Real Analysis
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Lecture - 48
Uniform Convergence and Integration

So, we have seen so far what is meant by saying that sequence of functions or a series of function converges either point wise or uniformly and we have seen that uniform convergence has many desirable properties. For example, what we have proved in the last class is that when each f n is continuous the limit and if f n converges to f uniformly then f is also continuous.

We have also seen that if a sequence converges uniformly it also converges point wise, but the converse is false and there is one theorem which puts some additional conditions along with the point wise convergence. That implies uniform convergence and that is a theorem which we shall discuss today, before proceeding further it is a fairly well known theorem.

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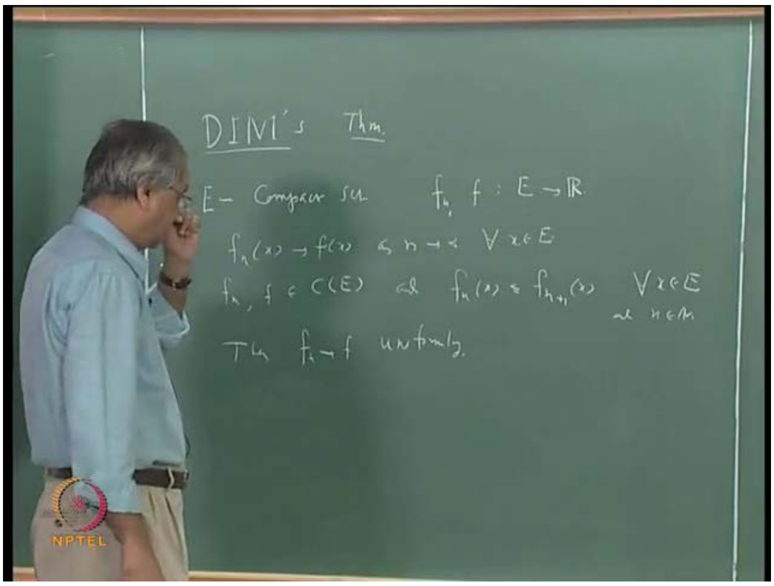

It is known as Dini's theorem and, so there are few additional things till, now whenever we talked about convergence of a sequence of functions either point wise or uniformly we did not say anything about the set E on which the functions were defined. But, now we shall require those things we will have to assume that E is a compact set in some metric space, E is compact set in some metric space. You can take it as a compact metric space and f n and f these are functions from E to R and f n converges to f point wise that is will say f n x converges to f x as n tends to infinity for every x.

So, we want to say under what additional conditions f n converges to f uniformly of course this one additional condition is already there compact set. But, that is not sufficient we will have to put a few more things first thing is that we have to assume that f n and f all of them are continuous that is, that is f n and f those are continuous function. That is f n each f n is in continuous, f is also continuous and this f n is a monotonically increasing sequence or monotonically decreasing sequence. It has to be a monotonic sequence if we assume that, then we can show that f n converges to f uniformly.

Let us take one of the cases suppose it is monotonically increasing and let us say that f n x is less not equal to f n plus 1, x for every x in E and n in n then f n converges to f uniformly. This is the theorem that is what are the additional requirements, E is a compact set f n as well as f they are all continuous functions and f n is monotonic. Here, we have taken monotonically increasing sequence, but a similar thing can proved if you take a monotonically decreasing sequence.

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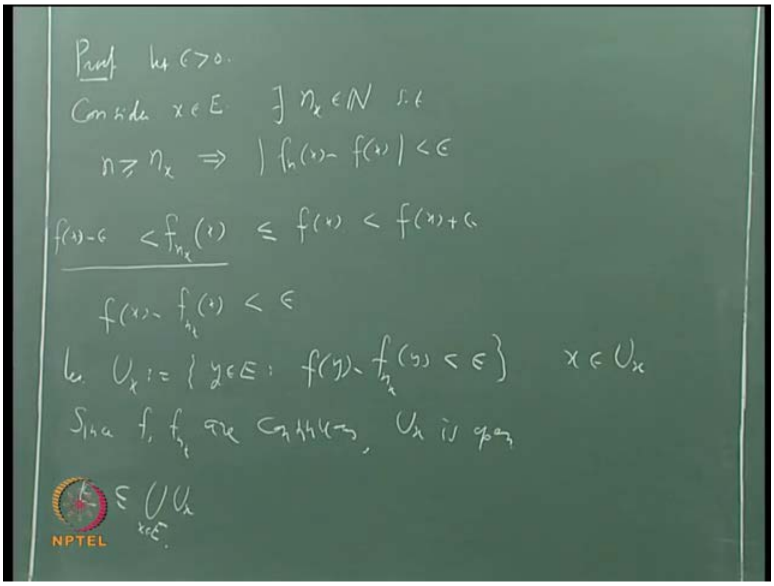

Let us, now look at the proof, let us take some x in E, now again you want to proof f n converges to f uniformly means by definition what we have to do is that given epsilon we have to produce some n 0 such that whenever n is bigger than or equal to n 0, mod f n x minus f x is less than epsilon. That should happen for every x in E that n 0 should work for every x that is the idea, so this such a proof has to start with let epsilon be bigger than 0, let epsilon be bigger than 0. Let us consider some x in E, consider some x in E then since we know that f n x converges to f x corresponding to this epsilon there will exist some n 0.

