Real Analysis

Prof. S.H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 7
LUB Axiom

We were discussing the structure of the real numbers system and in the last class we listed various properties of the real numbers. And let me recall that mention it one of the most important property of the real numbers is what?

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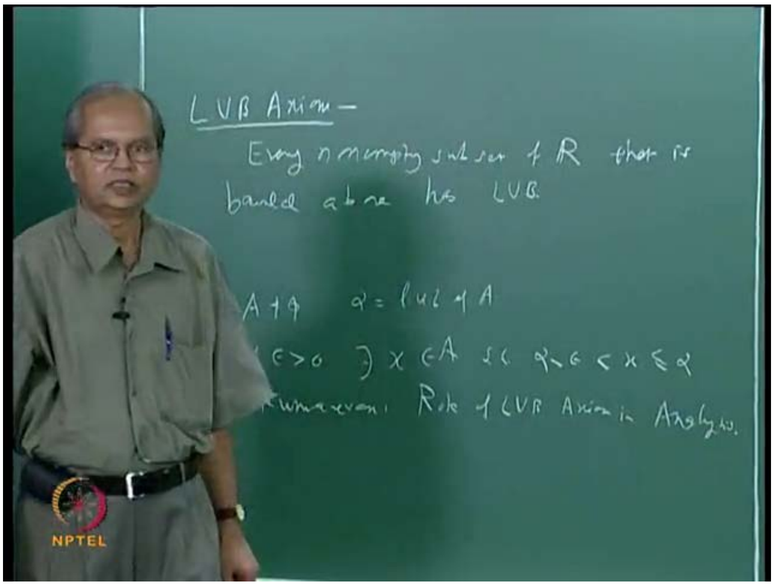

We had called the order completeness property or also known as LUB axiom. So, let me state that once again what it simply says every non empty subset of R that is bounded above this is bounded above has least upper bound. And we have already seen that once we say it has a least upper bound then in the least upper bound has to be unique. And by the similar argument we have we can also see that if this is accepted then we can in a similar way using. This we can also prove if a set is non empty and bounded below then it has greatest lower bound or what we call infimum also the way in which this LUB or supremium comes into picture in various proofs is as follows.

That is suppose you say that a is non empty and suppose alpha is the LUB of a then what we are seen in the other classes is that for every epsilon bigger than 0 alpha minus epsilon is not an upper bound. So, you can always find element in A which is bigger than alpha minus epsilon. So, for every epsilon there exist x in A this x may depend on epsilon this for different epsilon you may need to choose different x. So, if you want you can write x suffix epsilon which is was some books due, but not very important right now. So, their exist x in epsilon such that alpha minus epsilon is strictly less than x this means that alpha minus epsilon is not a lower bound.

And alpha itself is a sorry alpha minus epsilon is not an upper bound and alpha itself is an upper bound, so x less than or equal to alpha this will be always true. And this one particular epsilon is the one of the is the most important epsilon in the real number system which distinguishes real numbers from all other number system and we have also seen in the last class that.

Any with this axiom real number system because becomes what is called complete ordered field. And there is only one complete ordered field which has all these properties also the way in which the, this axiom comes into the picture in the various well known theorem analysis. Let me again remind you in the last class perhaps I have mentioned this article by professor S Kumaresom role of LUB axiom in real analysis axiom. In analysis we shall see some insanities of this in fact what you will notice at that is what we want to emphasize is that most of the important properties of the real line either followed directly from this axiom or in some chain of logic.

That is you prove some theorem using this axiom then something using this axiom then something else using that theorem etcetera and in last stage whatever we prove that particular property. So, every major property has a real numbers depends either directly or indirectly on this particular property ok.

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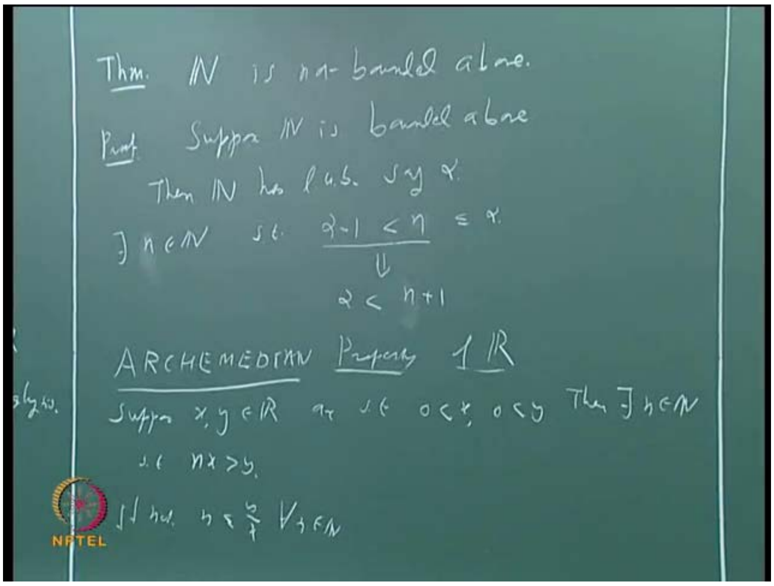

Let us see the few instances of that to begin with for example we can show this that this set of all natural numbers n is not bounded above not bounded above there is no real number which is bigger than or equal to all natural numbers. So, how does the we know that when suppose N is bounded above we shall see what happens we will get a contradiction. Suppose N is bounded above by the way instead of bounded above if we had thought below bounded below is that true it is bounded below that part is trivial. So, this is only important N is suppose N is above what should happen view of this it is already non empty we know several natural numbers. So, it should have a least upper bound it should have a least upper bound.

