Real Analysis
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Lecture - 5
Uncountable Sets Cardinal Numbers

So, we will continue our desiccation of countable and uncountable sets. So, let us quickly recall that in the last class we in the we proved five thinks about countable sets namely that first of all sub set of countable is countable. Then union of countable family of countable set and using that we also set the some of the familiar sets like this type of set integer is countable set of all set of all rational number is countable. Now, let us proceed with that I will know introduce one more notation.

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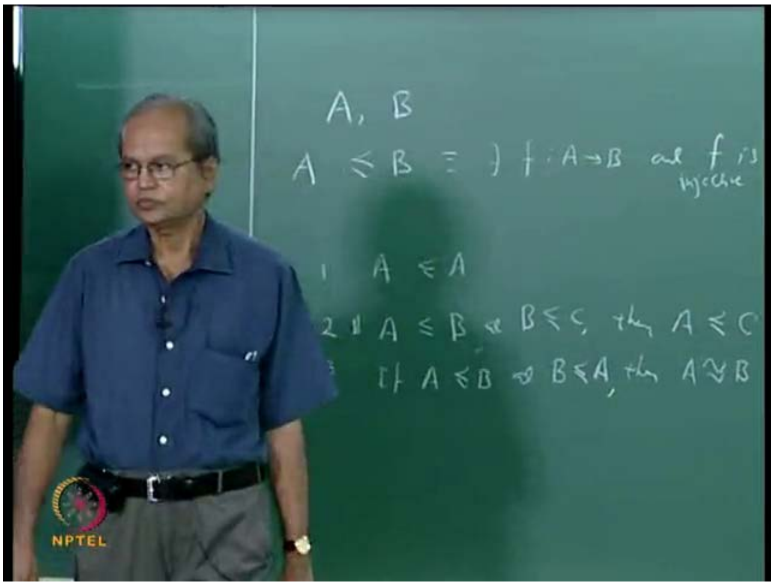

Let us say we have two sets A and B then we shall denote this lets say this is let us A is dominated by B. So, A is dominated by B, so or if this means there exists f from A to B and f is injective. Injective means as you know one one if may or may not be on to if may or may not be on to f is on to also. Then we know this means a numerically equivalent B. So, what so you can intermit this as number of element a number of element in a less than or equal to number of element in B that is the rough determine. Now, what can see about this relation properties of this relation first of all we can say this relation obviously A is dominated by a you can take f is anti function.

Then second think we can say that if let us say if A is dominated by B and let us say B is dominated by C then A is dominated by C that is also clear. So, if f is function going from A to B which is one one and if g is function going from B to C which is also one one just g from B to C to compose with f that will be function from A to C also one one. So, that clear right third property if A is dominated by B and B is also dominated by a then yes same body say some think then kind of A and B. That is then right A is a numerically equivalent to B. So, this last think not obvious what is mean suppose you have injective function have going to A to B, and similarly injective function going from B to A close to f just not same.

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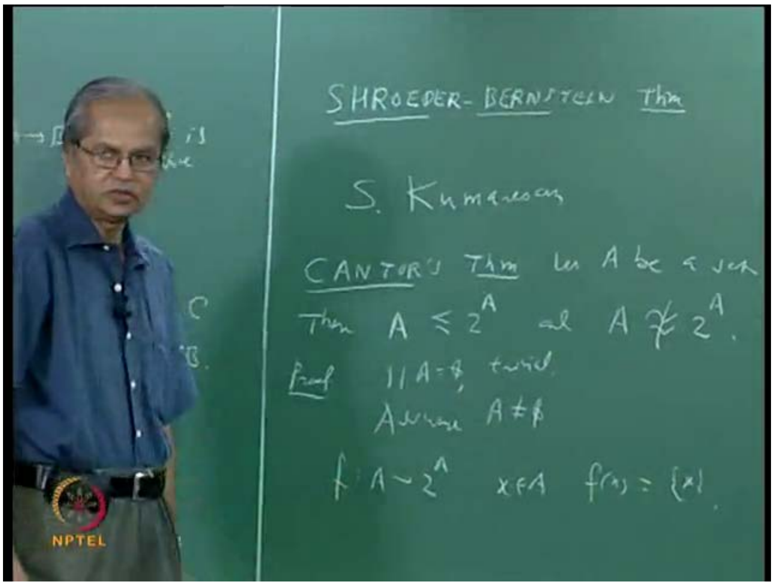

Then there exist some bijective function from A to B that is something not obviously us at all and in fact there is that is very well on the theorem in set theory its very famous theorem it is Shroeder Bernstein theorem. And what Shroeder Bernstein theorem what I said just now if you take two sets A and B if there is one one function from A to B and if there is one one function from B to A. Then there exist one one and on two function from A to B that is Shroeder Bernstein the proof this is same what lengthy. So, we will not discuss here those who are interest you can see proof of it in semen book I have mention semen book in beginning. It is even there because one more source in which you can find fairly good proof of this.

