Real Analysis
Prof. S.H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 46
Sequences and Series of Functions

Well we shall close this discussion about the integration and will move to the next topic.

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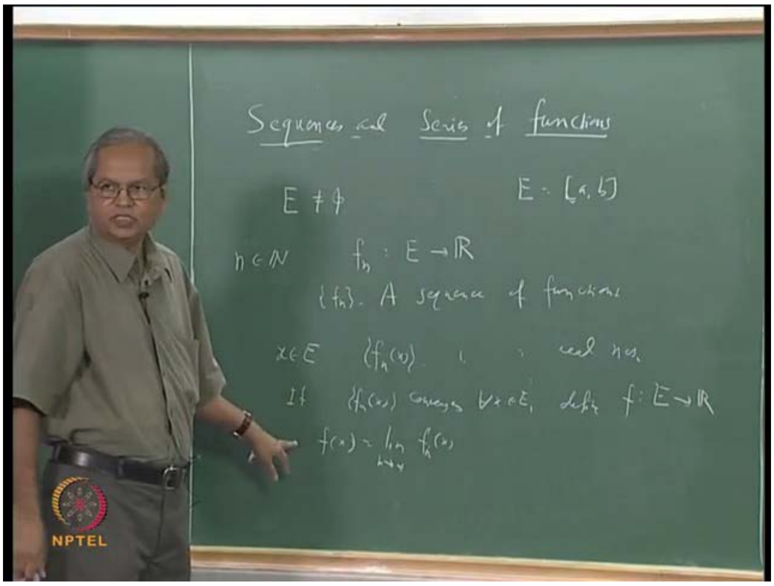

Namely that we will start discussing what is called sequences and series of functions again you are already familiar with sequences and series of numbers. So, all that we do, here is that replace the numbers by functions as you know let us begin with the definition. As you know sequence is a function whose domain is the set of all natural numbers codomain can be anything when codomain is a real number we call sequence of real numbers. If it is complex number it will be sequence of complex number, so similarly you can talk of sequence of anything sequence of matrices sequence of vectors and also sequence of functions. So, sequence of, now functions define were of course you can in to be very general, we can say that each function is defined on different sets, but with that kind of thing we will not be able to do further, so we shall just fix one set.

Let us say E its sub known non empty set and we shall assume that all functions are defined on this set E quite frequently. This set E will be subset of some metric space and in most of the application it will be some interval in the real line closed in bounded interval on the real line. So, for each n in N we have function f n or let us say real valued function f n from E to R then this f n is a sequence of functions, f n this is a sequence of functions. Now, what do we do with this sequence of functions, several things once you have sequence of functions then if you take any x in E if you take any x in E then this f n, f n of x is a real number, f n of x is a real number.

So, if you let this small n very over N then that will be a sequence of real numbers, so f n x is real sequence of real numbers this is sequence of real number and you already know what is meant by saying that a sequence of real number converges or diverges etcetera. Now, if it, so happens that for each x in E this converges then it will be in that and suppose it converges to some real number, then that will define a new function from E to R because for each x you have a new number which is a limit of this sequence. So, let us say if we can say that if f n x converges for each x in E then we can define a function f from E to R, function f from E to R given as follows that is f of x, f of x is nothing but limit of f n of x limit is n tends to infinity and when this happens we say that the sequence f n converges to the function f point wise.

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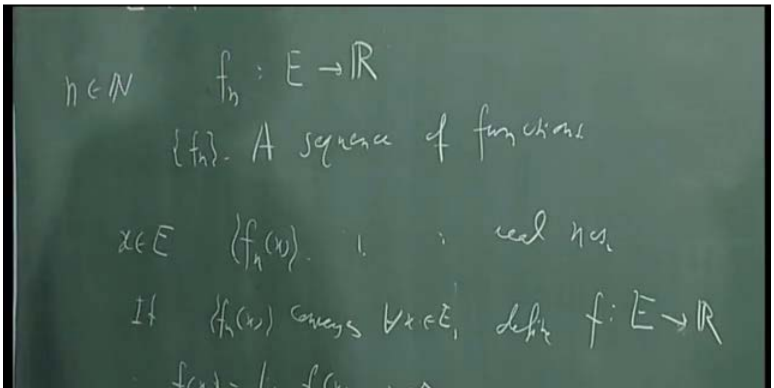

Then that means for each x that is for every x in E we describe it as follows.

