. Let us say that x n is just a bounded sequence, what does it mean? It means the sequence, which is bounded above and bounded below. Since the sequence is bounded above, it will have a least upper bound. Since it is bounded below it will have a greatest upper bound. Let us do some notations for this, let us say that let alpha is l u b of x n and let us say beta be g l b of x n. So first of all, it will mean that beta is less than or equal to x n less than or equal to alpha for all n. Just to keep track of what is happening, let us keep some example in mind.

(Refer Slide Time: 40:55)



Consider this sequence 1 minus 1 1 minus 1 etcetera. How did you describe this sequence? I can write x n by x series is equal to minus 1 the power of n plus 1. If I take minus 1 to the power first will be minus 1, so minus 1 depart n plus 1, so what are the alphas and betas here? So, in this case alpha will be 1, beta is minus 1. Of course this is a trivial example, but it just helps to do what is going on here. Now, next what are all to do is the following.

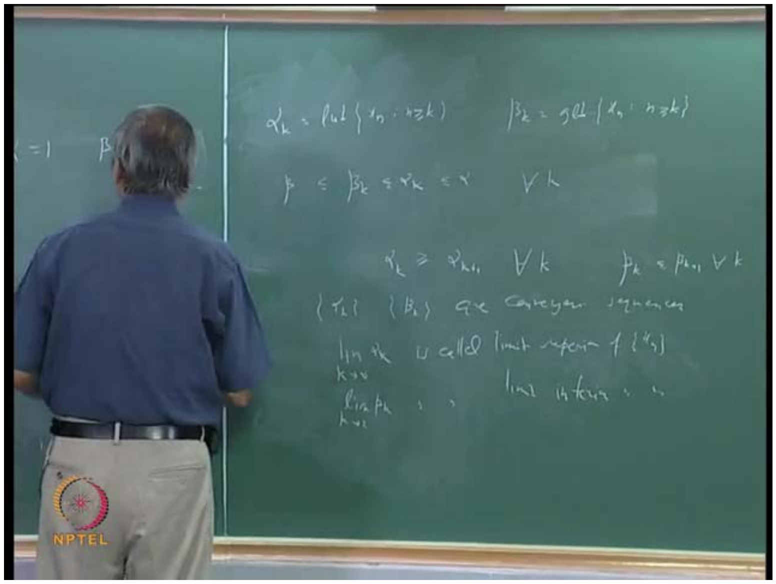
(Refer Slide Time: 41:52)



For example, now if I take any let us suppose I take l define alpha 1 also l u b of this whole sequence x 1, x 2 etcetera and similarly beta 1 as g l b of x 1, x 2 etcetera. Of course then you will see, this is the thing new because then alpha 1 is same as alpha and beta 1 is same as beta. But still I am using this notation because, there are all define a few things subsequently for which this is broken assisted this. What will I do is that, I will drop this first term x 1 from this sequence and then take the l u b or remaining for this sequence and call that alpha 2.

Instead of taking x 1, x 2 let me take this sequence x 2 x 3 etcetera and take the l u b of that and call that number alpha 2. And similarly beta 2 is g l b of x 2 etcetera and then proceeds in this panel. So, next I will be take alpha 3 as the l u b of the sequence x 3 x 4 etcetera. In general I will say that you take alpha k as l u b of x k plus 1 etcetera and similarly beta g l b of x k etcetera. I think once you understood what these etcetera are; let us now write in pen size notation.

(Refer Slide Time: 43:48)



I will say that what is alpha k? Alpha k is least uppers above of x n for any bigger than or equal to k. And similarly beta k is g l b of x n for any bigger than or equal to k that is, you are just taking the part the sequence x k, x k plus 1 etcetera. And obviously clear that all of these are also bounded above and bounded below because it is a part of the original sequences, they distribute. And so, each of this alpha k plus b less than or equal to alpha, each of this beta k also plus b less than or equal to beta.

And what is the relationship between beta k and alpha k? Beta k is less than or equal to alpha of k for all k and each of this alpha k is less than or equal to alpha. And each of this beta k is bigger than beta. Now, let us come to very important question. What is the relationship between alpha k and alpha k plus 1?

Student: Alpha k bigger than or equal to alpha k plus 1.

Alpha k bigger than or equal to alpha k plus 1, why? You are taking supreme no more over smaller set. You are taking the least upper abound of a smaller set so, that should be less than or it may equal to, but it should be in general. So, alpha k plus 1 is bigger than or equal to, so remember this alpha k itself is also see alpha k and beta k are all is the sequences. So, this is true for all k, so what follows alpha k proton ally decreasing. Now, once the sequence is going to be decreasing to know whether it is a composite or not what we should know, where this bounded below?