So, that where ever is n is bigger than n 0 whatever happens we say will happen, but that n 0 will depend on x because we have only assumed the point wise convergence, so let us call it n suffix x. So, we can say that there exist n suffix x in fact our whole point is to find n 0 which does not depend that is the whole idea of the proof. So, starting initially we do not know that, so let us take there exist n suffix n in such that n bigger than or equal to n suffix x this implies mod f n x minus f x less than epsilon. Now, let us also use the fact that this is monotonically increasing sequence for each x f n, x is a monotonically increasing sequence of real numbers.

We know that a monotonically increasing sequence converges to its supremum monotonically, so each f n x is less not equal to f x and f x is the supremum of all those f n x. So, let us rewrite this and I will use for this number n suffix x, I will take is equal to n suffix x, so what do we know this n suffix x we of course we know that f n x at x this is always less not equal to f x. But, because f n x minus f x is less than this, what this means is that f n x lies between f x minus epsilon to f x plus epsilon. So, this must be f n x also must be bigger than this is f x will obviously less than f x plus epsilon, but this will be f n x will be bigger than f x minus epsilon f n x is bigger than f x minus epsilon I will just look at this part of the inequality.

That is, that is the one which is useful, now and what it means is that f x minus this f n suffix x at x, remember everything is happening at x this f depends x I am taking that particular function f suffix n suffix x evaluating that at x, evaluating n at x, evaluating f at x. What follows from this is that the difference between this is less than epsilon, difference between this is less than epsilon. Now, we know that each f n as well as f is a continuous function each f n as well as f is a continuous function. So, what I will do is that I will take the set of all those num points in E for which this happens that is what is by I will, I will call that and that will of course, depend on this x that will of course depend on this x.

So, suppose I call that let us say U suffix x, what is this U suffix x, this is defined as let us say set of all y in E, set of all y in E such that f at y minus f suffix n suffix x at y. This should be less than epsilon, is it obvious that x belongs to this x because that is what this says if you take. So, first obviously that x belongs to U suffix x is it also clear that U suffix x is open this y is less than epsilon that is it is inversely match as under the continuous function f minus f n x of the interval.

You can take minus infinity to epsilon of course actually 0 to epsilon, so it is an inverse image of an open set under a continuous function. So, this is an open set, so U suffix and that is where we are using a continuity of f n and f suffix n x. So, what do we say that since f n f suffix n x are continuous, U x is open, U suffix is open and non empty of course because x belongs to u suffix x.

What did we do given a point x in, we have constructed an open set containing that open or we just call open neighbourhood of x. Then what is to be done after this because this is a standard compactness argument of that is we have started with e as a compact set cover E with all such sets. So, that, so E is contained in, E is contained in union of U suffix x union is taken over x in E, in other words what it means is that.

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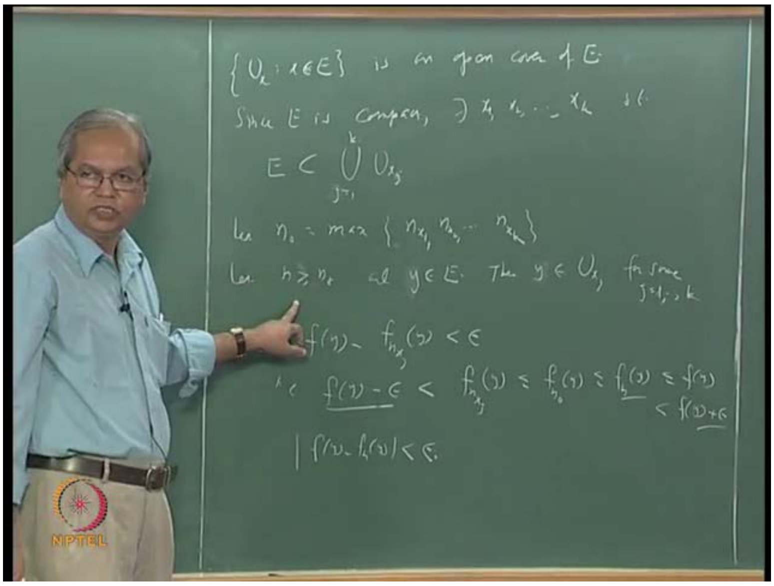

This suppose it take this family U suffix x, x belonging to E this is an open cover of E, this is an open cover of E, yes why it is an open cover obviously each U x is, U x is open set and every x in E is contained in some U x. So, you collect all such U x, E will be a subset of this union, so that means is open cover of E you can make a slight change in this definition. Suppose, E itself is you are taking as a metric space then you can take these, otherwise you can take suppose E is a subset of some metric space x instead of taking y belonging to E you can take y in that x, you can take y in that x. But, usually for compactness argument that does not matter, if E is a compact set in any metric space then it is a compact set regarded if you regard E itself as a metric space.