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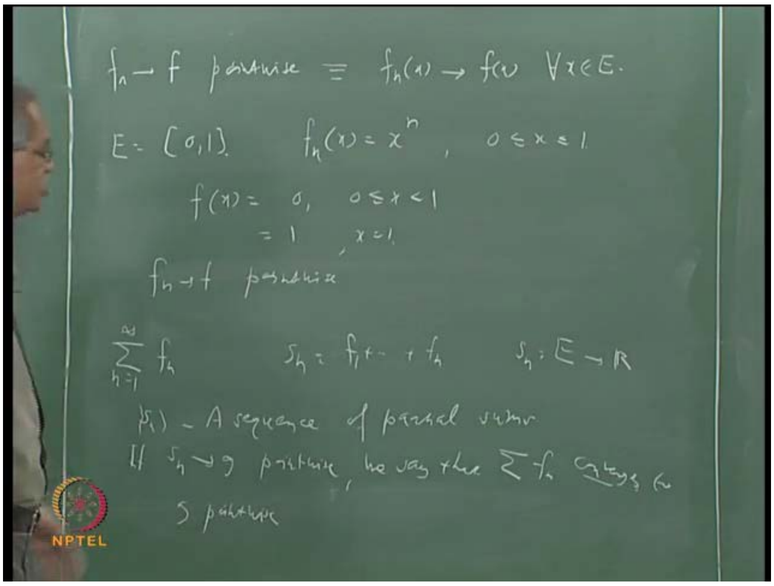

We, say f n tends to f point wise, what is the meaning of this, it means for each x the sequence f n x of real numbers converges to the real number f x that is, that is the meaning of this is the following that is f n x converges to f x for every x. Similarly, one can, now before proceeding for that just to, in fact you have you have come across such sequences of functions and consider their point wise limits also, but may be perhaps without explicitly mentioning that that is the sequence of functions.

For example, let us let us just take this case to begin with suppose we take E as let us say interval 0 to 1 and define f n x at to begin with let me say x to the power n, so 0 less not equal not less not equal to 1, these are very standard and well known example. So, what happens in this case when you fix x, when you fix some x then f n x is the sequence x to the power n consider various cases for x equal to 0. This will be 0 if x is less than 1, that limit will go to 0 if x is equal to 1 it becomes a constant sequence 1.

So, we can say that suppose you define f x equal to 0 for 0 less not equal to x less than 1 and equal to 1 for x equal to 1 then f n converges to f point wise that is about the convergence of the sequence of functions point wise. Now, just as we can talk about the sequences, in a similar way we can talk about this series as just as in the case of real numbers you know that everything that you want to say about the series is said in terms of the sequence of partial sums. So, do the same thing, here given suppose I want to talk of a series let us say sigma f n then look at its partial subs look at its partial subs, so suppose this is s n, s n will be, now f n plus f 2 plus f n.

Remember each s n is a function, each s n is a function from E to R see by the way, here going back to this example, here since you have taken the set E as 0 to 1 that is why f n converges to f point wise. Suppose we take some other set it may not happen, for example suppose I take E from 0 to 2 then for x bigger than 1 the sequence f n does not converge. So, it is it is not necessary that given any sequence it should converge to some function point wise or even if you take say minus 1 to 1 what will happen is that it will converge for minus 1 less than x less not equal to 1. But, if you take x equal to minus 1 then the sequence becomes 1 minus 1, 1 minus 1 etcetera and then it will not it will be a divergent sequence.

So, it also depends on what is this set E the behaviour of the limit or whether the sequence converges point wise or not, depends on what is this set E coming back to the series given any such sequence. We can think of a series like this sigma f n, n going from 1 to infinity or in certain cases it is convenient to take n going from 0 to infinity it does not matter where you start it does not really matter. This is something you should know and perhaps you know about the sequences series as far as the consideration of the convergence or divergence of a sequence or a series is concerned, what happens to a first few terms does not matter.

A finite number of terms can be removed from a sequence or finite number of terms can be added to a, can be inserted in a sequence or a series the convergent sequence will remain convergent and divergent sequence will remain divergent. Of course, if it is the series the sum will change some of the series might change, but the convergent series will remain convergent and divergent series will remain divergent. That behaviour will respect to convergence or divergent will not change by changing a finite number of terms either in a sequence or in a series.

So, coming back to this, this is a sequence of partial sums, so this s n is a sequence of partial sums, this is a sequence of partial sums and it, so this is a sequence of functions each s n is a function. So, we have already defined what is mean by saying that a sequence of function converging point wise. So, if this sequence s n converges to some function suppose that function is g then we say that the series sigma f n converges point wise to that function g and in that function g is called the point wise sum of this series. So, that is the idea, so let us just if s n converges to g point wise, we say that sigma f n converges to g point wise we say that sigma f n converges to g point wise and g is called the sum of the series.