If that is the case, because each alpha k is bigger than or equal beta k and beta k is less than or equal to beta. So, this beta is lower bound for all of this alpha k similarly, its less than or bound for all of this beta k also, but that is ok. So, alpha k and beta k those the sequence, which there also bounded above and as well as below, but what is a additional thing here is that alpha k is proton ally decreasing.

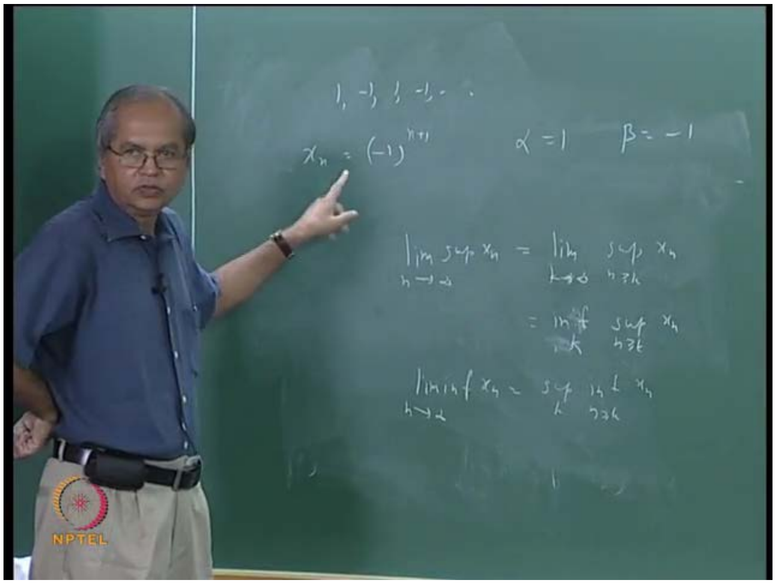
Similarly, you can say the beta k is proton ally increasing, so that is beta k is less than or equal to beta k plus 1 for all k. Because there are smaller set, for set lower bound should increase, so a given sequence x n may or may not be bound may not be prospering monotonic. It may or may not be increasing or decreasing, but this new sequences alpha k and beta k those are monotonic alpha k is decreasing beta k is increasing. So, this sequence will always have limit and what will be the limit?

Its limit will be nothing but for example, limit of the alpha k will be nothing but of course that limit may or may not be same as alpha or that limit may or may not be same as beta, that limit may be different. So, that limit as a special name, it is called for example this limit of this alpha k, this limit will always exist. Let us let us just told this k this alpha k and beta k are convergence sequences always.

Of course, we cannot say that alpha k converges to alpha because alpha k is boarding decreasing sequence. And so we monotonic decreasing sequence will converges to its greatest to lower above and that may not be alpha in fact it will not be injury, it will not be alpha. So, that may be some different number, so that number is called limit superior of this sequence x n. So, limit of alpha k as k tends to infinity, that is called limit superior of x n and similarly limit of beta k x tends to infinity this is called limit inferior of x.

So, we can notice one more thing here, this is monotonically decreasing sequence, so it must converge to its inferno. So, it must converge to its inferno, so what we can say that this limit superior what we have called here, let me write it here.

(Refer Slide Time: 49:42)



So, limit superior as this is a limit superior denoted like this. Limit superior as of x n as n tends to infinity, but this is nothing but according to us limit as k tends to infinity of alpha. What is alpha k? Alpha k l is nothing but least upper a bound of x n and n. So, we can say that alpha k is nothing but least upper bound same as superior this upper. So, this is supremum n bigger than or equal to k x n. And since this alpha k is a decreasing sequence its limit is nothing but inferno of that sequence.

So, I can say that this is nothing but infimum k bigger than or equal to n. Let us simply say infimum over k, supreme over n bigger than or equal to k x n. In this same way we can say that limit inferior of x n, let me just write in then will stop with that. Similarly, limit inferior of x n as n tends to infinity. There this inferior and inferior supremum will be inter-changed because beta k is not supremum, but they are infimum. So, this is supremum over k infimum over n bigger than k x n.

Since, beta k is increasing sequence, it will converge to its supremum and beta k itself is infimum for each k, so that will be the limit inferior of x n. So, this given sequence x n may or may not be convergent when the sequence is bounded. It may or may not be convergent, but it will always have limit inferior and limit superior. And what are the corresponding qualities for this sequence? For this sequence its 1 and minus 1. We will stop with that for today. We will see how to use this concept of limit inferior and limit superior subsequent.