So, suppose N is bounded above then N has least upper bound say alpha suppose I call that least upper bound alpha then use this bigger for an epsilon. This should have some x ignites the alpha minus epsilon less I will take the epsilon as 1 I will take the epsilon as 1.

So, suppose I take the their exist x in N such that alpha minus 1 is strictly less than x and this is less than or equal to alpha right. Now, take this particle if alpha minus 1 less than x from this can we say that alpha less than x plus 1 alpha less than x plus 1 in fact this is a natural number let us use the different notation. So, that it will understand what is happening instead of x say small n small n belonging to x such that alpha minus less than

n. So, this proves the alpha less than n plus one, but what is n plus 1 what is n plus 1 is natural number because n is a natural number and alpha is less than n plus 1.

So, that contradicts that alpha is an upper bound right, so that is not possible. So, N is not bounded above right alright. Then similarly there is another very well known property of real numbers which depends on or which relates on natural numbers and real numbers it is called Archemedian property of Archemedian. This name may have come from this greek mathematician Archenemies, but upper handedly he was not a one who discovered this anyway we will not go into historical periods. If you those of you interested you can look at the book what is properties say is the following suppose you take two positive real numbers.

So, suppose x y in R are such that 0 less than x and 0 less than y you to take two positive alter numbers. Then you will always find a natural number such that n times x is bigger than y that is given real two positive real numbers you can take any of those positive numbers and multiply that by a suitable natural number. So, that multiple becomes bigger than the other real numbers that is called Archimedean property then.

Of course, that number n may depend on both x and y that number n, so then existed natural number n such that n times x is bigger than y can you see that this follows the immediately from this see suppose this is false suppose this is false then what does that mean nx that means nx is less than or equal to y x is bigger than 0. So, does it mean than n less than or equal to y by x right see. If this is false if not n less than or equal to y by x for every n in n that is clear if there is no such n for every n must less than or equal y by x.

But, can that happen because that will mean again that will mean that n is bounded above right that is not possible alright. Now, many of you would have seen in this proof of famous of thing that route two is irrational right how does the proof go suppose it is rational then you say that route two is m by n.

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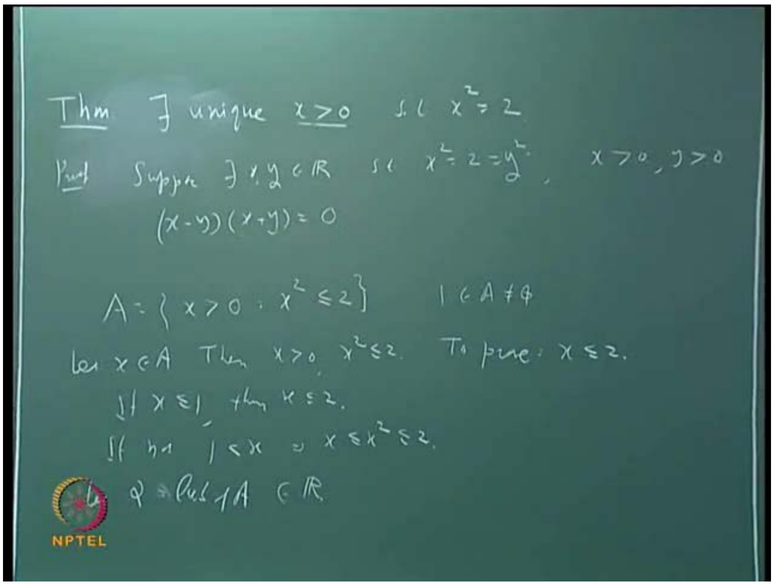

And then you remove the common factors from m and n then you will write than two is equal to m square by n square. So, that will give two n square is equal to m square etcetera and then since we have already this will be in the 2 divides m square that will give the 2 divides m. So, removing 2 again you will get the that will show that m is the even similarly it will project the n is also even by after we remove it. So, fine that proof is quite well known to you right, but does this say their exist route two as a real number how do you know that their exist in a real number. See all that this proofs that there is no rational number all that is proves that there is no rational number whose square is 2 right.

That is prove what that proofs right does it proof there is a real exist are real number whose square is true it does not right. It does not that needs a proof separately that they exist a real number which in fact without that. Even this notation route two is meaningless right without proving the, exist real number whose square is 2 even they have notation route two is meaningless. And to prove that you will need this to prove that you will need this well let us see let us see that how that can be done. So, let just take that as the theorem and then we will subsequent that is nothing particular about two. But, let us prove this is a similar proof I will say they exist x in fact one can say more there exist x unique x bigger than 0. This is unique x bigger than 0 such that x square is equal to 2 of course I can say it is unique because of this right.

Suppose, I remove this condition that is not unique why because minus x square is same as x square. So, as per as the uniqueness concert there is nothing much to prove right see there are two x and y we satisfy both you will get x square is equal to y square. And that will give x minus y x plus y 0 right see suppose let us suppose their exist x y in R such that x square is equal to 2 and y square is also 2. Then this will give that x minus y into x plus 1 is equal to 0 of course this is a important suppose their exist xy in R such x square is equal to both are positive, now if now if that is the case x plus y is also positive and.