But, else many of you heard Professor S Kumarasen quite famous for this MTTS program some you may have attended also. So, he is a professor in university of Hyderabad Mathematics department. Mathematics department Hyderabad you look at home page in university of Hyderabad and that home page contains some popular article and one of article is proof of Shroeder Bernstein theorem. You can find it there or also there is an MTTS home page, which also contains some popular mathematics articles there also you can find all right.

Now, let us again go to one another famous theorem in this here by the by can you see this three relationship if you look these three properties of this relation they look something like this is not equivalence relation. But, this is some think like partial order again not an exactly partial order because in partial we would require this happen then we should have A equal to B. So, it not A equal to B we have got A is numerical equivalent B. But, what we can say is that it is partial order see suppose you look at this equivalence relation among the family of all set that equivalent relation will split, we partisan the family of set in to equivalent class and what are the number of equivalence classes.

Those are the member which numerically equivalent each other right suppose we take set of all class all those classes on that class this is partial order because. Then the equivalence class contain this what is say this is equivalence class containing A is same as equivalence class containing B ok then that is go let go farther. So, far we have not seen any example an uncountable set to do that let us again go to famous theorem which know as Cantors theorem this theorem say let A B initiate A B initiate.

Then consider the power set of A that is set of all subset of A then first of all A is dominated by this power set A we shall use this notation two power A I have mention earlier 2 power A or script p of A. This two notation very commonly used the set of all subsets of a let use this and it is not equivalent that second part is very important first part is more or less trivial what does it happen function. Since it means that exist one one going to a to power set of A, but there is no bijection there is no bijection between these two set.

So, let us go to proof no first of all this is clear to at this whole thing is clearly if a is an empty set if a is an empty set or set will contain one element and set itself contains no element. So, there can be no bijection between and one one function will be a trivial function, so that case let us forget about. So, if A is empty it is trivial nothing to big, so assume A is non empty. So, if the first part we need to show that there exist an injective function from A to its power set that is given any element let us say x in A we want to construct function that is f going from its. So, suppose we take x in a this x f of x should be some sub set A f of x should be some sub set a can you see there is obvious choice for this single since containing element x right.

That is most obvious function think properly, so if f of x. So, this is function of A to x f power set that function is one one as clear is one one this part is proved. So, this part is proved ok. Now, we look at this here what is we have to prove that a is not numerically equivalent that means there cannot exist any bijection between this two set right.

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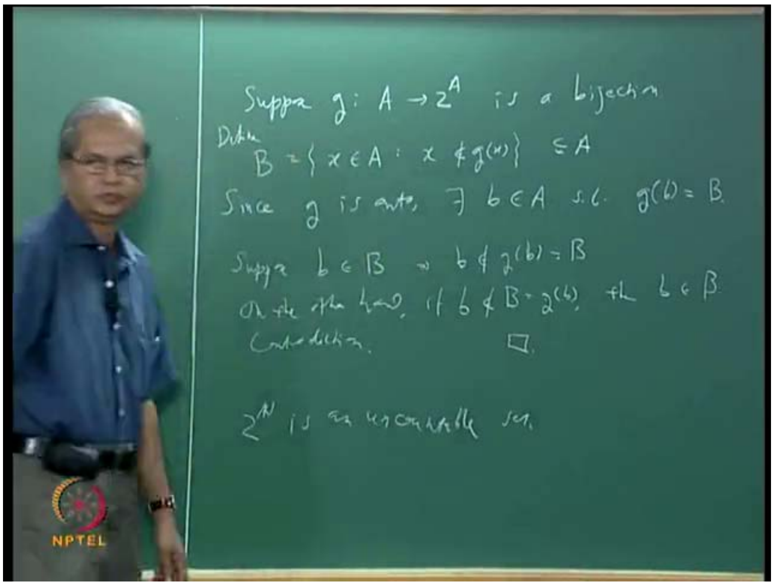

So, the proceed is fairly straight forward assume there exist bijection and get contradiction right. So, suppose g of A to power a is a bijection in fact show the cannot exist on two function from A to its power set of function. But, that all now we will by the by this proof is also given one more proof is also is given Cantor is a famous very famous German Mathematician who have done several think with analysis. And set theory you will hear this name again and again by the way Shroeder and Bernstein these two are also famous German Mathematician.

