Real Analysis

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Lecture - 6
Real Number System

In the last few lectures we discuss what can be called elements of set theory in fact we took a review of some of the properties of sets in this course of course. We shall be dealing with many sets and the most important of those sets is this set of all real numbers or also called real number system. So, let us discuss a few things about this real number system in today's and few more lectures.

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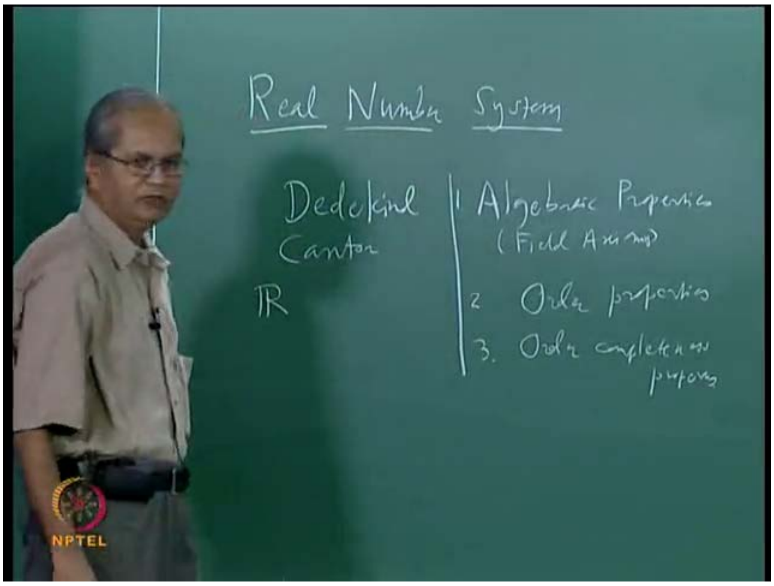

So, the idea is now we discuss real number system now first important question here is that how does one define real numbers or how does one construct this real numbers system. Well there are many ways one can start I mentioned till one of the lectures earlier that set of all natural numbers can be constructed using what is called piano actions. And once you construct natural numbers set of all natural numbers then that natural numbers along with this operations of addition is what is called as semi group. And there is a very standard way of constructing the group from semi group. So, that leads to the set of all integers then again algebraically speaking the set of all integers is a ring.

In fact something more than a ring what is called integral domain you would have heard of these terms in algebra. And there is again a very standard method of constructing a field from an integral domain you define certain equivalence classes define some operations on equivalence classes. And suppose you apply that method to the set of all integers you get this field of rational numbers the real question is how do you go from rationales to reals. There none of these algebraic methods work rational numbers is already a field there are two ways one is in fact a those two ways are due to two famous Mathematician.

One is due to Dedekind and second one we have already heard Cantor this procedure is what is called defining what is called Dedekind cuts those are subsets of rational numbers satisfying some property. And using those Dedekind cuts define the real numbers and second approach which is that of Cantor is using the sequences of rational numbers. And then define what is meant by again define some equivalences classes of sequences of rational numbers. And associate one real number to this this first approach Dedekind cut you can find.

Rudins book Rudins book first chapter is devoted to that rule, but, we shall not go into that kind of detailed study of constructing real number systems in what we plan to do is that. We shall just list some properties of the real number system which we shall be using very often and these are the properties which are called defining properties of the real number system. That is if you take all these properties together then there is a way of showing that real number system is the only system which satisfy all those properties of course, another way of doing this is, that as I said you can construct natural numbers by starting from piano axioms. Similarly one can also construct real numbers starting from axiomatic thing just state axioms.

So, our approach is something similar to that. So, though we do not call it as interactions because we are not going to follow that kind of logical regard here. So, what our plan is just to state the properties of real numbers at least there should be there together for the record and those are the words which we should be using very often. So, to begin with real number is a first of all this none empty set we will drawn by this. And there are actually I should say three sets of properties I will say first are what are called algebraic properties.

And in fact there is another way of dealing with this group of properties they are also called field axioms. This is one group second group of the properties is what is called order properties and the third group in fact this third group contains already one property which is known as order completeness property. Order completeness of sometimes simply called completeness property. And these properties all of these properties together we the set characterise what is called real number system I will come to that discussion also little later about this meant by characterising ok, so first of all.

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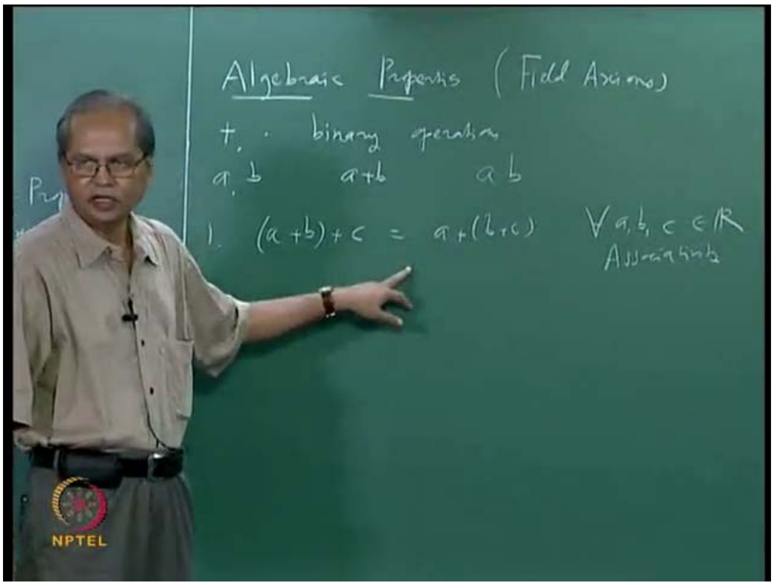

Let we go to this what are called algebraic properties what is algebraic properties say is the for algebraic properties or what are called field axioms what are also called field axioms. It is that first of all there are two operations on the real number system namely the addition of two real numbers this and also collect of two real numbers.