So, you can multiply both sides by 1 by x plus y you will x equal to y right. So, if we do not assume that x is bigger than or equal to 0 then x plus y is also be 0 and that will give y equal to minus x. But, if x and y are both positive x and y must be equal alright, now let us go to the existence let us say I will take the set a as the set of all x which are bigger than zero and x square less than or equal to 2.

Ok x square less than or equal to 2 is this a non empty set obviously one belongs to 1 square is less than or equal 1 belongs to A. So, that is non empty is that bounded above how do clear that is bounded above yes how do you show that how does one prove that. That will only assure that two does not belong to a how do show that is upper bound what do you is right 2 square that is not less than or equal to 2. But, that means two does not belong to a right how does it show a upper bound to show that something is upper bound what is related to show you need to show if any element in x. Then that means less than or equal to two right and how do you show that.

Let us let us x belong to a let x belong to a then what we know then these two we know x is bigger than 0 x square is less than or equal to 2 we want to prove that x is less than or equal to right. If we want to say 2 is an upper bound we want we want to prove that alright. Now, tell me how does one proceed after this there are no such taking square roots just use whatever we have said, so far about real numbers alright.

Let us hint once you realize it that will be clear suppose x is less than or equal to 1 if x is less than or equal to 1 then 1 is less than 2 there is nothing to be proved. So, obviously x is less than or equal to 2 if not one is strictly less than x right alright is 1 is strictly less than x can you say that will give x is less than x square remember x is bigger than 0 x is bigger than 0. If 1 is less than x or even less than or equal to is will to x is less than or equal to x square. And x we already know shown that x is less than or equal to we already assume that x square is less than or equal to right. So, that shows that a is bounded above a is bounded above.

And now use this LUB epsilon it has a least upper bound let us call that least upper bound as alpha write alpha LUB of a obviously alpha is a real number right. We have said it at every non empty set has bounded above has a least upper bound is a real number alpha is a real number. And what we want to say about this alpha that alpha square is equal to 2 this is what we want to say. So, first grade is equal to 2 alright now we use this property of this least upper bound alpha is the least upper bound of a rating then you should take any epsilon. And then alpha minus there should exist some x in A. So, that alpha in A epsilon is strictly less than x we shall choose the appropriate epsilon little later.

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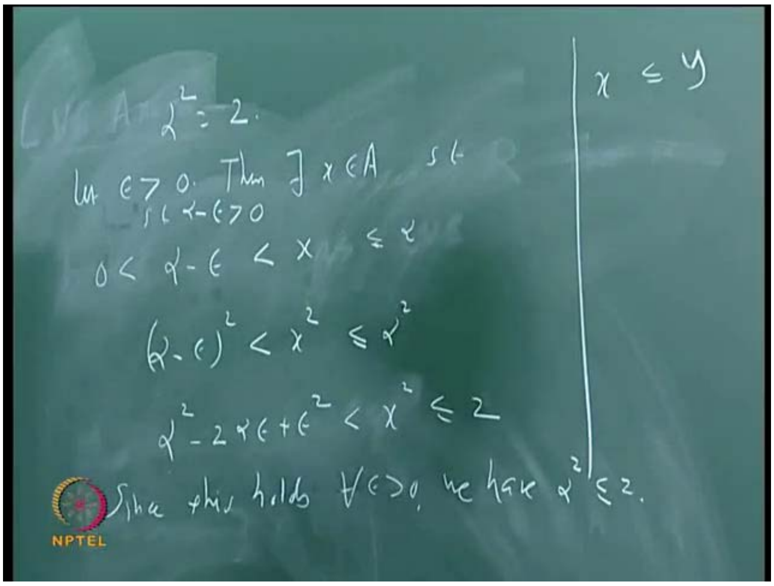

Let us start with let epsilon be bigger than 0 then their exist x in a such that alpha minus epsilon is strictly less than x. And of course this is less than or equal to alpha alright and we shall choose this alpha minus we shall choose epsilon that alpha minus epsilon is also positive right. That we shall always do that we can always do that is because suddenly alpha is in upper bound. So, alpha is bigger than or equal to any element here alpha is bigger than any element here. So, in particular alpha is bigger than 1 alpha is bigger than one for example, we can choose epsilon as half alpha minus epsilon will be strictly positive.

So, let me say here epsilon will be bigger than 0 such that alpha minus epsilon is bigger than 0 right this we can do always. So, this 0 less than alpha minus epsilon less than x and since alpha minus epsilon is less than x, now we can say that multiply this by alpha minus 0 in this inequality remain unchanged because both are positive numbers. So, what we get alpha minus epsilon whole square is less than x square less than or equal to alpha square. So, suppose we expand is what will get is alpha square minus 2 alpha epsilon plus epsilon square is less than or equal to alpha square right yes that is strictly in inequality.

So, what follows from this I want this also let me forget about this part this is strictly less than x square since x square is in A we can select x square is less than or equal to 2 x square is in A x that is what we want to use x square is less than or equal to 2 ok.

Now, does is follows from here alpha square is less than or equal to two remember this is true for every epsilon this is true for every epsilon right. So, I can make further choice of epsilon such that this number becomes this number becomes strictly positive. See this is something mentioned right in the beginning suppose you want be show that something is suppose x is lesser than or equal to y. You can show that x is less than y plus epsilon for every epsilon or which is same as when x minus epsilon is less than or equal to y for every epsilon. This is what we are sure here whatever epsilon you take whatever epsilon you take this is always going to be less than or equal to 2 right.