So, that really does not matter what follows after this because E is compact this should have a finite sub cover, since E is compact you can say its finite sub cover means there exist some finite number of points. Let us say x 1, x 2, x k there exist x 1, x 2, let us say suppose it happens up to x k such that E is contained in union of this U suffix x suffix k, sorry U suffix let us say x suffix j, j going from 1 to k, j going from 1 to k. Now, what you do is you look at the corresponding ns, for each x j you have n suffix x suffix j and then take n 0 as the maximum of all of them. So, let n 0 maximum of maximum of n suffix x 1, n suffix 2 etcetera n suffix x k then let us click, suppose we take any n bigger than equal to n 0.

Let us say what happens, let n bigger than or equal to n 0 then what we want to show is that for if you if you look at n f t for any n bigger than or equal to n 0, then we want to say that mod f n f n t minus f t is less than epsilon. Let n bigger than or equal to n 0 and t or x or whatever you take n bigger than or equal to n 0 and x in E if x is in E x is in one of this U x, 1 U x, 2 U x k because this is E is contained in the union of U x 1, U x 2, U x k etcetera. So, every point x in E is one of these open sets, so then x belongs to this U suffix x suffix j for some j, for some j, j going from 1 to K 1 of does, but if x belongs to U x j means, what it means.

That I think in order to avoid the confusion, I will change this notation, here instead of x let me take y, so that you do not confuse that with this x we start actually consideration of this x is over, here U suffix x does not matter. So, let us take some y in E then y belongs to U suffix x j, but what is the meaning of saying that y belongs to u suffix x j, look at what is the definition of u suffix x it set of all those y in E for which f y minus f n x y is less than now. Here, you have n suffix x suffix j, so y belongs to, y belongs to U x j this means f y minus f n suffix x j at y is less than epsilon let us rewrite this is, this is same as saying that that is f y minus epsilon is less than n f suffix n suffix x suffix j of y. Now, till now we have not used this fact only once we have used that f n is a increasing sequence, now for each n we know that f n x is less not equal to f n plus 1 x this is less not equal to f n plus 2 x. So, whenever n is less than m f n x is less than f m x, now what is relationship between this n x j and n 0 this is less not equal to n 0 is maximum of this two. So, I can say that this is less not equal to f n 0 of y and we have taken n bigger not equal to n 0 E have taken n bigger not equal to n 0. So, f n 0 of y this is less not equal to f n of y, can we always say that f whatever be y f n y is always less not equal to f y because again it is a monotonically increasing sequence, and f y is the supremum of all of all of those f n by and obviously f y will be less than f y plus epsilon.

So, what did we prove just look at this f y minus epsilon that is we taken n bigger than or equal to n 0 and y is in E, then f y minus epsilon less than f n y and then less than f y plus epsilon. So, suppose you combine these three inequalities is less than epsilon and this is true for every y and for the same n 0, remember we did not change n 0, this n 0 does not depend on y, once this n 0 is chosen as maximum of this that is fixed. So, if you take any n bigger than or equal to 0 and any y in E then this inequality holds f y minus f n, y less than epsilon and that is same as saying that f n converges to f uniformly.

This is what we wanted to show f n converges to f uniformly, so let me again remind you that what are the additional things that we required E is a compact set then f n and n. They are all continuous functions and f n f n is monotonically increasing sequence and each of these things we have used in the proof. For example, compactness we have used their f n is monotonically increasing that we have used, here and to show to say that this is an open set each of these are continuous functions all those things are used here. Now, if you have read the introduction of Seaman's book there is one remark that he has made, here that is whenever you learn a theorem each theorem has certain number of hypothesis.

Some conclusions that there will be a proof will have several steps and suppose you understand how n plus 1 step follows from n step. Suppose you understand this for each n that does not really mean that you have understood the proof completely. So, what is required is to understand what is the basic idea of the proof and what is the test for this test for this is that you ask yourself whether each hypothesis in the theorem is essential, whether it is required.

Now, how does one say that for example saying that this hypothesis used in this proof that is not a good enough answers it will only mean that this proof requires that. There may be some other proof without which does not use and still may be possible to prove the theorem without using the hypothesis. So, there is only one wants to settle such questions and what is the way you just drop that particular hypothesis written all other hypothesis and whether the conclusion still follows. If your answer no for that question there is only one way to settle this, you have to have an example, you have to have an example where all the hypothesis are satisfied.

Except the particular one which you are testing, which you are testing and the conclusion is false, and the conclusion is false. So, in order to under, suppose we apply that thing, here what are the additional things we have, we have I should t is compact of whether that is essential. We have to see, we have to do this by talking some non compact set whether f n and f continuous that is essential. Again, we have to see by dropping that, so certain examples we have already seen, let me just remind you.