So, suppose this is on two function think of I will think of set follows B as follows it is set of all in A since it is set of all x in A such that x does not belong to g. Remember g x subset of A x is function from sorry g is function from A to its power set, so every x g of x subset of a right, so x. So, given any x may not belong to g x, so you pick up those x for which x does not belling to g x I am not saying within this set is empty or not exist or not whatever is that take all x for which x does not belong to g x. You call that set B alright, now this g is bijection g is bijection and this B is sub set of a right. So, what follows from that there must exist element such that g of that element is this we can see that in fact for this all g is on two.

Since g is on two, so I will say there exist suppose I call element small b there exist small b in a such that g of this small b is equal to B that is fine ok. Now, we ask question what can we see about element b does this element b belong to B of course, b is subset of A and b belong to A. So, b has to b either b or outside b alright let us say what happen suppose b belongs to B that means what that means what that means what this implies B does not belong to g of B g of B right, but what is g of b. So, look at this what is what see if b belongs to B then b does not belong to B that is contradiction that is contradiction right what is other possibility.

Suppose b does not belong B suppose b does not belong B that will give b belongs to B because that is how on the other hand other hand if small bits does not belong to B which is nothing but g of b. Then the way in which you have define the b it means B must belong to B small b must belong to B then small b must belong to B that means B belong to small b B belong to b and small b does not belong to B both are in to contradiction. Such think cannot happen B is element of A and small b is an element of A big B is sub set of A.

So, small B has to be added either inside B or outside B and here we are seeing both the statement are let in to contradiction and again what is source of this contradiction this we assume that there exists bijection. So, that must be false, so this is contradiction. So, this completes the proof is this clear this is Cantors original proof of this theorem.

So, let me again compact to this set stamen of theorem that given any set a of course this is this is part A is dominate by x power set. But, A is set not numerically equivalent power set of A and it power set or there can no bijecton between any set and it power set does it immediately give example of uncountable set does. These have immediately yes what is that this do with this theorem right. That right that is suppose we take a asset of natural number suppose I take a asset all natural number then n and 2 power n there can be no bijection right. There can be no bijection and it means that 2 power n is uncountable set right.

So, this immediately give that 2 power n is uncountable set, so we have got example of uncountable set in fact it can be shown that this 2 power n is actually numerically equivalent to real number. This 2 power n is actually numerically equivalent numerically equivalent to real number. Let me give you some idea about this how show this I just give you some step in this may not be the whole think ok.

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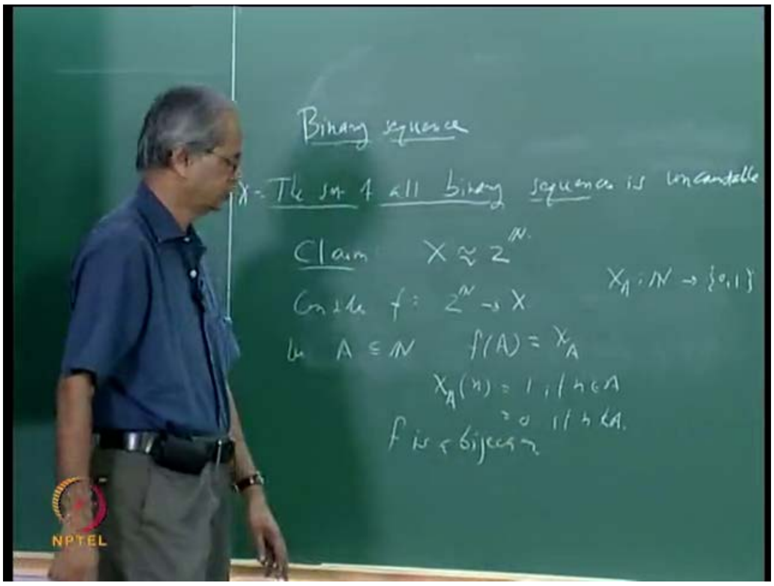

You have heard of this term binary sequence it is a sequence whose term 0 and 1 sequence is what sequence obviously function from the set of natural number. So, any sequence how term just 0 and 1 those are called binary sequence right. So, what I want to say is that suppose you take set of al binary sequence suppose you take set of all binary sequence then that it is uncountable, so set of all binary sequence is uncountable. Of course there are several way of seeing this, but one way of seeing that is that we can say that this set is nothing but this set this set is nothing but this set of all binary sequence nothing but in the since it is numerically equivalent.