For every epsilon that is important since the so the argument since this happens for every epsilon we must have alpha square less than or equal to 2. You must have alpha square less than or equal to we can say that since this holds for epsilon bigger than 0. Alpha square is less than or equal to is it clear alpha square is less than 2 one more thing we should notice right here. Since alpha is upper bound of a alpha is bigger than or equal to every element in a every element in a is bigger than 0. So, in particular alpha is bigger than 0 in fact we have also noticed alpha is bigger than or equal to 1, so alpha is strictly bigger than 0.

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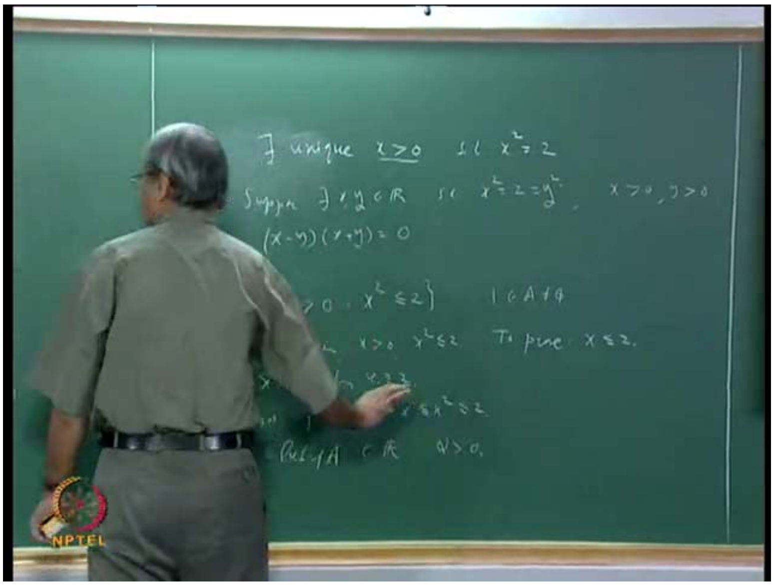

Now, we have proved that alpha square is less than or equal to 2 right, but what we wanted to prove we want to prove alpha square is equal to two right. So, that means what we what we need to prove next alpha square is equal to 2. So, that means what we need to prove next that is alpha square is also bigger than or equal to 2. Suppose we prove alpha square that is also bigger than or equal to 2 there it will in the alpha square or which is or other way is that we should show. That is inequality cannot be strict that is that is we have already shown alpha square less than or equal to 2.

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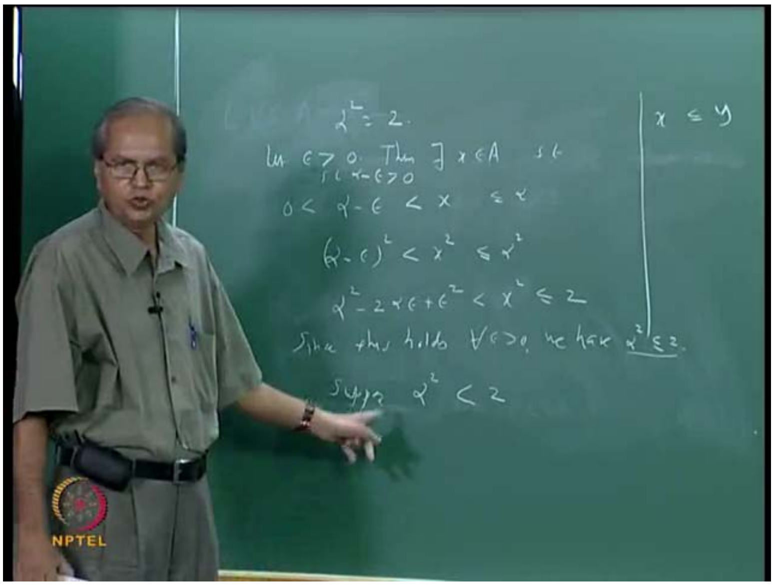

Now, suppose alpha square is strictly less than 2 alpha square is strictly less than 2 we have to show that this cannot happen our idea is to this cannot be happen. Then we will do something similar again here we have taken alpha minus epsilon will be. Now, consider plus epsilon we will choose an epsilon in a appropriate way and get some contradiction.

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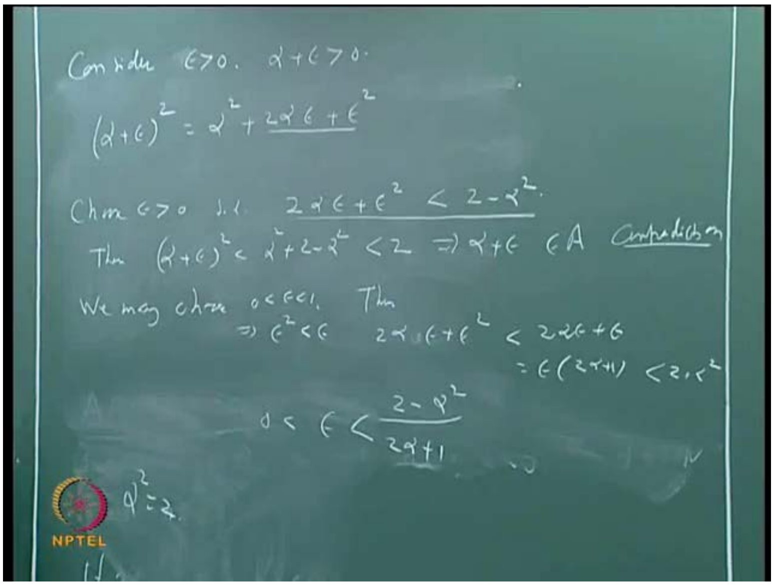