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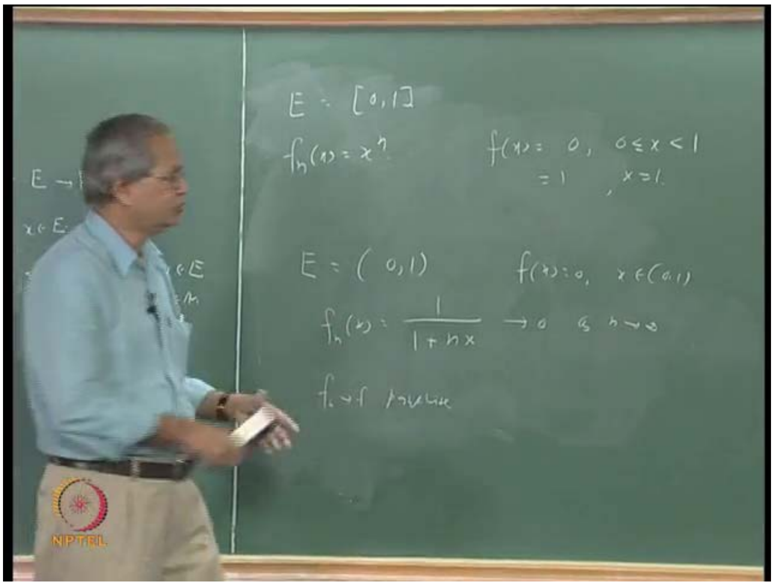

So, suppose I take E as 0 to 1, E as 0 to 1 this is in fact this is I think the starting example we have taken f n x as x to the power, n we have taken f n x as x to the power n and what was f x f x was 0 for 0 less not equal to x less than 1, for x equal to 1. Now, look at the theorem, here what were the hypothesis E is compact that is, now each f n is continuous that is fine f is not continuous, is f n monotonically increasing, is it not. But, it is monotonically decreasing that does not matter we have seen that whether it is increasing or decreasing, that does not matter is the convergence uniform we have seen, that it is not we have seen by several basis it is not uniform.

So, what does it mean that f n and f are continuous that f is continuous that hypothesis, you cannot draw that hypothesis you cannot draw. Let us take let us take one more example of similar type, suppose I want to check whether compactness can be dropped, so you take E as instead of this suppose I take E as open interval 0 to 1 then that is not compact. Now, suppose I take f n x of x as 1 by 1 plus n x, f n x as 1 by 1 plus n x those this converse point wise to what suppose you fix x and let n vary, let n go to infinity what will happen this will go to 0.

So, this goes to 0 as n tends to infinity point wise, so if you take f as a constant function 0 that is f x is 0 for x in 0 to 1.Then f n tends to f point wise f n tents to f point wise, is it monotonic suppose you fix x, how are f n plus 1 f n x related, it is it is decreasing, it is a decreasing sequence. So, everything else is satisfied except E is compact, is this convergence uniform, how do you settle that, I have told you one easiest way of settling the uniform convergence you look at M n as supremum of mod f n x minus f x in x n.

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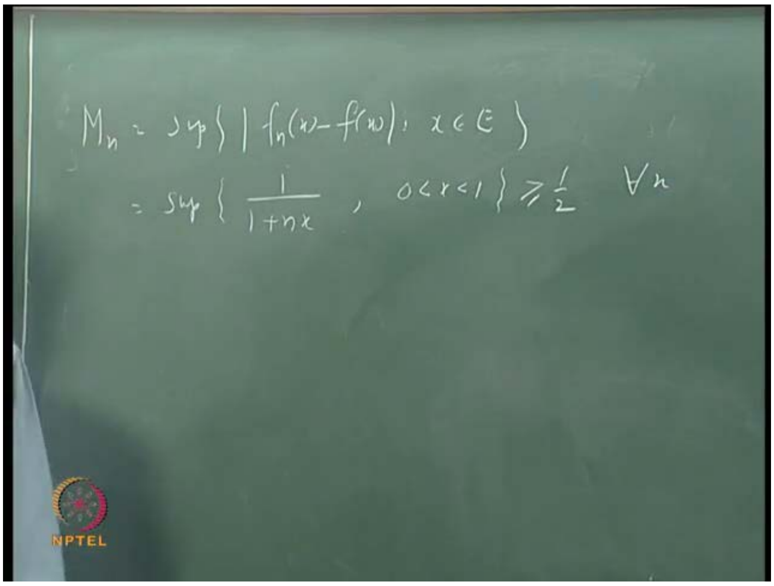

So, look at in fact that is the one which will work in most of the examples, so let M n be equal to supremum of mod f n x minus f x where x is in E. So, what does this mean in our situation everything is positive, so f x is 0, so mod f n x and f n x is it is just 1 by 1 plus n x, 1 by 1 plus n x for E is 0. So, 0 less than x less than one supremum of 1 by 1 plus n x and what is and what do we know about uniform convergence f n will converge to f uniformly if and only if M n goes to 0 M n goes to 0. Suppose you take n to be, suppose you take x equal to 1 by n can you do that 1 by n lies between 0 to 1 for every n, then what will happen to this it should be 1 by 2.

