Real Analysis
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Lecture - 8
Sequences of Real Numbers

So, yesterday we proved that if you are given a positive real number and any natural number n then there exist a unique number x unique positive number x such that x to the power n is equal to a right and. So, we can now denote this number x which satisfies this x to the power n is a.

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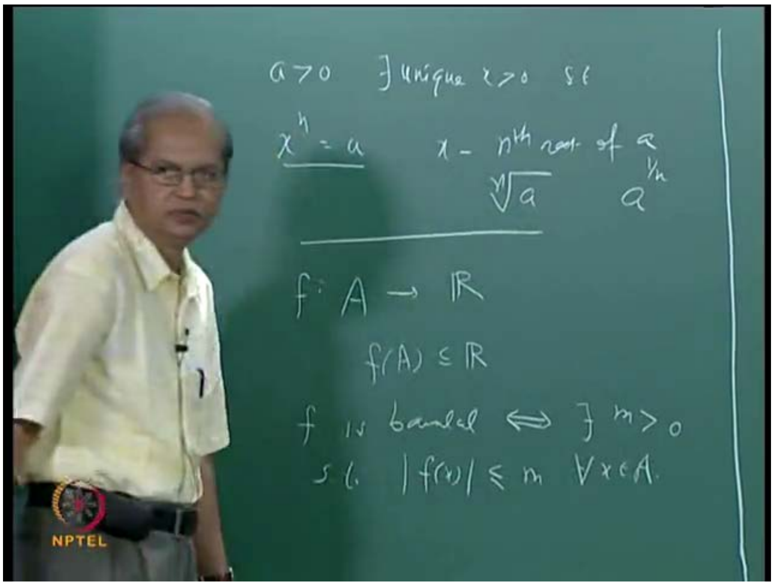

That is a is bigger than 0 and we have showed that there exist unique x bigger than 0 such that this happens. So, this x we can now call nth route of a nth route of a and we can have the usual notation and either this nth route of a or a power 1 by n. So, that is about the nth routes, now we have till now discussed what is meant by saying that say it is bounded above or bounded below or bounded, And what is meant by upper bound and lest upper bound etcetera, we have not discussed similar concepts about functions. But, we can now do that and the whole idea is that there is nothing really new involved here.

The reason is the followings we will talk about the real valued functions what is meant by the real valued functions that is it is range is in the real number that is or it is co domain is r domain can be anything. Suppose A is any set and f is a function from A to R when do we say that f is bounded that is this you look at the set f of a that is a subset of R that is a sub set of R. And we all ready defined what is meant by saying let us say it is bounded above bounded below and all those things. So, all those things we simply apply to those functions for example we say that f is bounded above if the set f of A is bounded above.

Then what ever is say the upper bound of f of A we will also call it upper bound of f. Similarly, will say f is bounded below f is bounded will mean it is bounded above as well as below right. And in particular for the sequence if a sequence is also function it goes from it goes from n to R. So, in a similar way we will say what is meant by sequence is bounded above bounded below bounded etcetera supremum of a sequence infrimum of a sequence supremum of the function etcetera. Since nothing new is involved here we shall not spend too much time in discussing those things all right what we now want to discuss next is.

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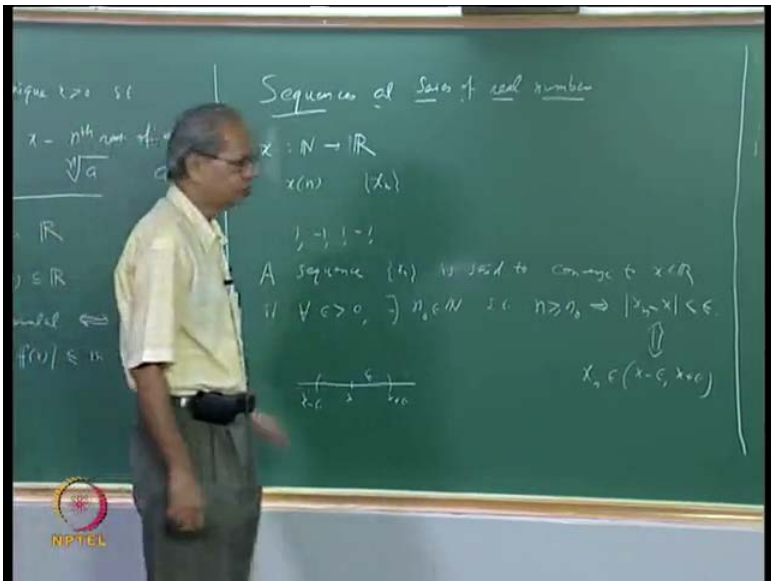

What is a broad topic sequences and series of real number, now before going to that let us also make one more observation here. And that is the following we will just try to do it on this one show that f is bounded if and only if and only if we can say that there exist some number. Let us say M bigger than 0 there exist a number bigger than 0 such that mod of f x less not equal to m for every x. In A this is something very well known fair fairly easy to prove and something that we shall be using very often one way is easy see mod f x less not equal to m this basically same as saying that minus m plus not equal to f x less not equal to m for every x.

So, that means m is an upper bound minus m is is a lower bound. So, that immediately show that f is bounded. Similarly if you assume that f is bounded that means it is bounded above as well as below then you can find such a number m that is that is well easy and the same thing about a sequence. If a sequence is bounded we can similarly found a find a number m such that mod of x n is less not equal to m for each n also. Since this topic of sequence essentiaries etcetera is something that you have already learnt in your under graduate courses. We shall not spend too much time in discussing many these things in detail.

We shall just quickly go through these things basically emphazing the role of LUB epsilons in various theorems. That you this something that you may not have realized while learning those theorems in the under graduate course and that is what we planned to do. So, first of all as you let me all ready seen that x n is the sequence is nothing but a function from f function from n to r usually for a sequence instead of taking the notation f. We usually take the notation x we usually take the notation x and normally we should have denoted image of this n under x as by this x n right. This is also quite convenient notation it is used in many books, but more customary is this x suffix n x suffix n and that is a sequence sometimes. This x suffix n is also used to denote the range of the sequence.