So, let us say consider epsilon bigger than 0 and there of course alpha plus epsilon is also bigger than 0 ok. And then see our idea is basically this if alpha square is less than 2 we bound we will find some epsilon such that alpha plus epsilon. Such that alpha plus epsilon whole square is less than 2 is that possible in fact that is what we shall verify those are the calculation those are the calculations that we verify. So, consider a epsilon bigger than 0 and alpha plus epsilon is also bigger than 2. Then alpha plus epsilon hole square this is equal to alpha square plus 2 alpha epsilon plus epsilon square alright. Now, alpha square is less than 2 alpha square is less than 2. So, which is same as saying that this if I take this number 2 minus alpha square that is a positive number 2 minus alpha square that a positive number.

Now, the question is suppose they choose an epsilon in such a way that this part two alpha plus epsilon plus epsilon square suppose that is less than this suppose this is less than this what will be the meaning of this then the whole thing will be less than 2 right. So, let me just say that thing first here choose epsilon bigger than 0 such that 2 alpha epsilon plus epsilon square is less than 2 minus alpha square how this to be done or how whether this is done not ensure that. Suppose this can be done suppose this can be done then what does that mean it will mean that then alpha plus epsilon whole square.

That is equal to that is less than alpha square plus 2 minus alpha square and that is less than 2 that is less than 2. And that means for this epsilon what does that mean that alpha plus epsilon is bigger than 0 and square is less than 2 that means this means alpha epsilon belong to A ok.

And this means alpha plus epsilon belongs to A, now with epsilon strictly bigger than 0 right can that happen right alpha is an upper bound alpha is an upper bound of a face of known number bigger than alpha can belong to x. So, this is the contradiction, so this is contradiction. So, only thing remains is that whether we can choose such epsilon right whether we can choose such epsilon now for this to happen what must be the case any way see if we choose any particular epsilon. Suppose this in equality what we did is equality right. If that inequality works in any particular epsilon it will also work any other epsilon smaller than that that is clear.

Suppose let us say it works epsilon equal to half it will also work epsilon equal to 1 by 4 1 by 8 or anything. So, I can, so it is because of this that I can assume right from the beginning that epsilon is less than 1 I can assume right in the beginning that I can choose epsilon smaller than 1. So, I can say that we may choose 0 less than epsilon less than 1 then the point to choose the point of choosing the epsilon less than 1 is that. This will also imply that square is less than epsilon if epsilon is less than 1 then epsilon square is less than epsilon square. Then the choice becomes very easy that is whole idea here then 2 alpha epsilon sorry 2 alpha epsilon plus epsilon square. This is less than it will be less than 2 alpha epsilon plus epsilon which is nothing but epsilon into 2 alpha plus 1.

Now, can we can we choose the epsilon such that this is less than 2 minus alpha square 2 alpha plus 1 is a positive number epsilon plus 2 alpha plus 1 less than 2 minus alpha square. This is said my saying epsilon should be less than 2 minus alpha square divided by 2 alpha plus 1 right that is. And of course epsilon, now 2 minus alpha square do you have a 2 alpha plus 1.

This is a positive number right can we always choose positive number positive number that is strictly less than right. We can always do that and also it we also wanted it should also smaller than 1 choose epsilon which is smaller than minimum of these 2. Take one 2 minus alpha square divided by 2 alpha plus 1. These are 2 positive numbers there minimum is also positive numbers there minimum is also positive numbers. For example you can choose an epsilon be half of that minimum there will be less than 1. And less than this right this is the kind of technique that you use in many proofs that is why I have done.

This is detail here is this clear whatever we have done now what is this shows the alpha square less than two this is not possible because we have contradiction to this. So, that means alpha square that has to be equal to two we already show that alpha square is less than or equal to 2, now we are sure it is not an strictly less than two. So, we must have alpha square where is equal to two alpha square is equal to 2 is that clear. Let us again take a look what did we prove that their exist of course uniqueness their exist a unique x strictly than alpha is x right that alpha is bigger than 0.

And alpha square where is two that is unique all that things are proved let me give you an exercise the following there is nothing particular about this 2 that 2. You can take any number here instead of two you can take any number instead of 2 and similarly you can take any natural number here instead of 2 right. After having proved this they know only this notation is winning root 2 or 2 power half or whatever it is.