So, I cannot say that M n than or equal to 1 by 2, always I do, I may not it is possible to convert the exact value of M n also, but that is not really essential. So, can we say that M n is bigger than or equal to 1 by 2 for all n, so what does it say if M n is bigger than or equal to 1 by 2 for all n, obviously it cannot go to 0? That means f n does not converge to f uniformly, f n does not converge to f uniformly, so that means the hypothesis of compactness also cannot be dropped, so what remains?

Now, f n is monotonically increasing or decreasing whatever it is I think I will leave that for you to as an exercise, check on your own whether this can be dropped check on your own whether that means what you have to construct an example. Where f n satisfies all other hypothesis except this and the convergence is not uniform if this is essential, if this is not essential then by with remaining hypothesis you should be able to proof uniform convergence, either you have to settle it. Now, let us look at one more property uniform convergence and till, now we discussed about the continuity, now let us next is differentiability and integrability. We will first look at integrability because that is little easier to discuss and to talk of integrability we cannot take any arbitrary set, now we have to take an interval.

So, let us say that a b is an interval and suppose you consider f n from a b to R converges uniformly to some function f from a b to R of course convergence uniformly on a b, convergence uniformly on a b then what we want to say is that each f n is integrable f is also integral and the integral of f n converges to integral of f. This can be proved for Remap’s integrals and this can also be proved for Remap’s theory integrals with, without any extra works. So, let us do it for the Remap’s theory integrals, so let us take a monotonically increasing function, let alpha from a b to R be monotonically increasing. Then if f n belongs to let us use this notation again R a b alpha that is, that means f n is Remap’s theory integrable with respect to this function alpha for each n.

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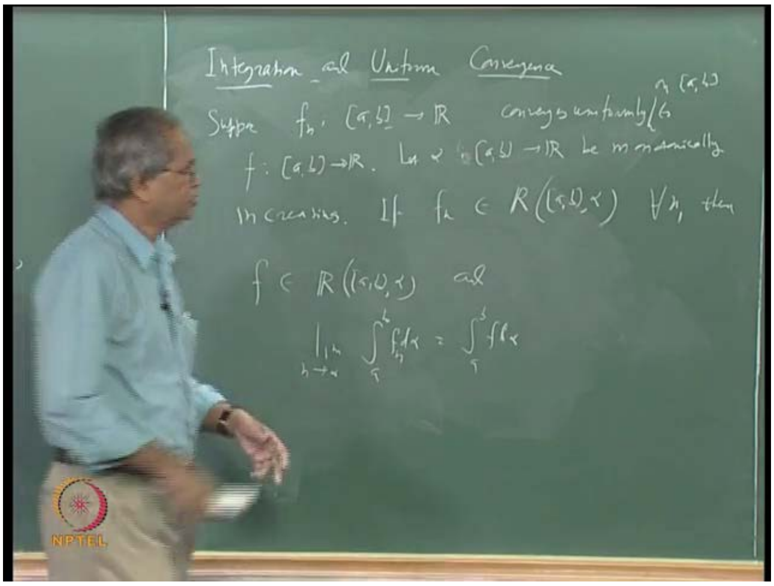

Then f is also Remap’s theory integrable and limit as n tends to infinity of integral a to b f t alpha, sorry integral a to b f n d alpha, this is same as integral a to b f t alpha. See we have seen an example where this is false under point wise convergence, we have seen example of a sequence f n converging to f. But, integral f n does not converge to integral of f, in fact for Remap’s integrals also this is not true if the convergence is not uniform, so this is a, this is a theorem on integration and uniform convergence this is what we were discussing, integration and uniform convergence.

So, there are two things to be proved, here first is we have to prove that f is Remap’s theory integrable and then we have to also prove that other thing we shall select to first to prove f is Remap’s theory integrable. We have known that one of the standard techniques is that given epsilon you should produce some partition such that for that partition U p f l alpha minus L p f alpha is less than epsilon that is what we shall try to do.