But, you should in general you should not confuse between the sequence and the range these two things are different because the range may be even a finite set range may even be a finite set. For example you can take the sequence 1 minus 1 1 minus 1 etcetera x 1 is 1 x 2 is minus 1 etcetera . But, if you take the range it is just two numbers 1 and minus 1 all right. Let us now go to the most important definition of what is meant by saying that a sequence x n converges to a number x. So, a sequence x n converges is said to converge that while we are defining know is said to converge to some x in r. If for every epsilon bigger than 0 this is a well known definition, let us just recall it.

If for every epsilon bigger than 0 there exist a natural number which we call n 0 n 0 in n. In general this n 0 will depend on this epsilon. So, if you want you can denote it as n 0 of epsilon. But, where it is understood we did not bother about it right n 0 in n such that what should happen whenever n is bigger naught equal to n 0 corresponding difference between x and x n should be less than this epsilon right. That is we can say in this say we that is n bigger nor equal to n 0 implies mod x n minus x less than epsilon. Now, this is a definition with which you are all ready familiar we will just see some implications or what exactly this means.

Now, you all ready know that saying that mod x n minus x is less than epsilon that is equivalent to saying that x n belongs to for example. This is basically equivalent to saying that x n belongs to this interval x minus epsilon to x plus epsilon. That is clear to everybody mod x n minus x is less than epsilon that is same as saying that minus epsilon is less than x n minus x same that is less than epsilon. And that will give you this right or geometrically what does it mean that if this is the number x if this is the number x. And if you take epsilon this is an interval x, so this is the length epsilon.

So, this will be the number x minus epsilon x plus epsilon then what ever epsilon you take and corollary interval x minus epsilon to x plus epsilon except. The first few elements of the sequence that is x 1 etcetera up to let us say x n 0 minus 1. All the other elements of the sequence must lie inside this interval that is all except a finite number of elements must lie inside it is interval x minus epsilon to x plus epsilon. And whatever interval you take epsilon is just to say that you can take the interval of any size. Suppose you take the smaller inter smaller interval that is smaller epsilon you may have to change n 0 depending on the epsilon you may have to change bigger take might be bigger n 0. But, whatever it is that number is always finite let me also mention a particular terminology which is used in this context.

We say that something happens eventually when ever that we dealing with the sequence we will say something happens eventually means this there exist some n 0. Such that for example we say that a property holds for a sequence eventually that is if there exist some n 0 in n such that for all n bigger naught equal to n 0 whatever we have said happens for example. Here I can say that all elements of the sequence are inside the interval x minus epsilon to x plus epsilon eventually given any epsilon the sequence lies comp completely inside the interval x minus epsilon to x plus epsilon. Eventually means except a first few terms all the other terms lie inside that interval is that clear there are there a few things which follow immediately from whatever we have said here.

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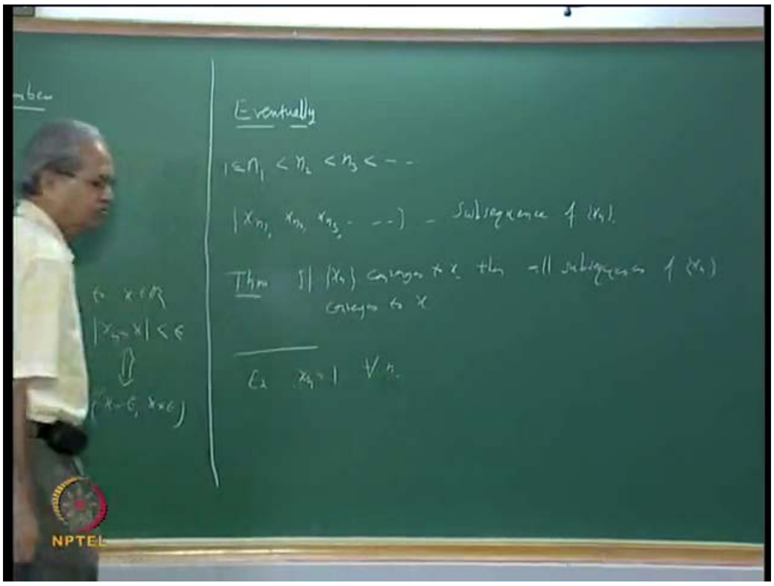

And let us just quickly see why that happens first of all you all know what is meant by a sub sequence. If you do not know let us recall it what is what is a sub sequence first of all we take what is called strictly increasing sequence of natural numbers that is. Take A for example take a sequence of natural numbers like this n 1 n 1 may be equal to or bigger nor equal to one. But, n 1 should be strictly less than n 2 n 2 should be strictly less than n 3 etcetera. As you know such a sequence is called strictly monotonically increasing sequence right. Now, given any sequence x n you consider the sequence x n 1 x n 2 x n 3 etcetera that is you are collecting you are taking some terms from the given sequence, but not arbitrary terms you are first you first fix a sequence.

Strictly clear sequence of natural numbers and take the corresponding sequence of corresponding terms of the given sequence. Then this is called sub sequence of x sub sequence of x for example. If I if I write some sequence like this x 2 x 1 etcetera this is not a sub sequence is that clear why this is not a sub sequence because we have taken n 1 as 2 and n 2 as 1 right. So, you cannot change the order in which the elements occur in the sequence we have to retain the same order and then pick up some elements from there that is important. Now, we have two sub sequences coming to picture here if a sequence converges to a number x all its sub sequences should also converge to the same element x.

So, that is the first theorem we will say if x n converges to x then all sub sequences of x n x n converge to x well how does this follow we can say that we all ready do that x n converges to x. That means we know that given any epsilon there exist n 0 in n such that either you can say. It this way or this way mod x n minus x is less than epsilon for all n bigger naught equal to n 0. Now, these suppose you take this whatever n j let us say if n j is bigger than n 0 bigger or whatever than 0. Then for all elements of this sequence which are for all indices which are bigger naught equal to that n j those x n j x n j also will lie in this mod n in the same interval after all x n js are coming from this x ns x n j are coming.