So, these are plus and dot these are two binary operations and so if you take two real numbers a b that their addition is defined as this a plus b denoted a plus b. And the product is throughout since this dot is an operation we should denoted by a dot b by despite customary to forget about writing the dot and just right a followed by b. So, a b and these operations satisfy some properties and those are the once we called field axioms. And as we go along this property you will writing this properties you will see that there are many other sets also which satisfy this properties and as we write more and more properties we will be see that certain of those sets get elaborated.

And real number estimates the only system which satisfy all this properties alright first of all let us start with the properties of addition. So, the first properties is the for lying what is of course all this properties you have already come across in some contrast to the other. But, since we are now discussing the real num esteems at least once they should be all together. So, first properties called associatively that if you should take three real numbers and take say a plus b and then at c to it with this should be same as a plus b plus c. And this should happen for every a b c in R and this is the property which is called associative I mean what is this associatively mean in practice that is six of binary operation.

So, it tells you given two real numbers what is meant by this some of this two. So, suppose you take more than two what do say does not matter whether you first add a into b and then add c to the sum or whether you first add b and c. And then add total a to that sum which ever away you group this two among this three element it does not matter.

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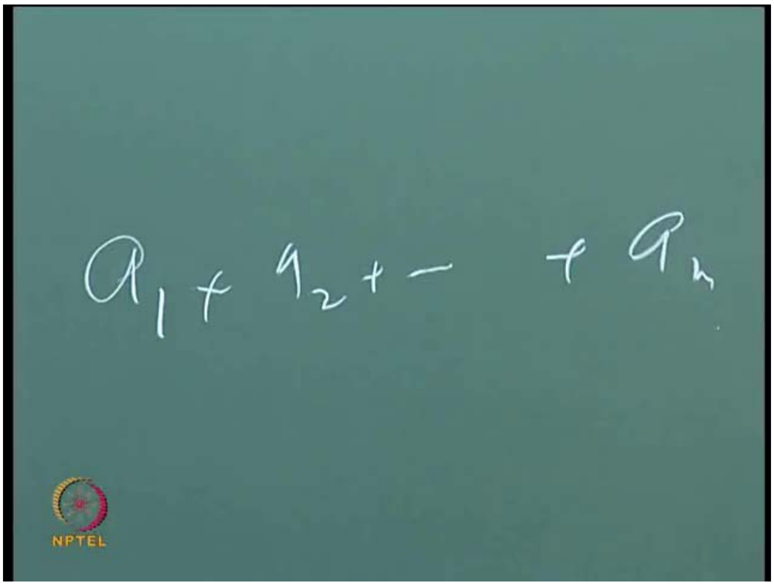

In fact it is very important when you consider more than three for example suppose you write something like this a 1 plus a 2 plus a a n excreta this something we will we required by veritable discussing the series of ((Refer Time: 08.16)) series. So, that kind of expression will be meaningless unless the operation is associative because there you have to say how exactly you are grouping the elements because of the associatively. It does not matter in what way you are grouped the elements alright. Now, of course this two operations of addition and multiplication they are defined on many other sets also for examples the set of natural numbers integers rational numbers excreta. And this first property is there followed by all natural numbers also this. So, just suppose real number as well just satisfy in this property then that does not distinguish between R and n, but now let us say the next will distinguish between R and n.

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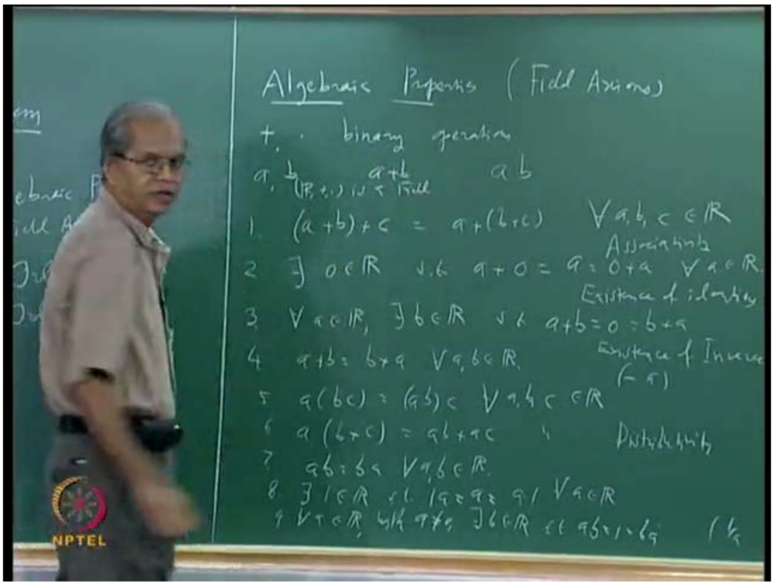

So, their exist an a element which is called identity with respect to addition that we shall denote by 0 that exist a number zero in R such that a plus 0 is a and 0 plus a is also a. This should happen for every a in R and this property says is called existence of identity existence of identity with respect to this first operation addition. Now, you can see already that this property distinguishes between R and n does not have this property though n has this property it does not have this property alright. Then the next property is what is called exist by the way this there are certain things which we will not go into detail prefer.