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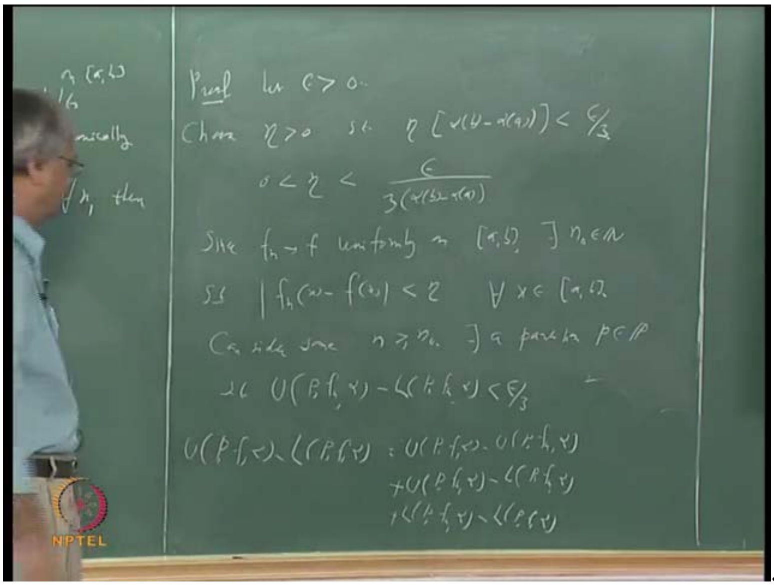

So, let epsilon be bigger than 0 then we know that each f n is integrable, so for each f n such a partition exist for each f n such a partition exist. So the idea is that we will take one of the portions and if n is large enough difference between f n and f is small, so we should naturally expect that difference between upper some of f n and upper some of f is small.

Similarly, difference between upper lower some of f n and lower some of f is also small and that is the idea that we are going to and since in doing this kind of things we will have to add and subtract certain terms 2 3 times. We shall use the usual technique of the, so call epsilon by three proofs that is let us say suppose we that is we if we have to add whatever quantity that we want to show as a small quantity we express that as a sum of the three quantities and show each of that as less than epsilon by 3 that is the idea.

So, first step is this since, here we have taken this alpha I will chose you can say something else, suppose I call it eta bigger than 0 such that eta into alpha b minus alpha a is less than epsilon by 3 or which is same as saying that 0 less than eta less than epsilon by 3 into alpha b minus alpha a. This is this is something we can always do then since f n tends to f uniformly what we can say is that we can always find some n 0 such that whenever n is bigger than or equal to n 0 difference between f n x and f x is less than this eta.

Difference between f n x and f x is less than this eta, so since f n tends to f uniformly on a b, there exist n 0 in n such that mod of f n x minus f x is less than eta, mod of f n x minus f x is less than eta. Remember that this is for every x in a b that is where the uniform convergence comes into picture, this is for every x in a b this is for every x in a b. Now, consider one such n you can even take n equal to n 0 or n 0 plus 1 or anything, so consider n bigger than or equal to consider some n bigger than or equal to n zero. Now, this f n is integrable, this f n is integrable, so there will exist some partition p such that difference between upper and lower sum for that partition is less than, whatever you want is less than let us say epsilon by 3.

So, there exist a partition p in script p such that U p f n alpha minus L p f n alpha is less than epsilon by 3, but of course this is not what we are interested, what we want to show is that the difference between U p f alpha and L p f alpha. That is small and as I told you the idea is that we shall use this that f mod f n x minus f x is less than eta, idea is the following. We want to show this, we want to show this is small U p l alpha minus L p f alpha, we want to show this is small, so what we will do is that we will add this terms U p f n alpha minus L p f n alpha that is we shall write this as U p f alpha minus U p f n alpha.

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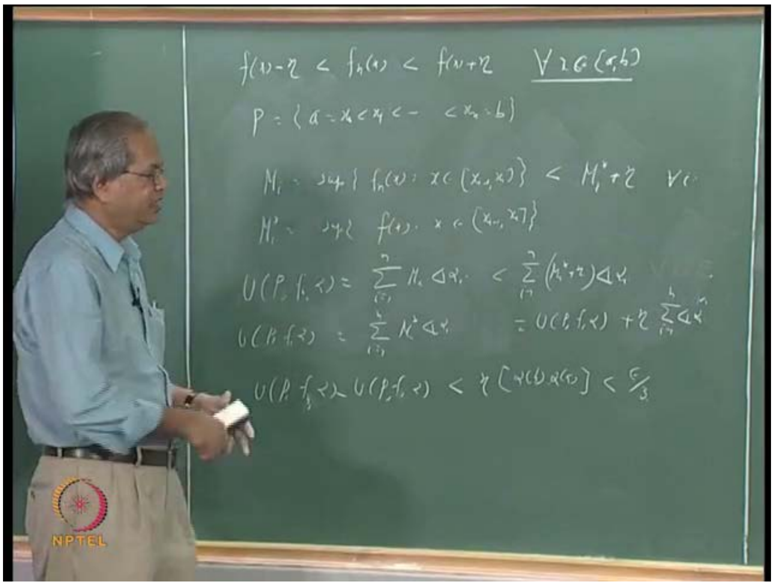

That is adding and subtracting U p f n alpha than plus u p f n alpha minus L p f n alpha and finally plus L p f n alpha minus L p f alpha and we will show that each of this less than epsilon by 3, each of this less than epsilon by 3. Out of which we already know about the middle term, here we know that the U p f n alpha minus L p f n alpha is less than epsilon by 3, only thing remains to be shown at these two are also less than epsilon by 3 that is where we shall use this, now to do that let us look at this partition p.