So, if something happens for all x n for n bigger naught equal to n 0 it should also happen for all x n j for n j bigger naught equal to n 0 right. So, there is nothing really great that is happening here. But, what can happen is that the original sequence may not be convergent, but it may have many convergent sub sequences.

For example here we will see that let us let us say this we have come to this point let us again let us go back to this in general how does one show that a sequence converges of course still now we have only seen this definition. So, as of now we can only use the definition and to show that a sequence converges subsequently we shall see many other methods we will we will use some of the well known theorems to show that some sequence converges.

But, using this definition what we can show for example suppose you take a constant sequence let us say let us take some examples here by the way the proof of this is clear. So, I need not I need not discuss it here I have I have mentioned you can write a proof on your own. So, take an example let us say constant sequence x n equal to one for all n when you have to show that a limit of by the way I did not mention that.

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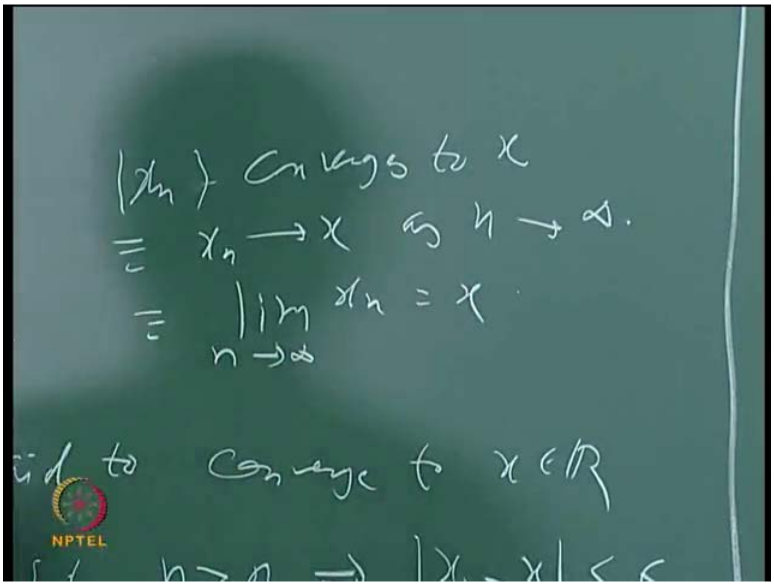

This x n converges to x this is invert x n converges to x there are a few other notations used for this one is this we also say that x n. Then right arrow x as again n right arrow infinity we say that x n right arrow is read as the tends to x n tends to x as n tends to infinity. That is a symbol used and there is another notation for this limit of x n as n tends to infinity is equal to x. You are all familiar with this all these three things mean the same thing it just different way of saying the same thing certain things are conveniently used at different times. For example in olden days when you needed to type things this was more convenient way to write because those arrows and all were not available.

But, now those kind of things are those kinds of problems are not there any way. So, suppose this is a sequence you for this sequence you can set first I want to show that a sequence converges. You have to have some idea of what this number x is because a sequence a given sequence x n has to converge to some real number which we call limit of that sequence x n. So, you have to first of all have some idea of what that limit is then only you can think of showing or showing that a sequence converges to that that limit. Now, in this particular case what it is clear that the limit has to be one each element x n is one and. So, suppose I want to apply this definition then you can say that given any epsilon you have to take x equal to 1 and mod x n minus x is 0 only for all n.

So, this is trivial there is nothing, so every constant this is a constant sequence even if you take some other constant here that is that is not going to change. So, every constant sequence converges right. Let us take one more example which is again very popular and with which we shall be using very often having that sequence and certain other sequences depending on that for example you take.

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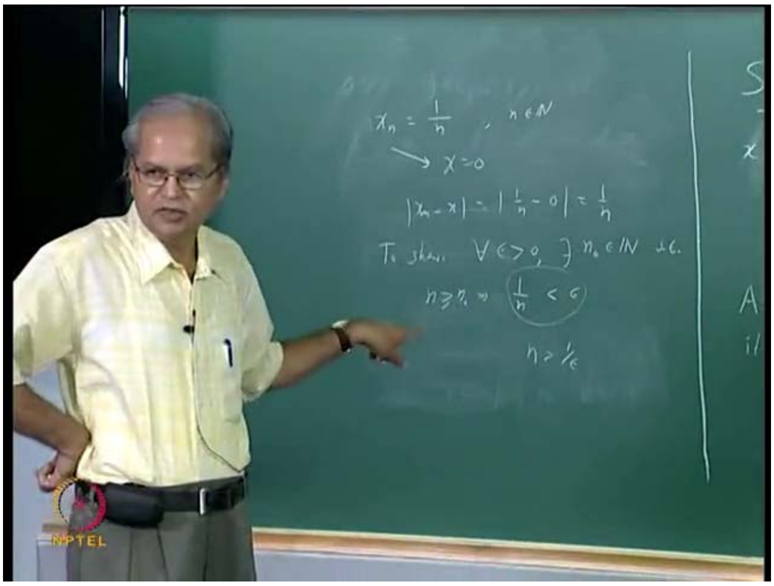

Now, x n is equal to 1 by n, now to show that this sequence first of all I believe that all of you know that this sequence converges and it converges to 0. So, the whole question is how does one show this and what is actually involved in showing that ok. So, in this case we have to take x is equal to 0, so this x n converges to this x 0. Now, to do that what we will have to show we will have to look at mod x n minus x. So, mod x n minus x that is the thing, but mod 1 minus n minus 0 and that is the thing. But, 1 by n right whatever be n, now what is the idea given epsilon given epsilon we should find an n 0. Such that for n bigger naught equal to n 0, let us say what is what we had that is to show this for every epsilon bigger than 0 there exist n 0.

In n such that for all n bigger than that is n bigger naught equal to n 0 implies 1 by n less than epsilon right 1 by n less than epsilon. Now, you can see that saying that is 1 by n less than epsilon that means n greater than 1 by epsilon right. Now, if you find n 0 which is bigger than 1 by epsilon if you find 1 n 0 which is bigger than 1 by epsilon. Then whenever n is bigger naught equal to n 0 that n is going to be bigger than 1 by epsilon.