And let us give some name to this suppose X is set of all binary sequence let X denote the set of all binary is sequence you want say X is uncountable. So, I want say this right this is claim mean this some think I want to show claim X is numerical equivalent to 2 power n i I will take I will take map from here from here to here.

So, consider f from 2 power n 2 X that is given sub set of n I want contract A binary sequence, so let A be a sub set of n and define f of a define f of a you all heard by what is mean by characteristic function of set. So, f of a define as characteristic function what is this mean that if it is it is one if number belongs to a there is remember this let we repeat this X and sub natural number right. Its function see its function from n to n because its its function from n to n and not n to n, n to set 0 1 binary. So, we define n is equal to 1 if n belong to a 0 otherwise that is 0. If m does not belong to n for example if a is set of all even number then the corresponding sequence is 0 1 0 1 extra similarly we can see. So, is it clear that this characteristic function.

Basically binary sequence characteristic function is function if function going from n to see each characteristic function is function going to this set zero and one it value is zero and one. So, its binary sequence is this function one one that is that is given sub set let us say if there are if the characteristic function of two set co inside right. That is when does something is one one suppose f of A is equal to f of B that I same as say in the characteristics function co inside imply that A is equal to B. That is clear is it on two that means given binary sequence given binary sequence can we construct sub set of n who characteristic function is given by binary sequence. That again clear you take those n for which that that is look at those n for which the value of the n collect those n.

And take that is set a take that is set A that will be sub set of n. So, this map which take A to its characteristic function is bijection. So, f is a bijection that true is this right, so f is bijection we already seen that this is this is an uncountable set. So, x is uncountable set also see one more proof of this the set of all binary sequence is uncountable. That is again that discuss it because one of the well known method of something is uncountable that is if I say this countable we already know it is infinite then you can arrange it element in the form of sequence right.

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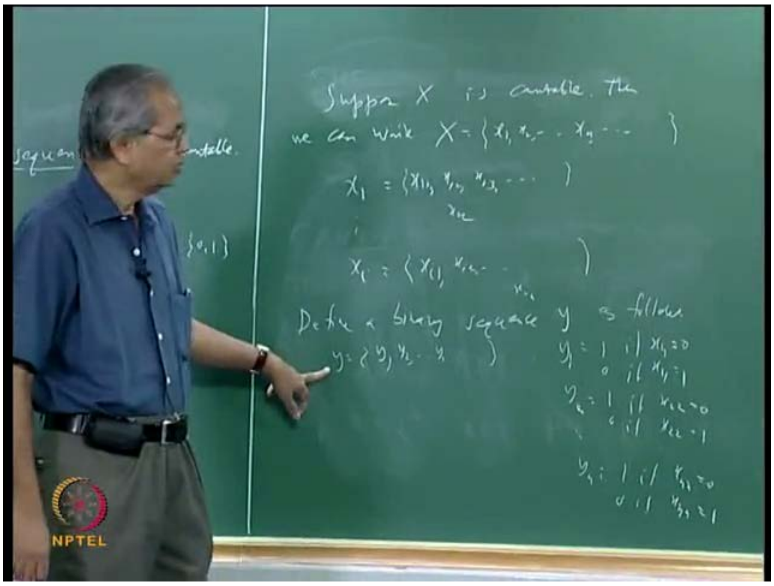

So, suppose X is countable suppose X is countable then we can write then we can write X as x 1 x 2 extra. Remember each of this x 1 x 2 is sequence right each of this x 1 x extra is 2 is sequence. So, let us have some rotation for this for example what is sequence x 1 the sequence x 1 I will denote it by denote x as x 1 1 x 1 2 x 1 3 etcetera remember each x 1 1 x 1 2 as an 0 or 1 each of this element extra either 0 or 1.

So, similarly let us say I sequence x i, I can denote it x i 1 x i 2 extra and show that this it suppose this will case it will be in that you can list all by binary sequence that is in this fashion. And what we want show is that cannot done. So, suppose I construct sequence which I different from all this then it will be in that X is not countable right.

Now, define binary sequence X as follows binary sequence x as follows I will use some other notation because X is. So, I will call that sequence Y define binary sequence Y as follows Y equal to y 1 y 2 extra and I should say what is y 1 what is y 2 what is y 1 what is y 2 extra what I say a as follows take y 1 you look at x 1 1. If x 1 1 is 0 you take y 1 as 1. If it is 0 then y 1 as 0. So, y 1 as 1 if x 1 1 0 and 0 if x 1 1 is equal to 1. Similarly, you take y 2 same way look at this x 2 second sequence x 2 x 2 1 x 2 2 extra look at this entre x 2 2 here if x 2 2 is one you take y 2 as 0. If it 0 you take y 2 as 1. So, you take y 2 as 1 if x 2 2 0 0 if x 2 2 equal to 1 and now it clear how to proceed in this way.