So, suppose p is let us say as usual a is equal to x naught less than x 1 less than x n is equal to b and what is U p f alpha, for U p to consider U p f n alpha or U p f alpha you have to look at the corresponding supremum in the sub interval x i minus 1 to x i. So, suppose let us say that M i is equal to supremum of let us say f n x for x in x i minus 1 to x i and let us say M i star is equal to supremum of f x for x in x i minus 1 to x i.

Look at this what this says is mod f n x minus f x is less than eta for every x in a b, let us rewrite it what this means is that f n x lies between f x minus eta f x plus eta that is, I will just rewrite this inequality what does this mean. That f x minus eta is less than f n x is less than f x plus eta this is true for remember that is important that is true for x in a b because that is what follows from that is what follows from uniform convergence for every x in a b. Hence, for every x in this x i minus 1 to x i also that means each f n x, here will be less than f x plus eta for every, so can I say from here that M i must be less than M i star plus eta M i.

So, I will say that, so what I will get is M i is less than because remember, here you are taking supremum of f x. Here, we taking supreme of f n x and f n x is less than f x plus eta, so we can say that M i less than M i star plus eta this is for every i, for every i then, so if you look at U p f alpha or let me write U p f n alpha that is nothing but sigma pi going from 1 to n. If it is f n it is M i, M i into delta x i, not delta x i delta alpha i, M i into delta alpha i and U p f alpha that is nothing but sigma i going from one to n M i star into delta alpha i.

So, if each of this M i is less than M i star plus eta I can say that this is less than sigma i going from 1 to n M i star plus eta into delta alpha i. But, what is this suppose I split this it is sigma M i delta alpha i that is nothing but U p f alpha, that is nothing but U p f alpha and plus eta times sigma delta alpha i going from 1 to n. This is something we have calculated several times sigma delta alpha i is nothing but alpha b minus alpha a, alpha b minus alpha a and, so this is we have chosen eta in such a way that eta times alpha b minus alpha a is less than epsilon by 3. So, that is the, so what is the that result, that if you look at the difference between U p f alpha and U p f n alpha that is in another words what I want to say is this U p f alpha minus U p f n alpha.

This difference is less than eta into alpha b minus alpha a, it is the other way less than eta into alpha b minus alpha a and this is less than epsilon by 3, this is less than epsilon by 3. Now, in a similar way if you look in a similar way, if you look at the lower sums instead of taking supremum we are taking infimum instead of taking lower sums. So, a similar relationship will be true, there a similar relationship will also be true there and with in a similar way we can show that the difference between the lower sum of f n and lower sum of f that is also be less than epsilon by 3.

So, suppose you combine all these you get finally that the difference between U p f alpha minus L p f alpha is less than epsilon that will show that each i, each f n is integrable f is also remains in this integrable again recall see what is the idea. Here, we are writing this U p f alpha minus L p f alpha as the difference as the sum of these three quantities and U p if for large n U p f n alpha minus L p f n alpha. That can be, that can be made less than epsilon by 3 because in, because f n is integrable because f n is integrable.

When n is large the difference between mod f n x minus f x can be made arbitrarily small and that is why the difference between the corresponding lower sums. The corresponding upper sums can be made arbitrarily small that is the idea that is the idea, to what remains we have to show this that limit we already know that this integral exist. Now, we have to show that limit of integral a to b f n t alpha limit of that is same as this, now to do that let us, let us recall something that we have proved earlier.

Do you remember we have proved this that if you look at any integral from a to b f d alpha we have prove that if f is integrable mod f is also integrable and absolute value of this is less not equal to integral a to b mod f d alpha this what we have proved. Then what I will say is that again consider a let us say this n 0 eta at everything is as already chosen, here keep this same things and again choose some n bigger not equal to n 0. Consider n bigger not equal to n 0, consider n bigger not equal to n 0 and look at the difference between these two integrals integral a to b f d alpha minus integral a to b f n d alpha I can write this as follows.

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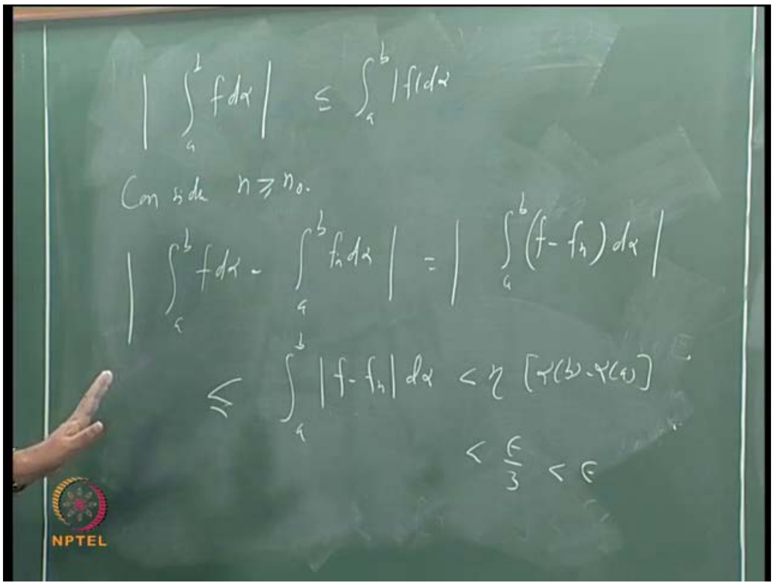