So, what is the issue now just a find n 0 which is bigger than 1 by epsilon after show that there exist n whatever epsilon is given to show that there exist n 0 such which is bigger than 1 by epsilon.

Now, does such an n 0 exist fine how does that follow right that is what we are we have shown that n is not bounded above. So, whatever epsilon is given 1 by epsilon is a positive real number we can always find some natural number bigger than that because n is not bounded above. So, why I have waving through so many details of this particular example it is that to show that 1 by n converges to 0. All that that is something which you have realized which you have taught to be very obvious all these days the idea is that the proof that 1 by n tends to zero involves the idea that n is not bounded above. And that involves the LUB epsilon if you remember the in order to show that n is not bounded above we have used LUB epsilon. So, to show that 1 by n tends to 0 basically involves LUB epsilon on the real number system all right.

So, that is about showing that a sequence is convergent suppose sequence, suppose you want to show that a sequence is not convergent. For example you know that this sequence is not convergent, now that is usually more difficult that is usually more difficult you will do. Let us let us first see how one can attempt to show that by using this definition right we shall see easier methods little later suppose you want to just use this definition. Then by this definition saying that a sequence does not converge means what it means it means no such real number exist right. No such real number exist it means whatever x you take whatever x you take for that x such a thing is not going to happen what does it mean.

See here you say that for every epsilon something happens ok, if for every epsilon something happens if that is false means there exist some epsilon for which it does not happen what does not happen that is that is that there exist n 0. Such that for all n bigger than equal to n 0 mod x n minus x is strictly less than epsilon. If this is false that means what for whatever n 0 you take whatever n 0 you take you can always find some n bigger naught equal to that n 0, such that mod x n minus x is greater than epsilon right. Now, this is something ah very useful and this something that you should you should may be record somewhere to show that. A sequence is not convergent using only the definition to show that a sequence is not convergent what.

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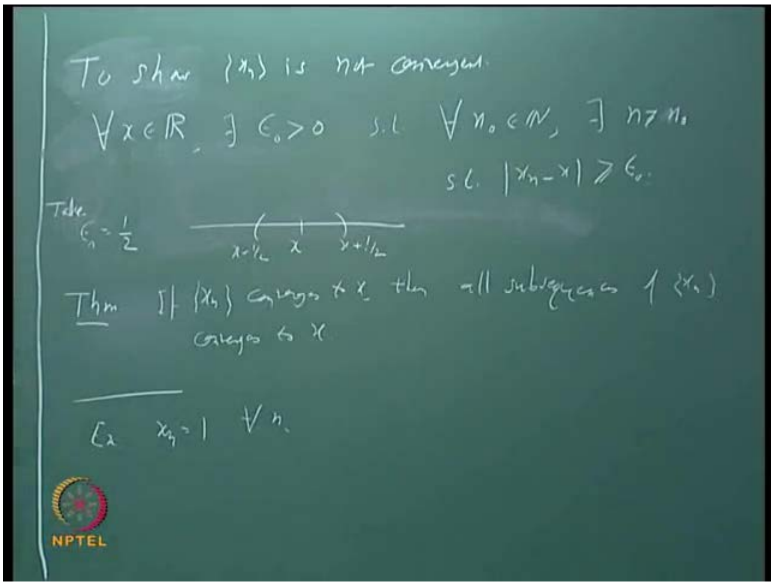

What is needed to be shown to show a sequence is not convergent n is not convergent what we did to show is that for every x in r. See there are two things here suppose you wanted to show that a sequence x n does not converge to x. Then we need not take for every x in r all right, but now we will just want to say this sequence x n is not convergent we can. So, what we have to say for every x in r there exist some epsilon let me call it epsilon suffix 0 because it is we have used this notation for general epsilon that exist. In fact that epsilon may depend on x that epsilon may depend on x there in such that for all n 0 in n that is whatever the n 0 in n there exist n bigger nor equal to n 0. Such that mod x n minus x is bigger than epsilon bigger not bigger naught equal to epsilon is epsilon 0 ok.

So, this is something you should always remember that if you want to show that a sequence is not convergent by using definition this is what you need to show this is what you need to show we will we shall see easier methods little later right. Now, coming back to this sequence suppose I want to show that this sequence is not convergent using that right, then I should show that whatever x is there will exist some epsilon can you tell me some obvious choice for this epsilon here. For example you can take 1 by 2, now why we say this the reason is the following saying that a sequence x n converges to x. If you look at this definition geometrically it means that all the terms of the sequence go close to x roughly speaking it means this that is given epsilon.

For every epsilon mod x n minus x there exist n 0 such that for all n bigger nor equal to n 0 mod x n minus x less than epsilon means after n 0 all terms of all terms of x n are close to x. Now, if all terms are close to x it means the question is can there exist some x such that 1 is also close to x and minus 1 is also close to x ok. Now, that obviously cannot happen because for example like somebody said suppose I take epsilon be equal to half. Suppose I take let us say take epsilon 0 equal to half ok then what ever x you take what must happen is that suppose x. Suppose it is here you take the interval if epsilon 0 equal to half this is x minus half that is x plus half.

It is x minus half and that is x plus half then what we if this sequence x n converges to this x in a sequence say. Then all terms of that sequence must lie in the interval x minus half to x plus half right. But, whatever n 0 you take whatever n 0 you take you can always find some n such that x n is 1. Similarly, you can always find some n such that x n is minus 1 if x n is 1 x n plus 1 is will be minus 1 and both of them have to lie in this interval. So, minus 1 and one both of them must belong this interval, now can that happen because the length of this interval is 1 and if minus 1 and both of them have to belong they obviously that length will bigger.

So, this sequence does not converge, but of course you will see that this is it will be very difficult to prove this kind of thing for every non convergent sequence. So, proving that a sequence is convergent or not convergent using definition alone can be quite difficult task. And that is why we shall see some easier methods to show both of these things when as convergence and similarly, when as sequence does not converge ok. Now, let me come to the first method that is where this theorem is very important here what we have proved that if a sequence x n by the way before that we have mentioned here.