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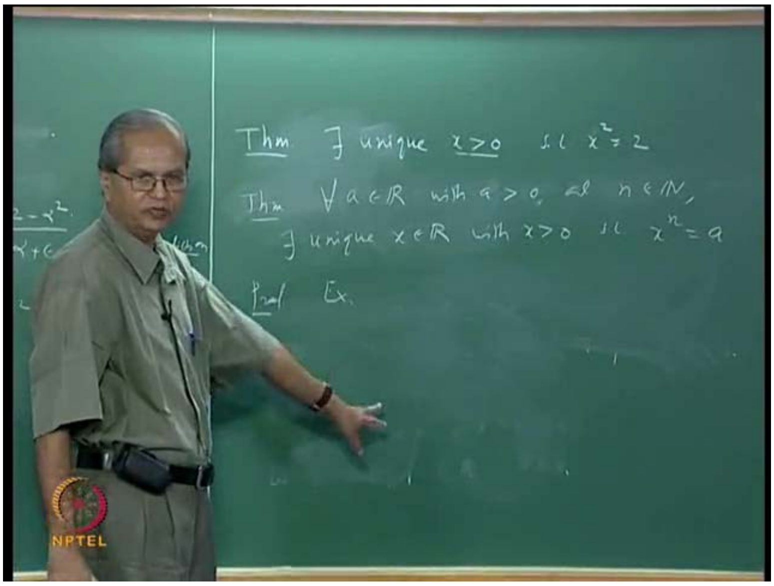

So, I simply state this theorem and proof will be left you other guesses. So, for every x in R with x bigger than let me let me take it as a for every a in R with a bigger than 0 and natural number N natural number N.

Their exist unique x in R unique x in r which is positive x bigger than 0 such that x to the power n is equal to A x to the power n is equal to A. And proof left to you as an exercise do it on your own whatever we have done for two you do a similar thing for n again uniqueness will be straight forward to two you. Similarly define a set a with a appropriate modifications here whatever set we have defined here you define set with appropriate modifications. It will be set of x bigger than 0 with x to the power n less than or equal to a there is shown that x is non empty bounded above etcetera take it is supremium and there is a similar way show that x to the power n is equal to A.

So, whatever we have done this square this n power everywhere and come across appropriate any qualities prove the appropriate in any qualities. Now, let us go to one more very important of real numbers see till now we have seen how natural numbers and real numbers are to each other. Now, we look at how rational numbers and real numbers are read together this important is sometimes express by saying that rational are dense in real numbers what we want to say that. Given any two real numbers their exist rational number between those two real numbers.

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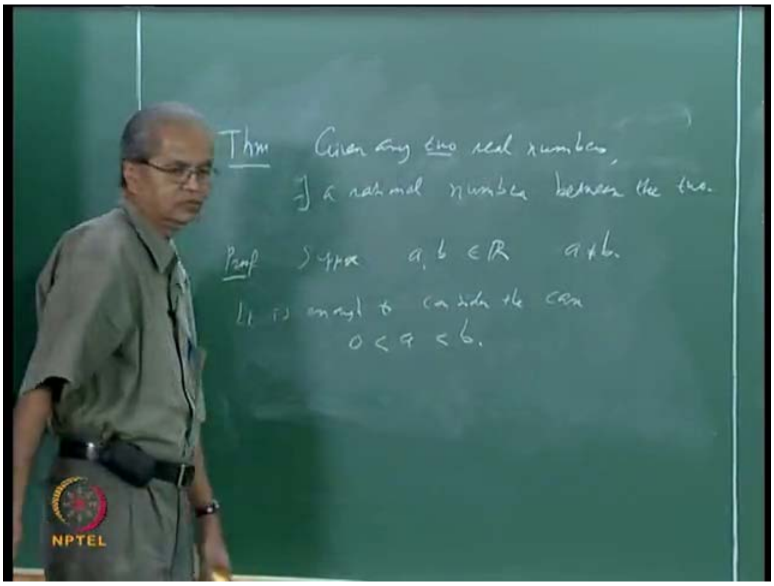

So, given any two real numbers their exist rational number between the those real numbers. So, suppose x and y are real numbers and we are saying that two real numbers it means they are two different real numbers. So, suppose let me say a and b suppose a and b are two real numbers and a not equal to b. So, first step I want to say that it enough to consider a case in both a and b are positive and both a and b are positive we can also assume that a is less than both they are different means either a is less than b or b is less than a. So, a and b are rotational we can assume that a is less than b. So, what is I am about to say is enough to prove it is enough to consider the case consider the case let me say 0 less than a less than b. In other words I want to say it is enough to consider the case when both are positive why it is enough what are the other possible cases we are in this negative.

And other is positive right one is negative and one is positive yes right 0 is there is nothing to prove if both are negative see. Suppose we have proved this case because q is the rational number if a is less than q less than b we are also saying that is same as that is minus b less than minus q less than minus a. So, suppose you take two negative numbers you take their minus of those two numbers those are positive. Find the rational of those two positive numbers minus of that will be rational between the two negative numbers right, so it is enough to consider this case that is clear. Now, let us how we go about this see after what is our idea we want to find the rational numbers.

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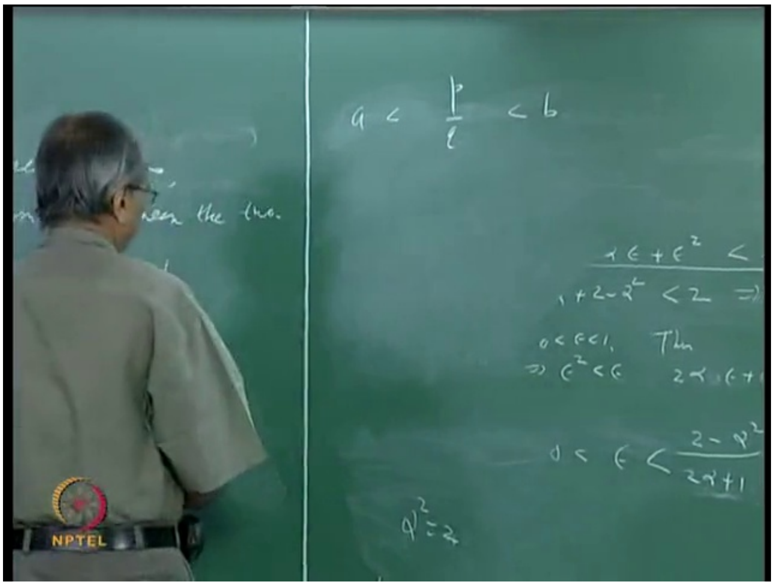