Integral of this, as integral of a to b minus f n d alpha that is from the properties of integral, now I use this, so what is means is that this is less not equal to integral a to b mod f minus f n d alpha. Now, again look at what we have said here for every x in if n is bigger not equal to n 0 for every x in a b mod f n x minus f x is less than eta mod f n x minus f x is less than eta. So, what follows from that this function if you look at this function mod f minus f n x it is value is less than eta in every x in a b. So, what can we say about the integral it will, it will integral will less than eta into alpha b minus alpha a, so this will be less than eta into alpha b minus alpha a.

We have already seen that this must be less than epsilon by 3, than epsilon by 3 and of course epsilon by 3 is always less than epsilon, so what did we show that given any epsilon bigger than 0 there exist n 0. So, whenever n is bigger not equal to n 0 the difference between these two numbers is less than epsilon that is same as showing that this, that is same as showing this. So, again recall what we have shown that if f n converges to f uniformly and if each f n is integrable then f is also integrable and integral of f n converges to integral of f.

Now, let us write an implication of this or the series because that is something, that is something more useful in practice and that is what you require very often and what is that. Suppose sigma f n, n going from 1 to infinity is uniformly convergent let us say this is converges uniformly, converges uniformly.

Since we use f already we will use something else, since converges uniformly let us say to some function g then what does this mean that if you take the sequence of partial sums s n then s n converges to g uniformly. Now, let us assume that sub, let us assume the same suppose each f n is integrable that it will mean that g is integrable, g is integrable at then integral of s n will converge to integral of g but, what is integral of s n.

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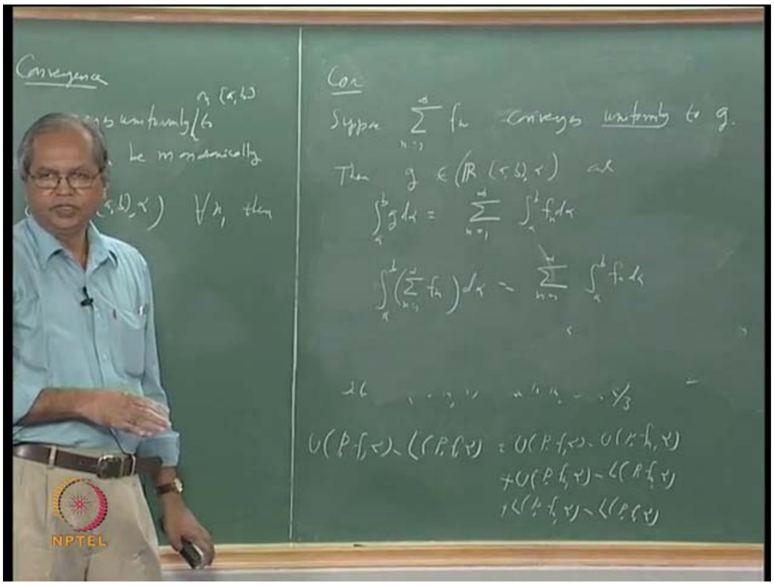

Let us say what is s n, s n is f 1 plus f 2 plus f n, so integral of s n integral a to b s n d alpha that will be same as integral a to b f 1 d alpha etcetera plus integral a to b f n t alpha integral a to b f n t. So, in other words what I can say is that if sigma f n converges to g uniformly than g is if g also belongs to R a b alpha that means g is also integrable and integral a to b g d alpha is same as sigma n going from one to infinity integral a to b f integral. That is integral of the sum is same as sum of the integrals of course this is something we already know the finite sums, the question is about the infinite sums.

In case of infinite sums it is in general false it holds only when the convergence is uniform that is important it holds when the if the series converges uniformly then the series form by taking the integrals converges to the integral of the sum. In other words this is roughly express by saying that you can integrate this series term by term that you can interchange the operations of integration and summation. Suppose you write this in the full form what is the meaning of this, this g is nothing but integral summation of suppose I write in the full form.

This will mean integral a to b sigma f n d alpha, this is same as this sigma this is, here sigma n going from one to infinity sigma n going from 1 to infinity integral a to b this should be f n, f n d alpha. In other words what this means is that the operations of integration and summation can be interchanged, can be interchanged and this can be done provided the convergence is uniform that is what this theorem says. So, we will stop with this, in the next class we shall discuss the differentiation and uniform convergence relationship between differentiation and uniform convergence.