So, this is what x n and does not converge suppose I simply want to show that a sequence x n does not converge to a number x then as I said earlier. Then this for every x that will go we have simply show that there exists epsilon 0 bigger than 0 such that whatever I have written after that it happens that is for that particular epsilon 0 for every n 0. You can find some n bigger nor equal to n 0 such that mod x n minus x is bigger nor equal to epsilon 0 right ok. Now, look at this theorem here what we have showed that if x n converges to x then all subsequences of x n should also converge at same number x all right now before going to that comment which.

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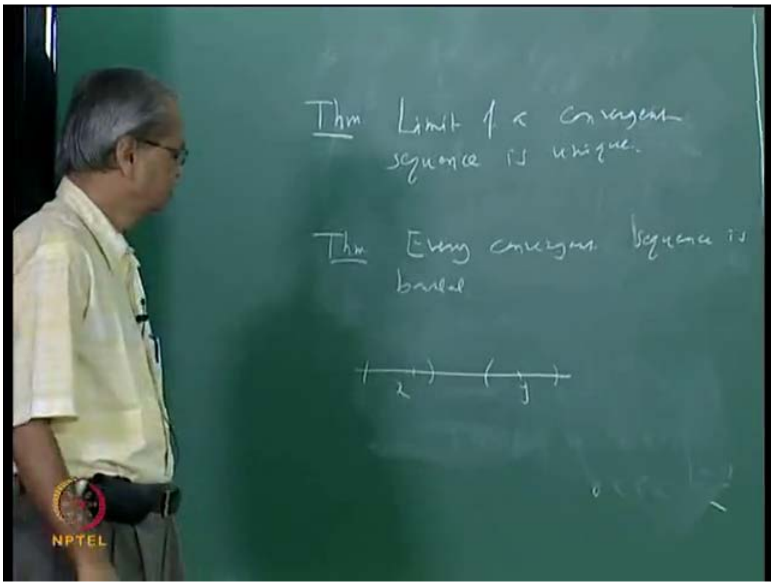

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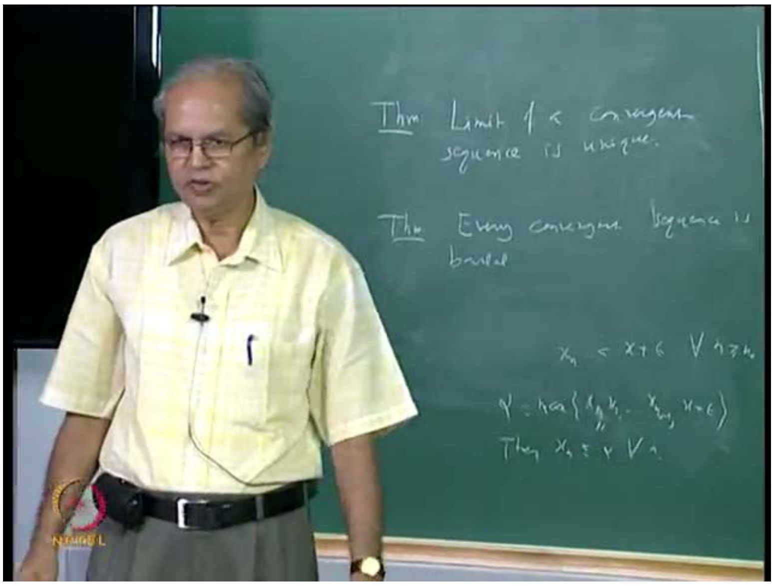

I have in mind let me also mention 1 more thing again a well known theorem that if a sequence converges it has to converge its limit is unique a sequence cannot converge to two different limits. So, let me simply say this limit of convergence sequence is unique all right, now how does one show that something is unique you assume that a sequence converges to two different limits and greater contradiction right. So, suppose x n converges to two different limit suppose limit n tends to infinity. Let us let me use that suppose x n converges to x and x n converges to y now if x is different from y right x is.

So, we can assume that x is we can assume that x is less than y we can assume that x is less than y I think I will just explain the proof geometrically you can write the proof suppose this is the case x is less than y all right. Now, can we find an epsilon I think it is better to look at this way of defining can we find an epsilon.

Such that the interval x minus epsilon to x plus epsilon and y minus epsilon to y plus epsilon are disjoint that is possible what is the obvious way of choosing that any you take any number which is such that 2 epsilon is the this is the length epsilon. The total length will be 2 epsilon you choose any epsilon such that 2 epsilon less than y minus x. In general we can choose any epsilon satisfying this 0 less than epsilon less than mod x minus y by 2 that will take care of x less than y or y less than x right. Now, since x n converges to x I will just tell this you can find some n you can find. Let us say some n 1 let me take up such that for n bigger than or equal to n 1 all of these x ns lie in this interval.

Since x n also converges to y you can find some let us say n 2 such that whenever n is bigger nor equal to n 2 all those all those x n lie in this interval right. Now, but you can always find n which is bigger nor equal to both n 1 and n 2 right. You can always find n which is bigger nor equal to both n 1 as well as n 2 right. Then for that n x n should lie here also and here also right that is not possible right. So, that shows that a limit if a sequence may not converge, but it certainly cannot converge to more than 1 number whenever it converges its limit is unique all right.

Now, let us come back to this theorem here we have proved that if x n converges to x all the subsequences of x n should also converse to the same limit x. Now, suppose you find a sequence and suppose that sequences let us say two subsequences ok. One subsequences converges to one number and second subsequences converges to some different number. Then what can you conclude from that obviously sequence because we have shown that if a sequence converges all its subsequences must converge to that same number same limit. So, obviously if that two subsequences converge to two different numbers then the sequence cannot be convergent right.

Now, does that give an easier method to show the sequence does not converge right for example look at these sequence right take x 1 x 3 x 5 etcetera right that is a constant sequence one. So, here you have a subsequence which converges to 1 then take x 2 x 4 x

6 etcetera that is also a subsequence that is a constant sequence minus 1 right. So, that constant sequence converges to minus 1. So, you have two subsequences converge at two different numbers.