Take the general entry by n as follows by y n equal to 1 if x and n is 0 that is suppose this is x n may be x n n and 0 if x n n equal to 1 then Y is binary sequence. But, since y i s binary sequence it must be one of this x 1 x 2 x 0 right. But, you can see that it can be x 1 because y 1 is different from x 1, but it cannot be x 2 because y 2 differ from x 2 it cannot be x n because y n different from x n. So, y cannot be any of this and still its binary sequence right.

So, this is contradiction and this contradiction we have got because of what because we assume x is countable and read the element of this form x 1 x 2 that cannot be done this is also fairly standard technique of proving that is say. It is not countable and this method is known as diagonal method of proof and you can see reason why it is called diagonal method. That we are arranging the element in same think like in the form of matrix and looking at diagonal entry and then constituting new sequence which differ each of diagonal entry that is why it is called diagonal method. Diagonal process is this clear now you all know that every real number has decimal expansion can be express in terms of decimal explanation you also it can also be express using binary expansion right.

Decimal exposition is just one chose it can also express using binary number just 0 and 1. So, what we can say is that set of all binary sequence is nothing but the set of real number you take any real number and take its binary expansion that binary sequence, so for right. So, each binary each real number you can associate binary sequence which is nothing but binary expansion right. And similarly if you are given any binary sequence you can associate real number with that only problem is that each of these entry you take integer part each of this you take fractional part that will problem to decide. But, let us say we take only those number lying between 0 and 1 let us just take the number lying between 0 and 1. Then this problem will not be there is there is no integer part.

So, you can say all, so suppose you are given any binary sequence like that you let us say some sequence 1 0 1 0 you can take that number 0.1010 etcetera that is the binary expansion of the given number. So, in other words this sequence x is it is numerically equivalent to X this sequence x is numerically equivalent to X.

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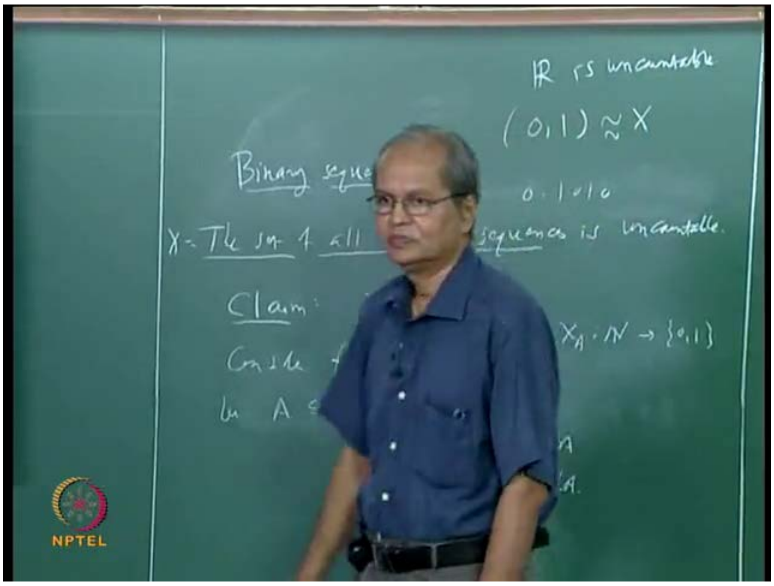

So, what does that proof the interval 0 1 uncountable it is numerically equivalence of all set of binary sequence which we have already show uncountable. So, that is uncountable all right now we have proved in the last class that is sub set countable set is countable does it also follow from the immediately that if A is set. And it is an uncountable sub set then the A itself is uncountable right.

Suppose set contains an uncountable sub set then the hole set itself is uncountable right basically same statement say in different method different lounge. Now, if this is uncountable that R is uncountable it means R is also uncountable in fact you can show some think bore if imply to say that R is uncountable. Then this is in ff pick up same subset that is uncountable one show that. But, we can show same think more it is following I will give that you as an excises you take any interval of arc then you show that that arc open numerically equivalent to whole of part it is in particular as an exercise.