This because this something you also learn in many other places the axioms itself does not say that the identity element is unique that is this 0 element is unique. But, there is a standard way of proving that once you assume these two properties there it can be proved that this only one element satisfying this property. And hence we can call the identity the identity alright. Similarly now we shall say that given for every a in R we will say that they are insist b in R such that a plus b is equal to 0. And b plus a is also 0 and this is what is called existence of inverse with respect to the addition again at this similar comment with this to meant you i have not studied here at this b should be unique.

But, you can prove this is there are standard ways of proving that I we will not gone to that kind to prove because something like this we will learn algebra also this element b is unique. So, we can denote that we can give some notation for this and the standard notation will be this minus a we should denote this b as minus a called addictive inverse of a. And as you know any set with one binary operation which satisfies these three properties is called a group and. So, I could have just said simply say said it R plus is a group, but again as a set that is not something that is something special about R there are many other groups other than R.

And there is one more property that this group what is called commutative or obedient group that is a plus b is equal to b plus a for every a b in R coming back to my comment about natural numbers we have seen that natural number satisfy only the first properties. It does not satisfy any of this of course it satisfies this, but it does not satisfy any of these two properties. But, you have take this set of all integers or you have take a set of all rational numbers its satisfies all this properties it satisfies all this properties. So, we these properties distinguish between R and n b, but yet do not distinguish between R and z or R and q alright or for let matter R and c.

Even the complex number satisfy all this properties our idea is that once we write down all these properties we will we want to say that real number estimates the only system which has all this all this properties alright next. Now,0 this is about the operation with properties of this first operation, now let us look at this second operation that is also associative. So, a into b c or this is same as a b into c for all a b c in R again this is same as this is called associativity of this second operation and the same comment here. So, I will not go in details of that then we also want to say how this two operation are related to each other that is suppose you take three real numbers.

And suppose you take a and multiply that by b plus c then this should be same as a b plus a c this should be same as a b plus a c. Again this should happen for all a b c in R and this property is described by saying that multiplication is distributive you over addition this is called distributives alright. And perhaps many of you know that any set which has two operations and which satisfies this six properties is called a ring. It is called a ring and R is one example of a ring you just not a unique ring z is a ring q is a ring there are so many other examples of rings alright then. But, this ring are has some extra properties namely that is also commutative ring this also that is the second operation is also commutative.

So, a b is equal to b a for every a b in R again this also is there that properties also there. With z z is also commutative ring then that of by the way not all rings are commutative there are like well known in samples of non commutative rings. Then not all rings have identity element with respect the second operation, but this second operation also has identity here. So, that element as you know we will call denote by one their exists element let us say one in R such that one into a is equal to a and that is same as a into one for every a into R. So, that is a commutative ring with identity and again the same comment about this identity we can show that this identity is unique though we shall look for something similar to this. Just as we talk about the inverse with respect to this operation similarly we want to say something about inverse with respect with second operation.

And we cannot simply imitate this property here we cannot say that every element has an inverse, but every element other than that zero which is identity of the addition that has an inverse. So, that is the last properties in this field axioms for every a in R with a not equal to 0 because that their exists b in R such that a b is equal to 1 and b a is of. So, of is equal to one and as a set about that addictive inverse in a similar way this b is called multiplicative inverse. And in a similar way we can show that this multiplicative inverse is also unique given a there exists only one b with satisfy this property and hence we can device some notation for this.

And that notation as you know is 1 by a 1 by a are also a inverse that is the notation for the multiplicative inverse you can see that this last property is not true about z this last property is not true about z right. So, already we have distinguish R from n and z we are not yet distinguish it from q, but let us see let us do it for that. So, all these properties together I could have described in just simply once and tens by saying that this R plus dot is a field that is why those are called field axioms. And again field is one of the algebraic structures which will learn in algebra. But, I have prefer to write down in all details because these are the properties we shall be using in subsequent. So, that is about this algebraic properties, so this is over now and.

So, what are the sets with which we could distinguish R using algebraic properties n and z but, q and C will have all this properties those are all. And not only that there are several other fields all now let us go to the second group of properties namely this order properties ok alright.

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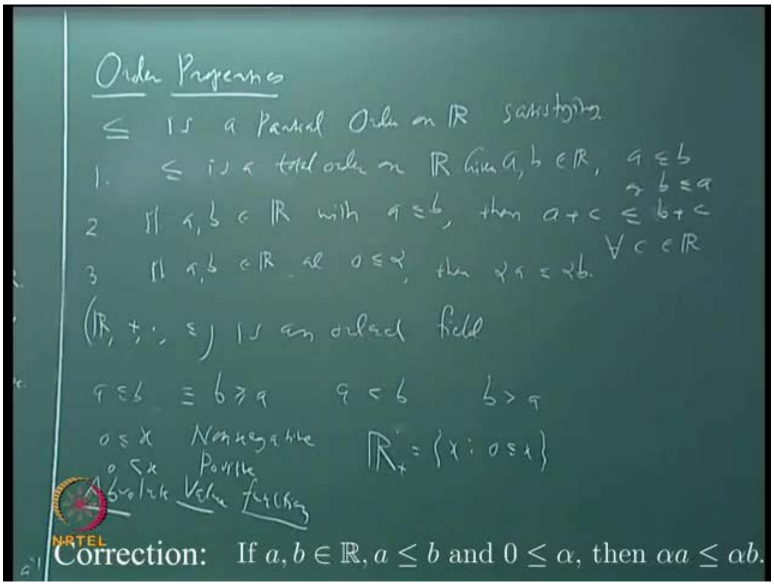

So, so not only that R is a field, but you have also has one more property that never there is an order that is a natural order defined between real numbers and that order is what we have define as less nor is equal to. So, first of all less nor is equal to is a for the time be let me say it is a parcel order on R less nor is equal to is a parcel order on R satisfying for a. In fact I could have said here itself a total order. But, let us go in systematic way first of all this parcel order is a total order that is given any two a at b in R either a less nor equal to be or b less nor equal to a. So, it can say that this less nor equal to is a total order let be again they call one less be that is given a and b in R given add to real numbers a is less nor equal to b or b is less nor equal to a.