So, from that we can conclude that the sequence does not converge right. So, that is an easier way of showing that a sequence does not converge right. Now, one more property of convergent sequence every convergent sequence is bounded every convergent sequence is bounded. Again that is state forward nothing must be related because how does to show that a sequence is bounded you need to show that it is bounded above. And bounded below or you can show there exist some m such that mod x n is less not equal to m for all we can we can choose any way look. Suppose we can take some epsilon bigger than 0 we can take any epsilon it does not matter.

You can take for example epsilon equal to 1 then we know that for that epsilon there exist some n 0 such that all x n bigger not equal to n 0 for all n bigger not equal to the 0 x n must lie in this x n must lie in. This means what it means x n is less than x plus epsilon for all n bigger not equal to n 0 right. Now, what are the elements that are left out for which this may not be true x 1 to x n 0 x foot they are finite numbers there are finite you take those how many are there are n 0 minus 1. You take those numbers plus these number x plus epsilon and take the maximum of all of them. Let us say suppose that maximum I call alpha is maximum what is that x let us say x n or let us let me say x 1 x 2 etcetera up to x n 0 minus 1.

And x plus epsilon then I can say that x n is less not equal to alpha for all n right. So, x is x n is bounded above this is an upper bound. Similarly we can say that it is bounded below you can take minimum of x minus epsilon and these numbers that will be a lower bound. So, every convergent sequence is bounded, so these tells you one more way of showing that the sequence is not convergent right for example suppose you take a sequence suppose the sequence x n.

Let us say x n is equal to n then obviously this sequence is not bounded and hence not convergent. So, whenever you learn a property of a convergent sequence if that property does not hold it will mean that the sequence is not convergent ok. Now, let us recall some of the well known theorems about the limits of sequences and as I said since those are the things with which you are already familiar we will not go through go through those proofs in very much detail.

First of all we want to discuss how the various algebraic properties of real numbers are related to this concept of limit of a sequence. In other words what we want to say is that suppose x n converges to x and y n converges to y then what can we say about a sequence x n plus y n x n into y n etcetera, so let us let me just write that ok.

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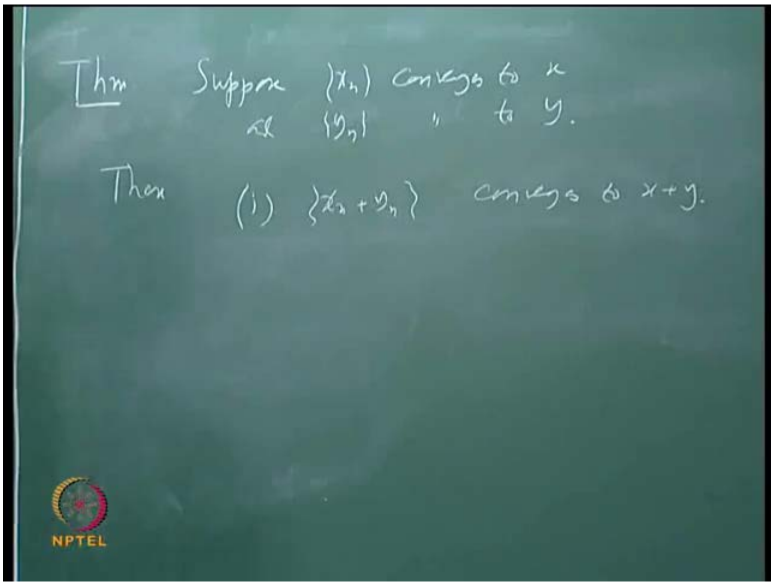

Suppose x n converges to x and y n converges to y then we want to say, so many things. So, I will just write here then first of all we will say that x n plus y n this sequence converges to x plus y and you would have seen the proof this several times I will just quickly recall this. And all other things I will state without proof to show that x n plus y n converges to x plus y we use this standard definition. So, suppose we take some epsilon bigger than 0 ok.

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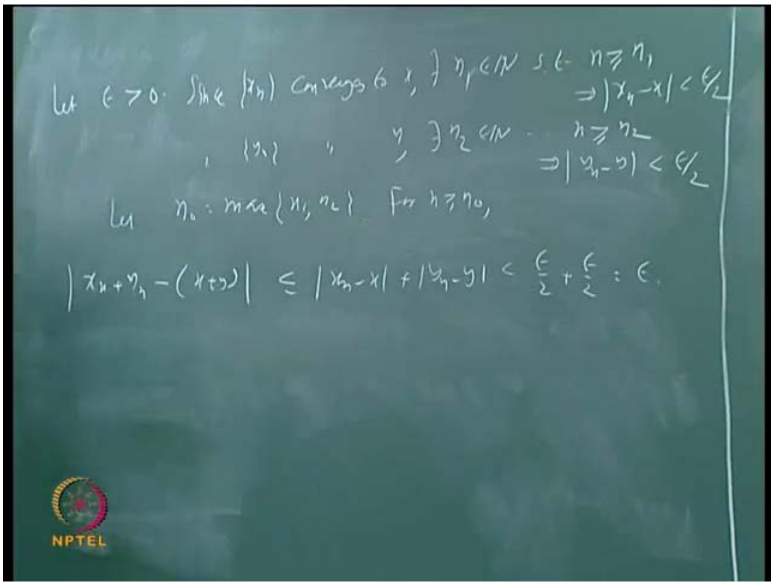

Then for these epsilon we have to find some n 0 such that whenever n is bigger not equals to n 0 mod x n plus y n minus x plus y that is that is less than epsilon. Now, all that one has to notice here is the following that is if you look at mod x n plus y n minus x plus y that is the nothing but x n minus x plus y n minus y. And because of the properties of the absolute values which we have seen, so this is less not or equal to mod x n minus x plus mod y n minus y right.

And then we know that x n converges to x and y n converges to y, so we can choose n 0 or n 1 or etcetera sufficiently large. So, that each of this can become less than epsilon by 2 right this is called as standard epsilon by 2 proof two epsilon 2 epsilon proof. So, whatever you call the idea is that to make each of these things sufficiently small to that some becomes less then epsilon. So, we can say that first of all we can say that since x n converges to x there exists n 0 in n all right. We call it n 1 n 1 in n such that n bigger not equal to n 1 implies mod x n minus x less than epsilon by 2 all right.