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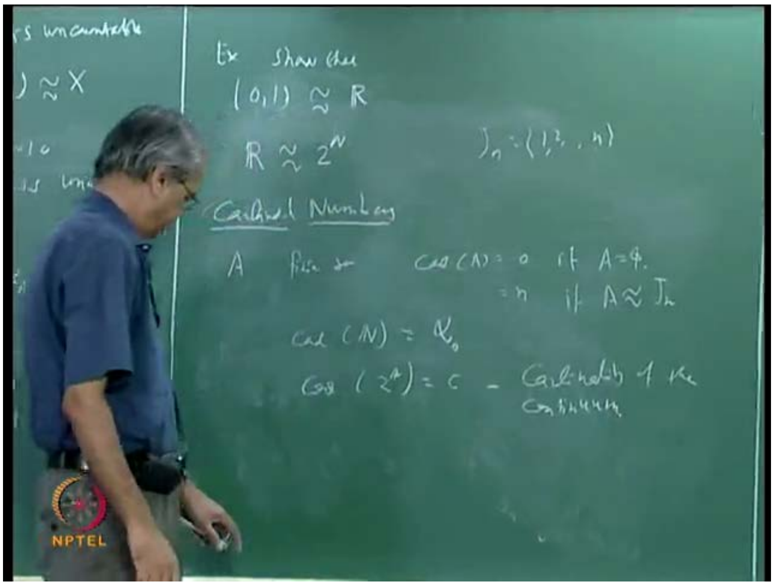

Show that if you can do this for this interval you can do it for any interval. So, that what does it mean that show that there exists bijection from open interval 0 to 1 to the whole of R element exist try find this function on your own. At build you can do this you can show can show that nothing particular about 0 and 1 you can take any open interval and that any open interval is numerically equivalent to R. An in particular any to open intervals are numerically equivalent to each other ok. But, once you show this it will be in that R is numerical equivalent 2 power n that clear right we have already show the this is numerically equivalent to x and x is 2 power n and this numerically equivalent to R.

So, comparing all this you can say that R is numerically equivalent to the power set of a all right. Now, I will just make five comments of what are known as cardinally number and then we shall close these dictions about countable and uncountable set excreta. Cardinal number I think I have mention it earlier also if a is finite set let us take if A is finite set Cardinal number is nothing but a symbol which we associate with every set. We call it cardinal number of that set cardinal number of set. So, suppose A is set let us say suppose A is n finite set we shall say cardinal number of A is 0 if A is empty.

And this is equal to n equal to n if A is numerically equivalent this j suffix n we said A is finite that is a is either empty or A is numerically equivalent to remember what was j sufficient. It was this set 1 2 3 up to n segment consist of first natural number in other word this n coordinal number of finite set is nothing but the number of element in that set it nothing but number of element in that set. Then Cardinal number of all this set of natural number that is denoted byte this symbol it is called alpha naught it is radius Alpha naught Aleph suffix 0 Aleph. And then cardinal number associate this set with any this set r 0 1 or 2 power n extra whatever I will take this 2 power n that is usually taken as symbol small c and that is called cardinality of the continuer that is called cardinality of the continuer. Now, if you take various Cardinal number then we define relationship between them suppose A and B are cardinal number ok.

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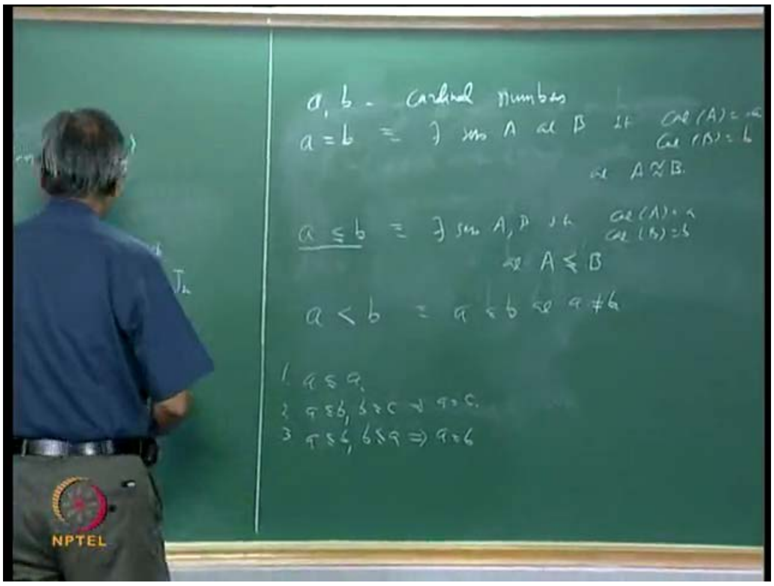

Focuses one thing is clear A is equal to B happen which case A equal to B means those are associated with two set which are numerically equivalent to each other that is A equal to B means let us write it in full form it means there exist set A and b. Such that cardinal Carnality of A is small a carnality of B is small band a is numerically equivalent to b. So, if the two set are numerically equivalent the cardinal number associate with them is same. Let us now also see what is being of this a less than or equal to b again it means that there exist set A B such that cardinality of A is small a cardinality of B is small b.