Then what we also want to do is to relate this order with respect to this algebraic operation we want to know what is the connection between this order relations and this binary operations. So, in other words what we want to say is that suppose you are given two real numbers a and b and suppose a is less nor equal to b and suppose you add some real number to both. Then is that order preserved and that is one with I show naturally a procreate if the order has to have some relationship with the algebraic operations some of these properties are natural to expect. So, let us say if a b belongs to R with a is less nor equal to b then a plus c is less nor equal to b plus c for every c in R that means addition of some real numbers does not change.

The existing order between the two real numbers if a is less nor equal to b and to both of those number is u and some real number there some a plus b will also remain less nor equal to some b plus c. But, you can see that something like that is not true about the multiplication right suppose a is less nor equal to b. If I multiply both of these by sub number let us say alpha then you cannot anger and say alpha a less nor equal to alpha b. But, you but we can say that in some extra condition and that is when alpha is bigger nor to equal to 0. So, we can say that let we write that if a b is belong to R and let us say 0 less nor equal to alpha 0 less nor equal to alpha.

Then alpha times a is less nor equal to alpha time b, now any field on which such an order is defined and which satisfies these three properties it is called an ordered field. So, I could have simply described all this line plus this three this twelve properties in simply one sentence by saying this that R plus dot less nor equal to is an ordered field alright. It is a very well known property I mean those of you who have done a course in complex variables is there is one of the first thing that is proved in many of this complex analysis books. That it is not possible to define any order on the set of complex numbers which satisfies this three properties.

So, complex numbers is a field, but it is not an ordered field. So, this is what distinguish between R and c, so for till now what we have distinguish we have distinguish between R and n R and z and R and c. But, still rational numbers remain rational number satisfy all this properties ok. So, rational R plus is also an ordered field q plus dot less nor equal to is also an ordered field. So, the question is what is it the distinguishes between R and q it is this last property for this called order completeness property and we will come to that property little later. But, before that I want to say a few things about this order relation some of these things we have already discuss ok.

Once there is some parcel order in in any set you can talk of what is meant by saying that a set is bounded above what is meant by saying that an upper bound least upper bound. And all those thing similarly what is meant by saying that lower bound greatest lower bound or which also called in infimum excreta. Then there are other notations for also example we also denote that is a less nor equal to b this is basically same as saying that b is a equal to a this order relation is nothing but basically reverse of this. It is basically saying the same thing, but using this different notation since it is also very popular we will mention it because it is also quite customary to use this.

And also similarly what is bend by saying that a strictly less than b or b strictly bigger than a again as we have seen earlier this means a is less nor equal to b but, a is different from b and similarly, over there right. So, what we can do is that we shall we shall also see that those numbers let us say suppose 0 is less nor equal to x right. Then as we as we know such a number is called none negative number.

So, we will select x is non negative and if 0 is strictly less than x such an x is called a positive number and say there is what is meant an negative if x is less than 0. It is called negative if x is less than or equal to 0 it will be called non positive it will call non positive and we shall also frequently use this notation R plus. Either this are some books also use this super R plus, let us also this is more popular let us this is this is set of all x such that 0 less nor equal to x that is set of all nonnegative number. By the way this real number system is also called real line.

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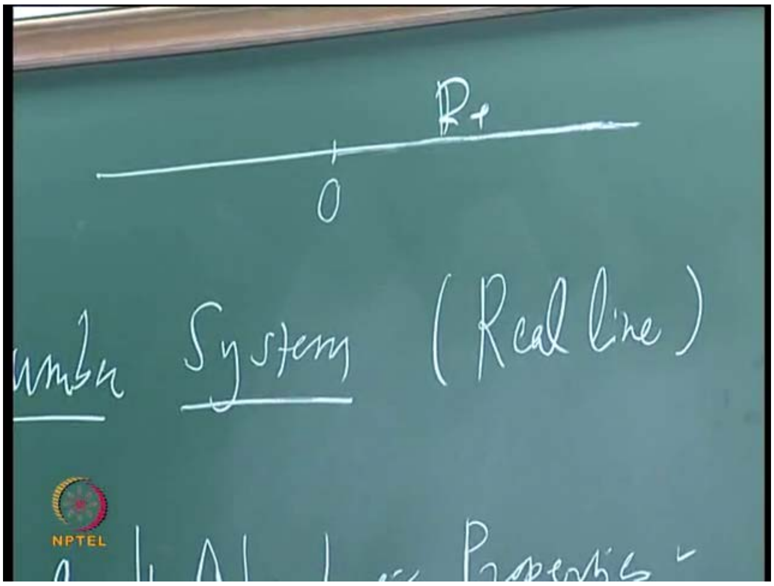

Reason being you can identity each real number with a point on a straight line that is you usually we look at this as a real line and suppose you take this number as 0. Then this part is what is R plus and similarly this part will be denoted by R minus this part denoted by R minus in the set of all non positive real numbers.