And similarly since y n converges to y I can say there exist n 2 in n such that n bigger not equal to n 2 implies mod y n minus y less than epsilon by 2. And we can say that take the maximum of n 1 and n 2 and call that number n 0. So, let n 0 in maximum of n 1 and n 2 then for n bigger nor equal to n 0 we can we can look at this x n plus y n minus x plus y that is less not equals to mod x n minus x plus mod y n minus y and each of this is less than epsilon by 2. So, this whole thing is less than epsilon by 2 plus epsilon by 2 that is epsilon all right remember the whole thing is that for large values of n x n is close to x y n is close to y.

And hence x n plus y n must be close to x plus y that is what we have been using and we were we have been proving right there is nothing very particular about this epsilon by 2. Here you could have taken this as a epsilon by 3 that as 2 epsilon by 3 epsilon by 4 or 3 epsilon whatever it does not matter epsilon by 2 is just convenient all right.

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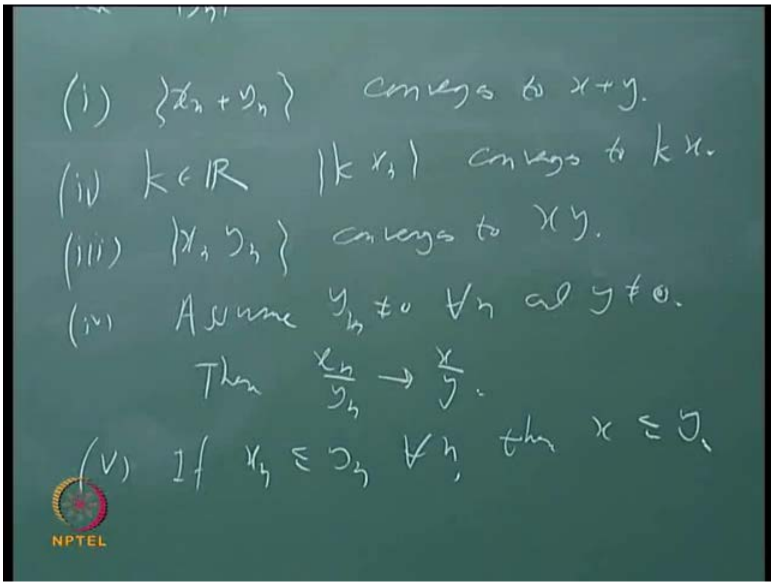

In a similar way you can prove that if you take any real number k then k times x n converges to converges to k x o only thing is in this proof you have to make two cases. If k is equal to 0 this becomes a constant sequence x n then there is nothing must to do there if k is not 0 you look at mod of k x n minus k x and, so if mod x. So, instead of epsilon by 2 here you take n 0 such that mod x n minus x is less than epsilon by mod k right and then k x n minus k x mod of that will be less than x n that is trivial. Similarly now for the product here do little more work x n into y n converges to x y again these also I will these proof I will not consider it discuss here you would have seen this. And you can try to reconstruct on your own for that you will have to use this property that every convergent sequence is bounded right.

Then little more involved is the proof when you would look a instead of x n y n if you look at x n divided by y first of all for that you have to assume that y n is non zero and also we expect to show that it is converges to x divided by y. So, y also should not be zero ok, so let us show that assume y n is different from zero for all n and y is different from zero. Remember simply saying that y n is different from 0 for all n does not ensure that y is not 0 it can happen that every term the sequence is non 0. But, the limit can be 0 we have seen that example 1 by n right. So, then x n by y n converges to x by y, so this is about the algebraic properties how the limit of a sequence behave as with respect to various algebraic properties of the real numbers system.

We also want to say something about the order properties, now what is meant by order properties that is suppose we know that x n is less not equal to y n for all n can we say something about the limit we should need to explain it x less not equal to y we. So, if x n is less than or equal to y n for all n then x is less than or equal to y. Again its fairly easy to prove this because it is less not equal to if false means what y is strictly less than x you can easily find some epsilon such that x minus epsilon is bigger than y. That is suppose this is false it means let us say this is x is strictly bigger than y bigger ok x is strictly bigger than y right.

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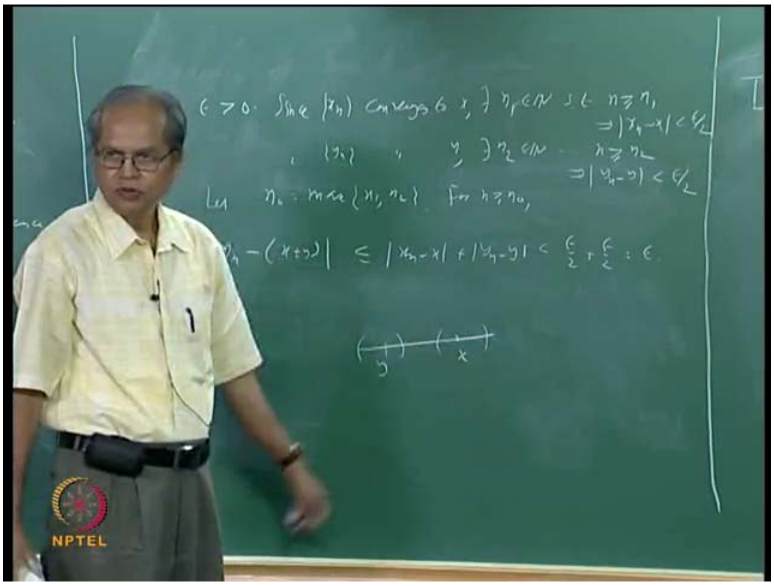

Then you can easily find some epsilon such that this interval y minus epsilon y plus epsilon is this and this interval x minus epsilon to x plus epsilon then find the corresponding n 0 n 1 or whatever etcetera. So, far large n is x n lie in x minus epsilon and x plus epsilon and y in lie in between y minus epsilon and y plus epsilon. So, you can find some value of n such that y n is in this interval and x n is in this interval. Now, that is not possible right because we assume that x n is less not equal to y n only thing. You should that you remember here is that even if the inequality is strict here even if x n is strictly less than y n here we cannot say x is less than y if x n is less not equal to y for all n.