We want to find numbers p by q we want to find the number p by q we want to find natural numbers p and q or integers p and q says q not equal to 0. And this should happen a less p by q less than b maybe one of this inequalities may be less than or equal to also that does not matter right. Of course a and b are both positive is it clear p and q must be positive right. So, we have to basically choose p and q in such a way that this happens this happens alright to do that we have to say how to choose q and how to choose p, now first consider the case.

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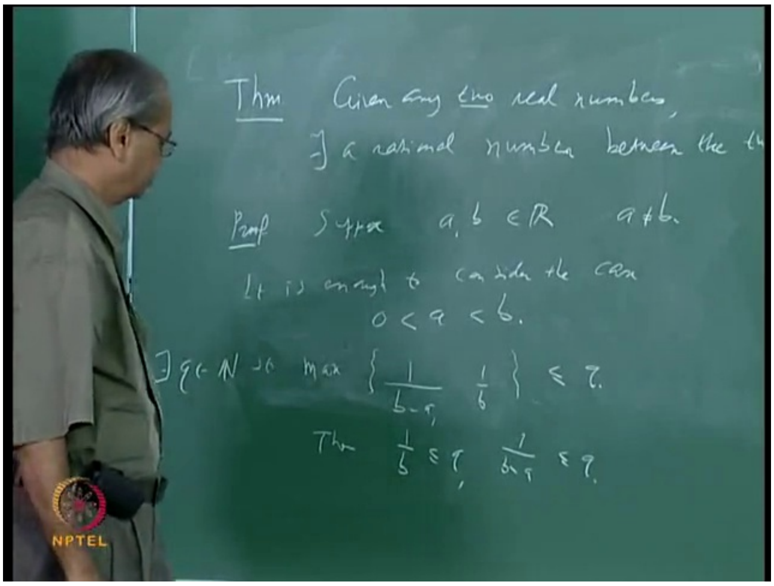

Let us consider that consider that number 1 by b minus a and 1 by b both of these are positive numbers you can take maximum of these they are also positive you can take maximum there is this also positive. Now, whatever is positive real numbers since we have already proved that the set of all natural numbers is not bounded above I can always find natural number which is bigger than the maximum of these two is that clear to you this is a real number maximum of these two numbers are real numbers. And I can always find a natural number which is bigger than this real number, so that number I will call q that number I will call q.

So, you can say their exist this q in n such that this maximum is less than or equal to q right that is 1 by b less than or equal to q and 1 by b minus a is also less than right 1 by b is less than equal to q 1 by b minus a. That is also less than or equal to q this something we want to prove let us just record what we got here. Then we will have these two inequalities 1 by b less than or equal to q 1 by b minus a, this is also less than or equal to, now let me take this set I will again call this set as A.

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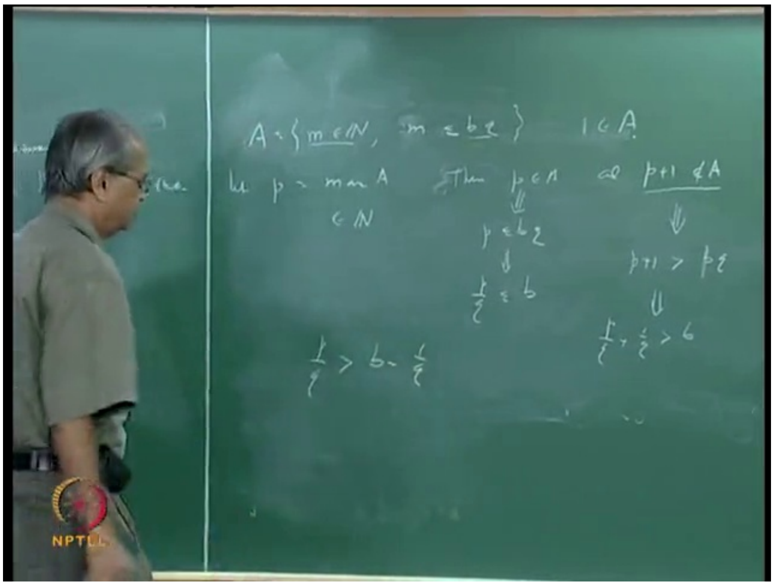

I will take this set as a set of all natural numbers m in n such that m by q is less than or equal to b m by q is less than or equal to b or that is same as m less than or equal to b times q is this set non empty right this is the obviously belong to it what is it. But, right one belongs to, so it non empty its obviously bounded above of course this b times q cannot be natural it is just a real number. But, we are taking all those natural numbers which are less than or equal to this real number right.