Then when we are read this discussion of this order properties there is one more important function that which will come across very often this define our real numbers. And this depends on the order and namely absolute value function. I think I will continue here let be write this absolute value function absolute value are maybe you are also use to call it modules absolute value function this denoted by this bond it goes from R to this R plus.

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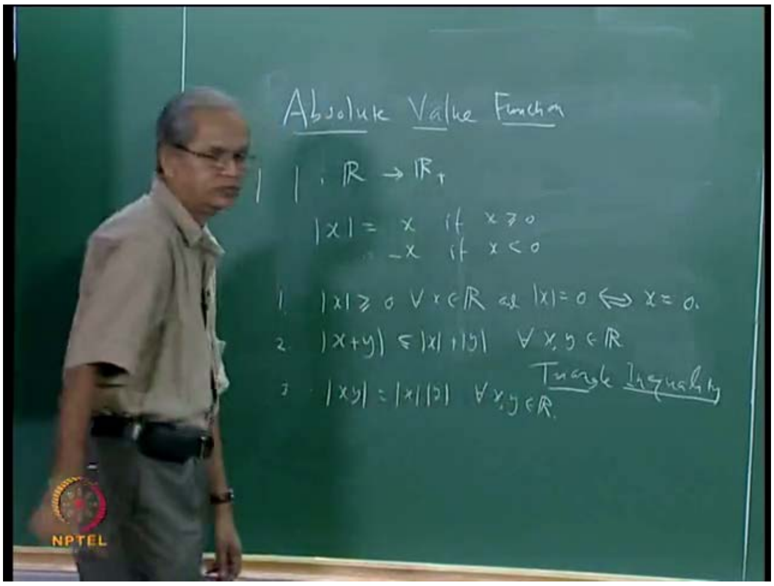

And denoted defined it as follows more of x or absolute value of x this is defined as x if x is bigger nor equal 0 and this equal to minus 6 if x is less than 0 other ways because of this total order property any x should be either bigger nor equal to zero or less nor equal to zero right, if it is if it is different from so far, so these two properties all x. So, that a function on x and then it is fairly easy to prove some of the well known properties of this absolute value function one is this absolute function is always bigger nor equal to 0. This is zero for and for all x in R and this becomes 0 when it will be 0 if n only if x in 0 right. This is no other positive because x is to be 0 or minus x to be 0 both case it means x in 0.

So, and x mode R x absolute value of x in 0 if and only if x is equal to 0 and second property is the following absolute value of x plus y is less nor equal to absolute value of x plus absolute value of y. And this is true for every x y in R in fact this property and properties like this we shall come across very often in this course this property all well known name it is called triangle in equality can you see why it is called triangle.

Inequality right I mean basically it says that if if x and y replace two sides of the triangle and if the third side is represented by z plus y. Then accepts then the length of the third side is less nor equal to some of the length of the other two sides that is what its. And this something we shall come across very often in this course alright how does one prove this well we have to make various cases for example. Suppose x and y both are 0 sorry suppose x and y both are both are bigger nor equal to 0. Then more x plus y is x plus y mode x is x not y is y. So, both sides it become x plus y if both are negative also same thing.

So, we love to similarly consider various cases what happens if x is positive y in negative x is negative y is positive excreta its actually there are three or four possible cases. And in all and in basically you just verify at in all cases in this inequality is true and similarly we should also know what is a relationship with this absolute value function with respect to the product. That is even simpler it is absolute value of x y is same as absolute value of x multiplied by absolute value of y for every x in y in R alright. Now, let us now go to this last property namely property of what is called ordered completeness.

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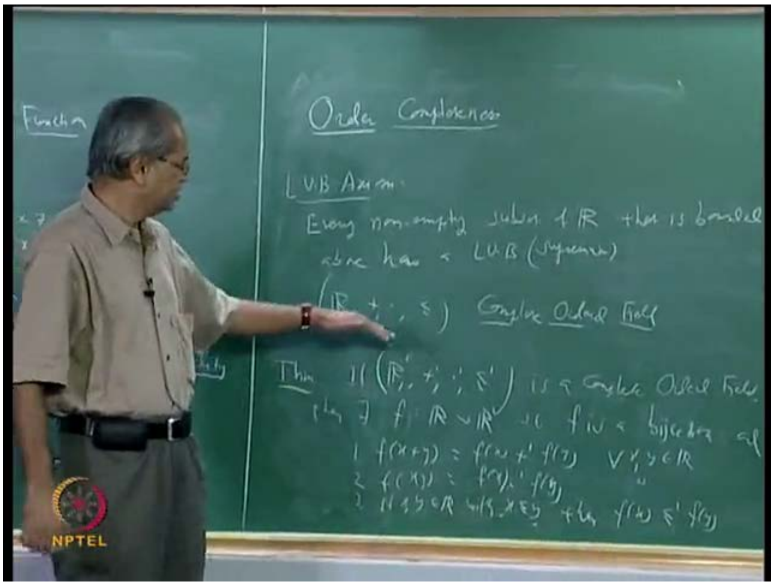

When we discuss partially ordered sets we discussed what is meant by set being bounded above what is meant by upper above what is meant by least upper above. But, we did not say that every set which is bounded above it has to have upper bounds. But, it may or may not have a least upper bounds in an orbit any partially ordered set when ever that happen that is called order complete right. And this real number esteem has that property what is that property that every non empty sub set of R which is bounded above has a least upper bound has a least upper bound it is also sometimes called L U B axioms it is also called LUB axioms.