We can we can conclude from that that x is less nor equal to y, but x n strictly less than y n or x n strictly less than y can you can you again give me obvious example of this why you can take the sequence 1 by n. And take the constant sequence 0 suppose you take x n constant sequence as 0 y n as 1 by n and x is strictly less than y n for all n. But, x is not less than y now there is one more very interesting that depends on order property, normally that is suppose you take three sequences x n y n and z n such that.

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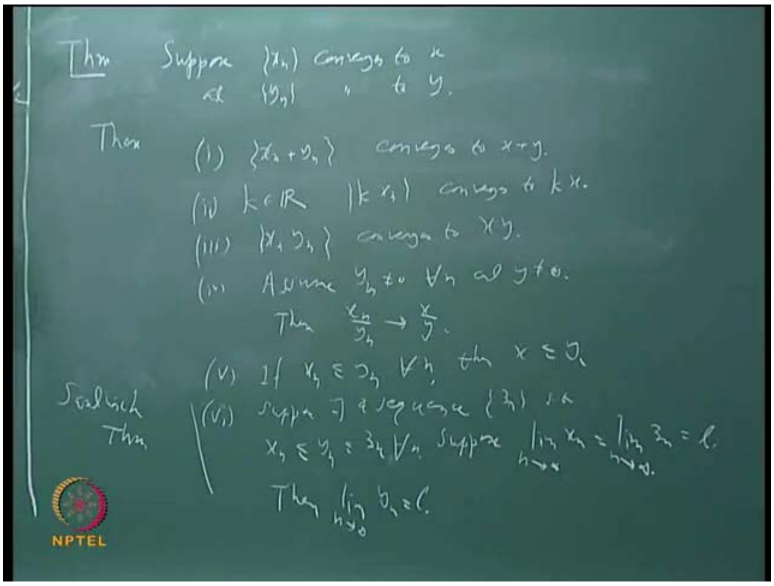

Suppose there is a sequence z n suppose there exist a sequence such that let us say this z n lies between x n and y n. Such that x n whichever x n is less than or equal to y less than or equal to z n x n is less than not equal to y n and z n, suppose x n converges z n also converges suppose both of them converges to the same way limit.

Suppose limit of x n as n tends to infinity its same as limit of z n as n tends to infinity remember this symbolist as limit exist and equal to some whatever number is there. So, say the power element as n that is x n converges to n and for each n this should be true for all for each n x n is less not equal to y n and less not to z n. Then what should happen and that is y n also converges and converges to same limit then limit y n tends to infinity is also y n. In fact this is theorem is very well known this is called sandwiched this last part that is called sandwiched theorem. I mean the idea is that sequence y n is sandwiched in between 2 converges.

Then that should also converge to the same limit and these are the theorems till now we have to show that a sequence is convergent and to get the limits of the sequences. These are theorems, which are used very often that is till now for example we have seen limits of only two sequences. One is a constant sequence and other is a sequence 1 by n then now using this theorems one can decide convergence or divergence of various sequences and also decide what their limits is for example something like this.

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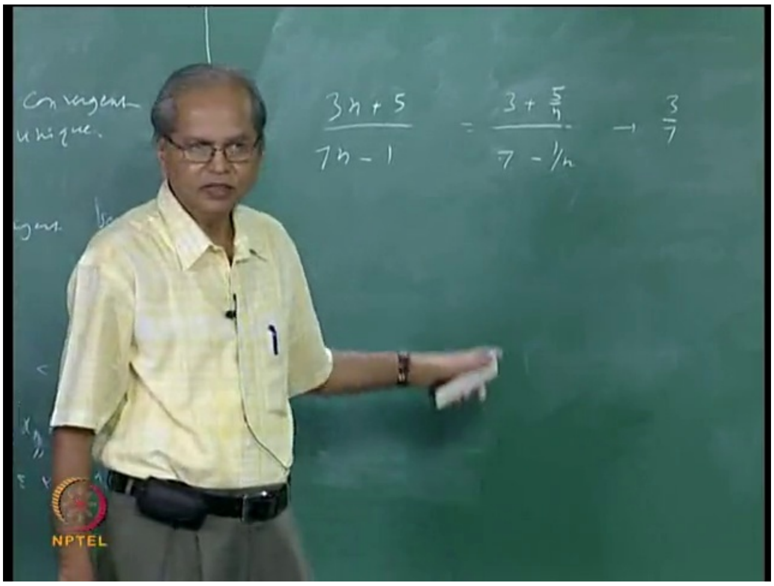

Let us say 3n plus 5 suppose x n is something like 3n plus 5 divided by 7n minus 1 right. Then you know that what is discussion write the same thing as n plus 5 plus 7 divided by n minus 1 by 7 sorry not this by divide everything by 1 this is 3 plus 5 by n and 7 minus 1 by n right. And then what are the theorems that are used to show that this sequence is convergent first of all we have shown that 1 by n is convergent sequence. That goes to 0, so 7 minus 1 by n that goes to 7 there we are using this fact x n then 5 by n that should also goes to 0 using this right.

Then 3 plus 5 by n that should go to three again using the first part x n plus y n and finally limit of this is 3 by 7 that follows by using this because using this because here y n is not 0 for any n and its limit is also not 0. So, x n by y n converges to x by y, so using all these things we are we can conclude this sequence converges to 3 by 7. This is something we are doing all along and perhaps not realizing that in concluding that this sequence converges to 3 by 7 you have basically used all these things right.

We can similarly see the illustrations of the in the fact the last thing sandwich theorem is also very useful in showing that all the sequences are convergent. For example one of the ways that showing that sequence is convergent is that you find two sequences one smaller nor equal to that and one bigger nor equal to that such that. And show that those two sequences converge at the same limit and conclude from the sequence y n also is the convergent and goes to the same limit. This is a the very standard technique in showing the sequence is convergent we shall see the examples, in next class.