And a is dominated by b that means there exist set A and B and there injective function f going from A to B which is one one and corresponding number are small a and small b. And similarly one will also like to define what is mean by a strictly less then b this again as usual we will say that a is less than or equal to b and a not equal to b in term of set what does it means. It means there exist set A and B such that carminative of big A is a carminative of big B is b and A is dominate. But, not numerically equivalent b in fact there exist there two set with this properly which are numerically equivalent to each other that is meaning saying that a is strictly less then b now if you look at this set of all cardinal number.

Can you see this is partial order for example, we can say see this property that is a is of course a is less than or equal to a that is clear a is less than or equal to a then second think is that a is less than or equal to b b is less than or equal to b. That implies a equal to c and lastly a is less than or equal to b and b is less then or equivalent to a now this time I can say a is equal to b. And then last follows from the Shroeder Bernstein thermo because a is less than or equal to b b is less than or equal to a what that means.

There exist two set this A and B that is carminative of A is small a carminative of B is small b and is dominate by b and b is also dominated by a then in Shroeder Bernstein thereon a and b must be numerically equivalent which is same as saying small a is equal to small b. Now, among the set of all coordinal number this relation less than or equal to this is partial this relation is partial order in those number.

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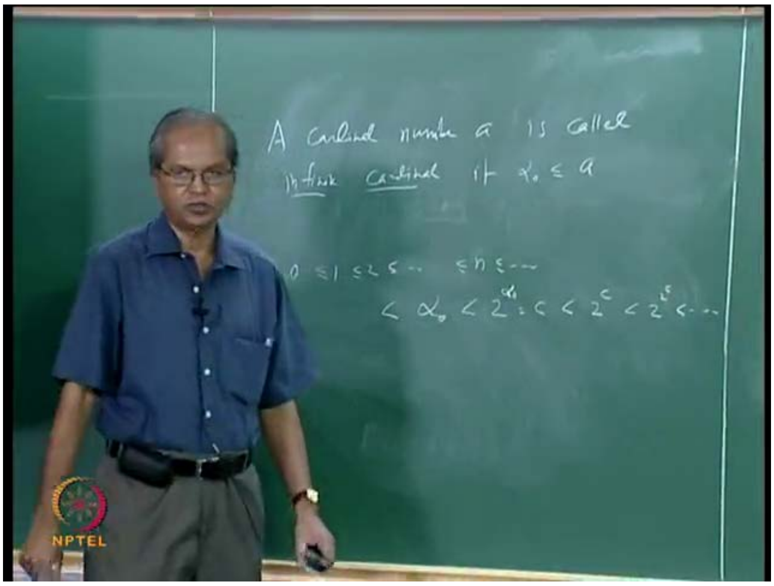

Now, let us take one more definition suppose A is a cardinal number a this is called infinite cardinal infinite cardinal if this number alpha naught less then or equal to is infinite cardinal. Now, coming back to this relationship for example alpha naught is itself is infinite cardinal C is an infinite cardinal n number because of equal to C is also any cardinal number because C is also infinite cardinal. One more thing is that here is that in view canards thermo for any cardinal number a is less than 2 power a and what is mean that 2 power a if this a is associated with set A cardinal number of a then 2 power a is the power set A 2 power set of a because that is what follow very us definition we have seen so far all right.

Now, we can see relationship between whatever number that we have see. So, it will try to arrange all of particular order they being seatrain particular order smallest cardinal number of course 0 then 0 that is less than or equal to 1. That is less than or equal to 2 extra n that takes into finite coordinate number those are same as the 0. And other natural number then all this finite coordinate number those are strictly less then alpha naught what it means given any finite set you can have an injective map that side to n. But, no on to map no bijection, so that is same as already finite coordinate strictly less then alpha naught all right.

And then what by observe there alpha not is strictly less than 2 power alpha naught right. But, 2 power aleph is same as c that is what we have see here that is what we have see here or easel R is numerical equivalent 2 power a cardinal number associated with n is alpha naught that is that associated with or aleph. So, which is safe same as this 2 power alpha naught same as c right. Then c will be again strictly less than 2 power c what is 2 power c it is you take set of all sub set of R whatever the cardinal you will associate with that that will be two power c. And similarly you go of, so 2 2 power c will be less than 2 power c sorry 2 power 2 power c extra then that will less than 2 power 2 power c that way you can go on no this lead to one very natural question.