Ok that is every non empty subset every non empty subset of R that is bounded above has least upper bound least upper bound or what we are also called supreme. And this is a property which distinguishes real number system from the rational number systems we can show that rational numbers do not have this property one can easily construct subsets of rational numbers which are which are bounded above. But, which do not have least upper bound of course by that we mean least upper bound in rational numbers least above because the rational numbers is also subset of y. So, obviously least upper bound will be there as a some real number will be a least upper bound.

But, that is not what we are looking for suppose we just take the set of all rational numbers then that does not have this ordered completeness property which means you can find non empty subsets of rational numbers which are bounded above. But, which do not have least upper above we shall come to that kind of sum little later by the way as I have said here any filed which also has this order and which satisfy is called an order field. And if it also has this additional property of order completeness it is called complete ordered filed.

So, what we have said is that R plus dot plus dot equal to is a complete ordered field and I have wanted to say further this is a property which correct rises the real number systems. It means that this is the only complete ordered field what does it mean you are already you heard of what is meant by what is called isomer forum between groups or between fields. That is those are the bisection those are maps which preservers the algebraic operations. So, similarly one can term of isobar fizzes between that two field suppose you are give two fields there will be a bisection. So, that addition here is preserved and similarly multiplication is also preserved when we ordered field we will also expect that an order is also preserve that order is also preserve.

So, what this means what I want to say is that if you take any other completely ordered field then that will be isobar fix to this that is in that sense this is a unique completely ordered field alright. Let me just state that more precisely in the that is let us state that is a theorem if R. Let us let us let me use some where notations let us say R prime plus prime say dot prime and this is less nor equal to prime suppose this are. Suppose this is complete ordered field complete ordered field that means this is on an empty set these two are binary operations these partial order satisfying all this things all this properties which you are varieties.

So, for if that happens then this must be basically same as R this must be basically same that means what I mean you may give different labours to the elements we may give different notations to the operations. But, they are essentially the same to write that mathematic in a precise mathematical language we use what is called language of isobar fesses. Then we can said in there exists f from R to R prime R to R prime such that f is a bisection and and first properly is f of x plus y is same as f x this plus prime f y plus prime is this operation here from all x y in R. And similarly for the product f of x y is equal to f x y again for every x y in R.

If it had satisfied only these two properties we would have called it as isobar fesses that is it that it preserver the algebraic properties of course. If it satisfies only these two properties it is called homo bar fesses. But, since it is a bisection that is called isobar fizzes any way this not very important this it should be f x f x in this dot prime f y alright f x that is it now it is ok. And similarly we have the order is preserved that is if you take two of different sets and y and f x is less nor equal to y.

Then f x less nor equal to prime f y. So, third property is this if x y belong to R with x is less nor equal to y then f x this less nor equal to prime that means addition is preserve multiplication is preserved at the ordered is also preserved. And it is a bisection that means x is there is no difference between this and this in that sense this is a unique completely real number system is the unique only complete ordered field. And that is the one this set of properties distinguishes real number systems from every other system every other number system we shall not go into the proof of there because it is somewhat lengthy. But, I but those of you who are interested I can give a reference you can see the proof of this theorem in that book in fact some of this thing about set theory also follow some treatment.

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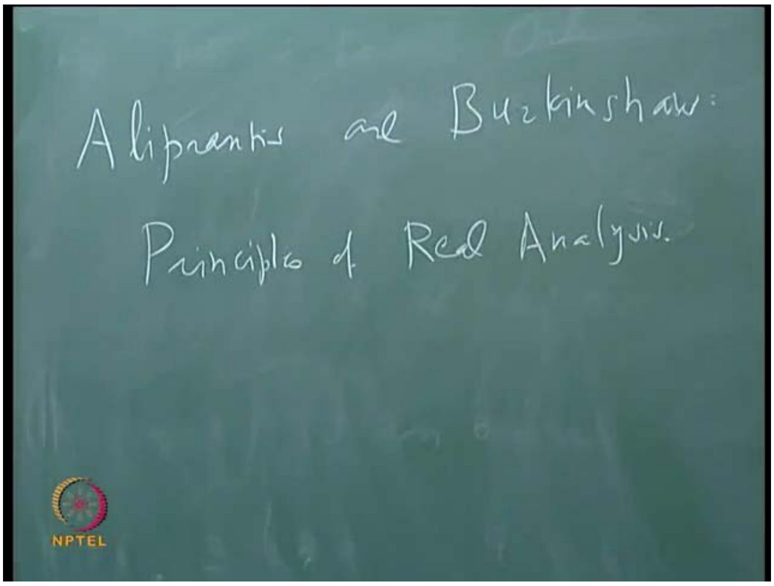

In this book because it is slightly better than the other books, now book author of Aliprantis and Burkinshaw principles of real analysis that is the title principles of really analysis. In this book you will find a detail detailed proof of this theorem one more thing as your as I said this is a axiom that is the most important axiom of the most important property of the real number system.

And in fact whatever we need to prove about real numbers well most of the well known properties of real numbers is subsets of real numbers follow from this axiom either directly or what happens is that. We use this property to prove some theorem in that theorem to prove something else and that to. So, suppose we simply trace back we can always trace it back to this particular axiom in fact this is something that is brought out very clearly in an article.