Now, since this is bounded above can we see that there must be maximum in this which is subset of natural number it not subset or real numbers it is a subset of natural numbers and that is bounded above. So, there must be maximum among this, so that maximum I will call p right. So, let p be equal to maximum of a what does it mean that p belongs to a and since p is a maximum p plus 1 does not belong to it right.

So, that is and not only that fine this is what we require that is. So, then p belongs to a and p plus 1 does not belong a we shall use both this facts. Now, p belongs to means what of course p is a maximum of a, that means see remember a is subset of the natural numbers we are choosing maximum numbers. So, this is also natural number, so this is a natural number and, so what does what does p belong to a mean p is less than or equal to bq right. This means p is less than or equal to bq remember q is also natural number. So, this implies p by q less than or equal to b right. This is one of thing what we wanted alright we also wanted p by q bigger than A.

That will follow from this part p plus 1 does not belong to a means what p plus 1 must be strictly bigger than bq right p plus 1 must be strictly bigger than bq. So, this means p plus one strictly bigger than bq alright. This also means p by q plus 1 by q means bigger than b right p by q plus 1 by q is bigger than b alright let me write it here. So, p by q is bigger than b minus 1 by q, now what do we know about 1 by q it is here 1 by b is less than or equal to q. So, this means 1 by q is less than or equal to b.

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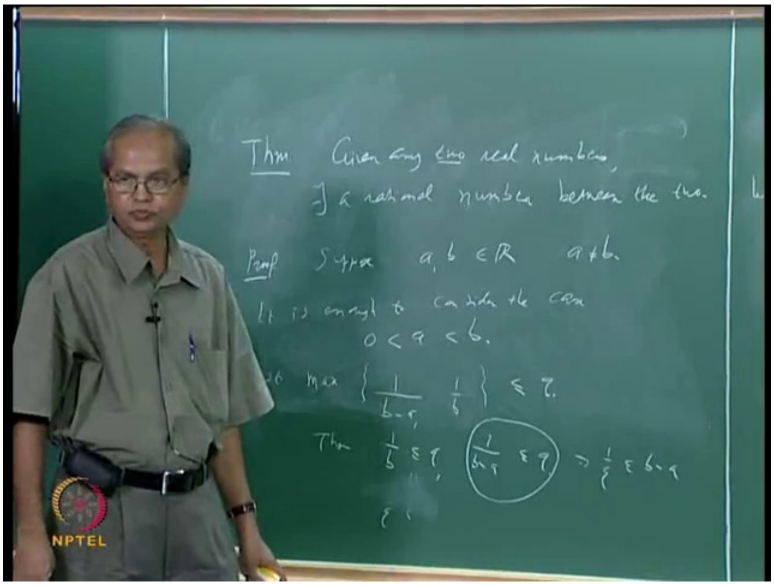

Sorry we will direct this is not this is not something that is useful now this part is useful. Now, 1 by b minus a is less than or equal to q, so this means 1 by q is less than or equal to b minus a right. And hence what can we say about minus 1 by q minus 1 by q is bigger than or equal to minus of b minus of a right ok.

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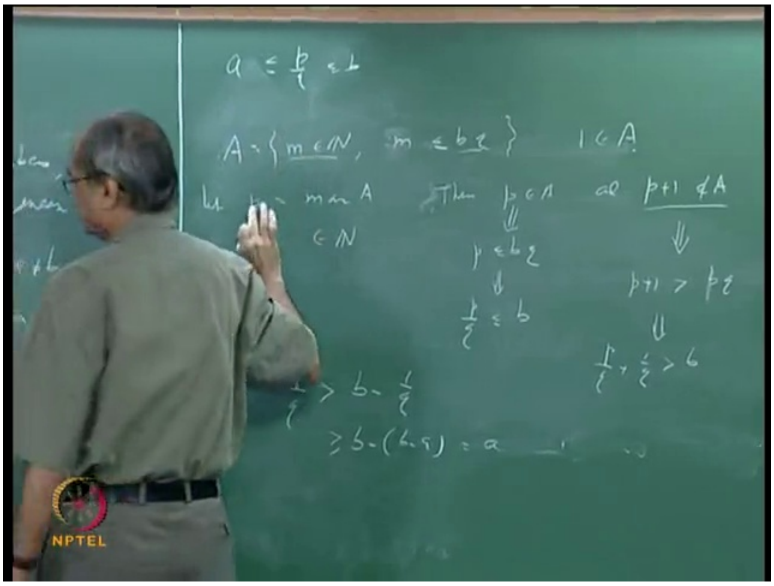

So, this is bigger than or equal to b minus b minus a that is same as a right. So, here we have proved that p by q is less than or equal to b and here we have proved that p by q is bigger than a p by q is bigger than a. That is what we wanted to remember we wanted to show this that is a less than or p by q less than or equal to b and we have proved that right. So, we have shown that between any two real numbers that exist a rational numbers and let me again remind you that this is what express by saying that rational are dense in reals. Now, remember in this proof we directly use the LUB axiom in the proof itself right in this proof it directly use LUB epsilon in proof itself.

In this we proof we did not use it directly, but did we use it indirectly where right for example here we say that exist q in n. Such that this is less than or equal to q there we use the fact that n is not bounded above right. This we can get because the set of all natural numbers is not bounded above and that was the property that we got from LUB axiom right. So, what I wanted to say again is that practically everything that you will prove about real numbers it will follow either directly or indirectly from LUB axiom an. We shall see that extents in the next few lectures when we discuss the real number system we will stop that firm.