Now, this is fine alpha naught less than 2 less than 2 power c that is c is less than 2 power c the also fine the question is do the exits any cardinal number lying between these 2 do there exist any cardinal number lying between this two. For example between alpha naught n c in between c and 2 power c, so let we make that question exercise that is of course that quarto is trivially take here if it finite cardinal then you can easily find number like n and 2 power if it finite cardinal. Then you easily find number between n and 2 n that is number it is not clear about infinite cardinal.

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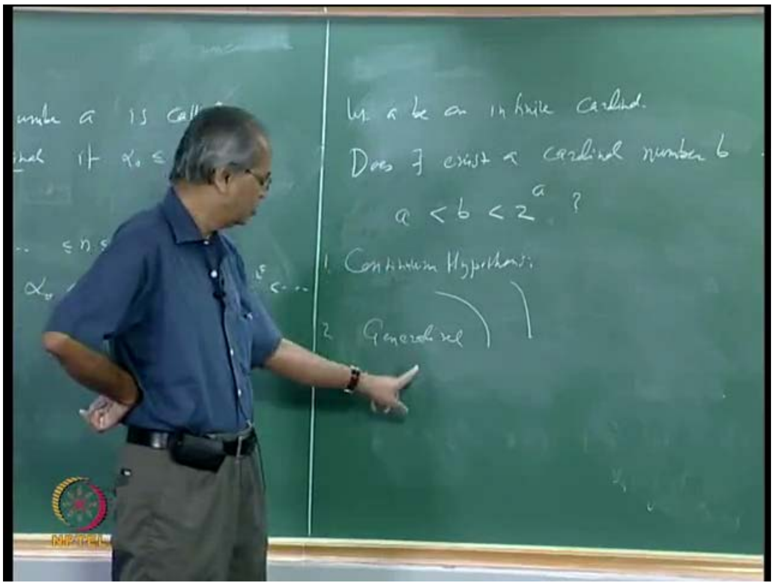

So, let me guess let a b infinite coordinate infinite coordinate does there exist does there exist cardinal number b. Such that a is strictly less then b then b is strictly less than 2 power a of course this is question in fact it is easier to understand. This question if you formulate it in terms of set suppose you are an infinite you are an infinite set. Suppose you are in infinite set contains constable sub set. So, the coordinate of infinite set will be bigger than bigger or equivalent to alpha. So, if you take any infinite set its cardinal number will be infinite cardinal. So, suppose you take any infinite set and you take its power set right then cardinality of that set will be a coordinate of power set is 2 power.

So, asking whether they exist B with property given infinite set and it power set can you find subset which is bigger than which is bigger than or equal to given set that means this can you find B, such that A is dominated by B, but not equivalent B and B is dominate by 2 power a. But, not equivalent 2 power a that is what in teams of set A and such nobody no the answer to this question till now this is open question in set theory. The answer is not known and there are some special case of this they have some very special name one of them is continuum hypotheses it is called continuum hypothesis.

Continuum hypnoses assumes negative answer for this when a is alpha naught in other word continuum hypotheses says there is no cardinal lying between alpha not. And see in term of set there is no set which strictly bigger then set of all natural number and strictly less then set of all real number that continuum hypotheses. And similarly there is another think which is called generalize continuum hypnoses.

It is let me simplify continuum hypnoses say the answer to be known for this hole question answer to be known for this question. And what is known about these continuum hypnoses is that using the other action slandered action this set theory you cannot prove disprove continuum hypotheses. Now, this some think you may be find it difficult to understand this is go to with logic what it means that. If you assume that continuum hypotheses is true that is quite consistence with all other action of set theory you can developed certain kind set of theory. And you can develop certain kind of theory assuming that continuum hypotheses is true at the same time if you assume that continuum hypotheses is false.

Then that is also consistence with all the other action of this set theory and you can develop some other kind of model of set theory assuming all other actions is false. Now, this is express by saying that continuum hypotheses in independent of all other action of set a that is using other action of set theory you cannot prove or disprove continuum hypotheses right that is the. So, that is the answer know about discounted, but even that is not known about this generalize whether even that is the case not is not known about this generalize continuum hypotheses all right. So, this I will close discussion of countable and uncountable finite set for the time being. So, the next class go to the next topic.