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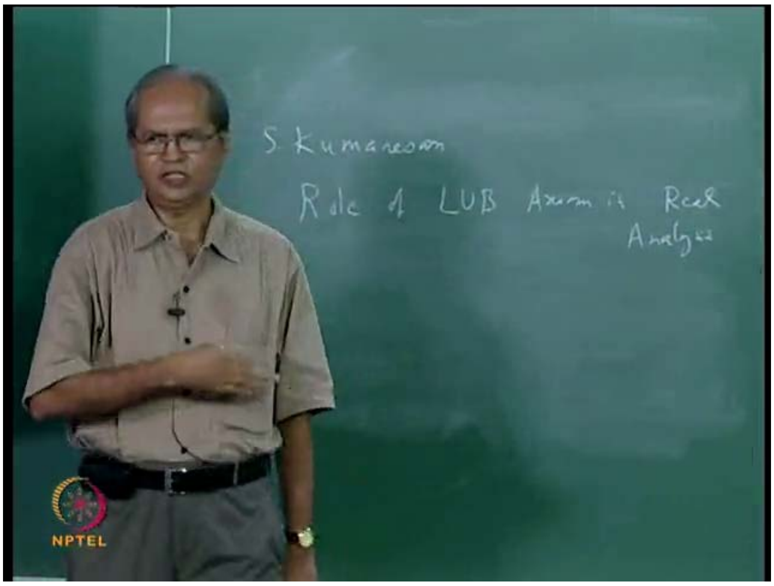

Yesterday I mentioned about professor S Kumaresom S Kumaresom he has written one more article about this, now it is title role of lube axiom role of lube axiom in real analysis. This is a interesting article it gives a detailed account of various properties of real number system and various subsets of real numbers. And how those properties follow directly from this LUB axiom, now let us see some of those things.

Now, there are few things which I have already mentioned now once we know that least upper bond exist. Then it is clear that a least upper bond is unique least upper bond is unique alright. Now, whatever we can say about the once we assumed that to least upper bond exist we can similarly say that if I say it is bounded below then it will have least upper that is lot about that will follow from this right.

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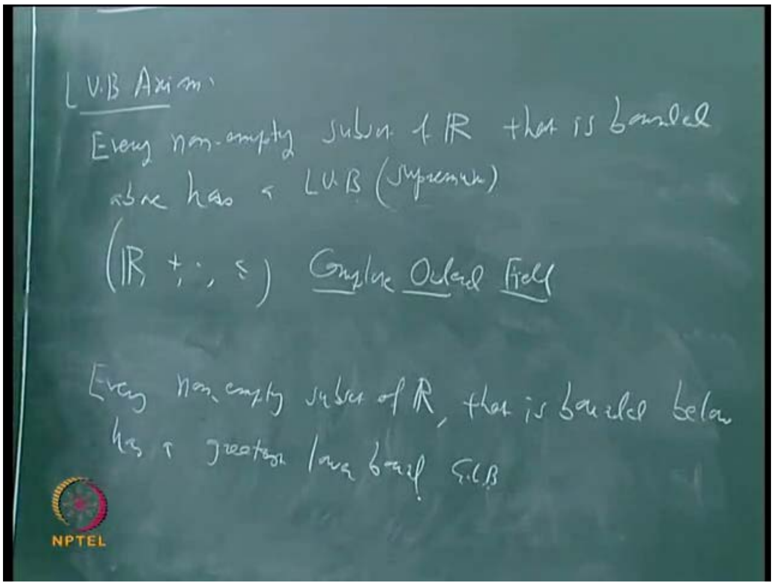

So, we can say that every subset of R every of course non empty subset of R every non empty subset of R every non empty subset of R that is bounded below. That is bounded below has greatest lower bound greatest lower bond or which we can be denoted by G L B greatest lower bond or ((Refer Time: 44.00)) bond. In fact when can show that these two things are equivalent that if you assume this you can prove this and if you assume this we can prove this right. And the whole idea is that the axioms about the order that we wrote here various axioms.

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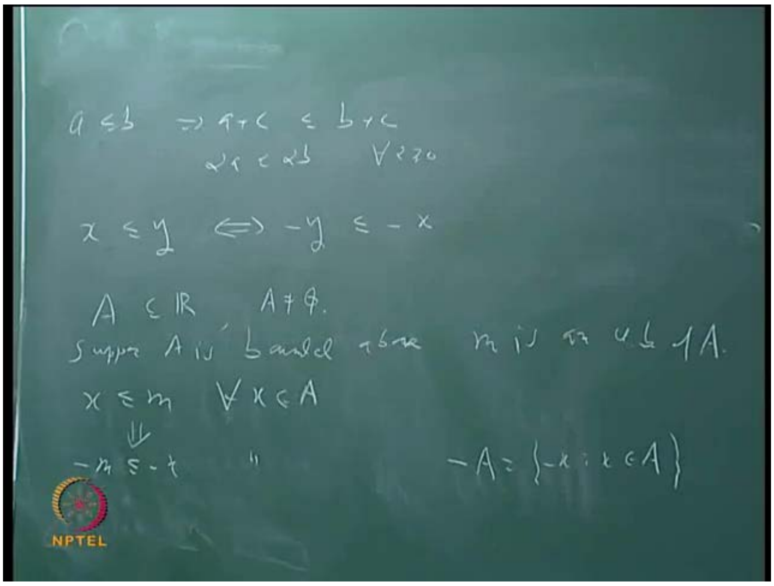

For example we said that if a is less than or equal to b then this implies a plus c is less than or equal to b plus c and similarly if alpha a is less than or equal to alpha b for alpha divide or equal to 0. Using these you can prove the following that is suppose you take say two numbers x less than or equal to y suppose you take two numbers x less than or equal to y.

Then this is equivalent to say that minus y less than or equal to minus x that is clear in fact that follows from those properties try to try to prove this yourself try to prove this yourself from the properties that we are listed. But, let us say suppose this is clear to you suppose this is clear to you. Then suppose say it is bounded above let us say let us say A is a subset of R and A is non m suppose A is bounded above suppose A is bounded above A is bounded above what does it mean. It exist some of our bound suppose that upper bond is a let us say a is bound above and m is an upper bond of A. Then this means that x is less than or equal to m for every x in p.

Now, this is same as now look at use this is this same as saying that minus m is less than or equal to minus x for every x in a right. Now, suppose instead of A I consider the set minus A minus A means what take minus x for every x I will just use this notation. So, suppose I look at minus A then it is take minus x for x in A it is clear if a is non empty minus a is also non empty because if some x belongs to A minus x belongs to minus A. If A is bounded above what will what you can say about minus A minus is bounded and if and if n is an upper bound of A minus n is a lower bond of A and. So, similarly if something is least upper bound of A.

You can similarly show that that is the greatest lower of bond of minus A and wise versa see suppose a set is bounded below. Then in the same way you construct minus A is bounded above for example here if I started m less nor equal to x then here we would have got minus six this is go to maximum. So, that if that minus m would have been upper bound of minus x. So, you just basically use that and get this from that suppose a is an non empty subset.

Suppose let us say A is non empty subset that is bounded below then minus A is bounded above use this to get this upper bound and minus of that will be the greatest lower bound for this right that is it or. Similarly, you can go from here to here. So, these two properties are essentially the same alright then I should also make one more comment this is a technique in the proof which we shall be using very often.

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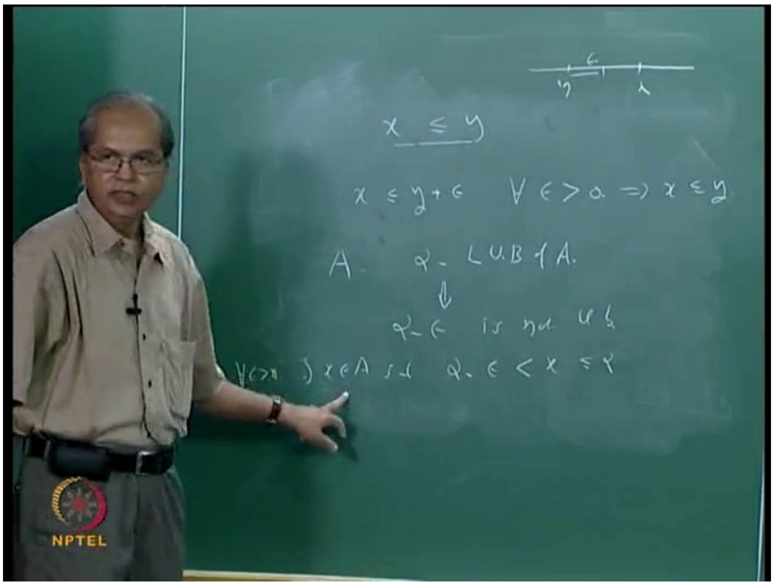

Suppose there are two numbers x and y x and y and we want to show this that x is less nor equal to y suppose we want say x is less nor equal to y and suppose. We cannot show it directly it is very difficult to show and suppose we are able to show this that x is less than or equal to or x is less then y plus epsilon for every epsilon bigger than 0. Suppose we said x is less than y plus x epsilon for every epsilon bigger than 0. Then will it imply this is it clear to everybody how does let us let us see I think it is not clear to many of you how does this follow see suppose this is false. Suppose this false then what should happen y is strictly less than x I thing let us use this joint that means his is y.

And this is x suppose this falls then y should be strictly less than x can I take some positive epsilon since that y plus epsilon is less than x foe example I can take epsilon x minus y by 2. Suppose I take this number as epsilon then this y plus epsilon will be less than x right. So, here in fact I can here even take this even if we are what is that x is less nor equal to y plus epsilon for every epsilon bigger than 0 it will imply x less nor equal to y right. So, this is a technique or the proof which is use very often in right then one will. So, this will imply x is less nor equal to y one more thing suppose a is a let us say A is a non empty subset.

And let us say let us say the number alpha is the least upper bound of a alpha is A least upper bound of A it is a least upper bound means what anything less than that is not an upper bound right if I take any numbers smaller than that is not an upper above. So, which means for, so alpha least upper bound this means alpha minus psi loam is not upper bound for any psi loan bigger than zero. But, if something is not in the performance is what, so is this some element in a which is strictly bigger than this. So, that means they are exists x in a such that alpha minus epsilon is strictly less than x and of course x is less nor equal to.

Alpha if alpha is a least upper bound for every epsilon we can set for every epsilon bigger than 0 there exists x in a such that alpha minus epsilon less than x in fact that this part is trivial because. Once it is an upper bound x is less than in that we are not saying anything new, but this is an important. And this what we shall be using in several proofs see if alpha is an upper bound of A then given any epsilon you can find some number in a which is strictly bigger than alpha minus epsilon.

And similar statement you can make about the greatest lower above for greatest lower bound exist any number bigger than that will not be an lower above. So, you can always faded number in a which is strictly less than that I will not go to that statement that statement you can write on your own. But, this is something which we shall be using very frequently in several proof in the analysis.