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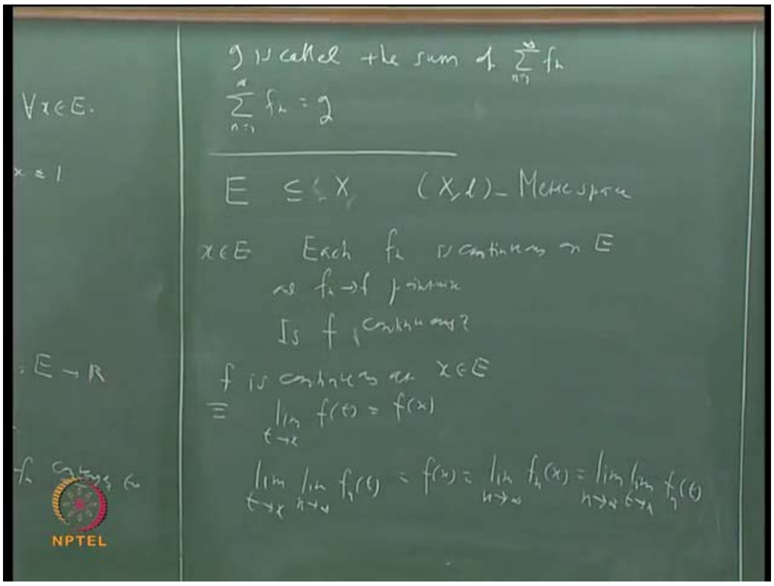

So, this g is called and divided by this sigma f n, n going from 1 to infinity is equal to g, so g is called the sum of the in fact one should say point wise sum because everywhere we are calling point wise convergence, so we should say point wise sum. But, is not very strictly as held to this sum of this series sigma f n, so this is as far as the definition, now what are the kinds of problems that we are interested the problems are as follows.

What we want to know is that given a sequence or a series suppose each f n has some property, suppose each f n has some property then does this sum also have that property does the limit also, have that property or in this case does this sum also have that property. What are the properties usual properties, that we are analysis calculus for example continuity the kind of question we are going to ask is suppose each f n is continuous does it follow that f is continuous or in this case. Suppose each f n is continuous does it follow that g is continuous and, similarly about the differentiability.

Suppose each f n is differentiable does it follow that f is differentiable another question about the series and suppose each f n is integrable on some on the intervals suppose E is the interval a to b. Suppose, each f n is integrable on a b can we say that f is also integrable on a b etcetera. Now, when we start analysing these questions, it turns out that this point wise convergence is not a very satisfied thing you already have one example. Here, each f n is a continuous function, each f n is a continuous function, but this f is not a continuous function.

In order to understand what exactly is happening, here you can see understand that this question basically means whether we can interchange certain limits. In each such question some 2 or more limits are involved, if they are taken in a particular order you get one value if taken some other order you take other value. The question is whether those 2 limits can be interchanged, let us let us illustrate this suppose we take this question that is f n converges to f let us, let us look at here. Suppose, so to talk of the continuity we have to take this set E as in some metric space for the, so let us say that E is in some metric space.

Suppose x d is a metric space E is subsets of some x where x E is some metrics space and let us say some x is in E. What is the question, if each f n is continuous on E suppose that is the question, if each f n is continuous on E and f n converges to f point wise then the question is the question is does it follow does it imply that is f continuous. But, this is the question which we have already answered by looking at this example what I wanted to say is that this answering this question essentially involves interchanging the order of some limits.

So, that is what I want to illustrate and what exactly it means let us take some point x E what is the point in saying x is continuous at E. Suppose you take some suppose x is, sorry f is continuous at E means you take limit of f, f of let us say f of t as t tends to x that should be same as f x. So, that f is continuous at x, at x what is the meaning of that, this means let us say this is same as saying that limit of let us say f at t as t tends to x is equal to f x. Now, let us also use the fact that f is a point wise limit of f that is for each x in e or each t in e f of t is same as limit of f n of t.

So, this left hand side f suppose I put that, here f of t will be nothing but is limit of n tends to infinity f n of t, so that is this is same as saying the limit as t tends to x and in limit as n tends to infinity of f n of t. This is equal to f x then if f is continuous at x means this right f is, but what is f x, f x is limit as n tends to infinity of f n of x, so this is same as limit of f n of x as n tends to infinity. But, we have assumed that each f n is continuous, so I can replace that f n x as limit of f n t as t tends to as t goes to x. So, I can say that this is same as limit as n tends to infinity limit t tends to x of f n of t, so just look at the first and last, just look at the first and last.

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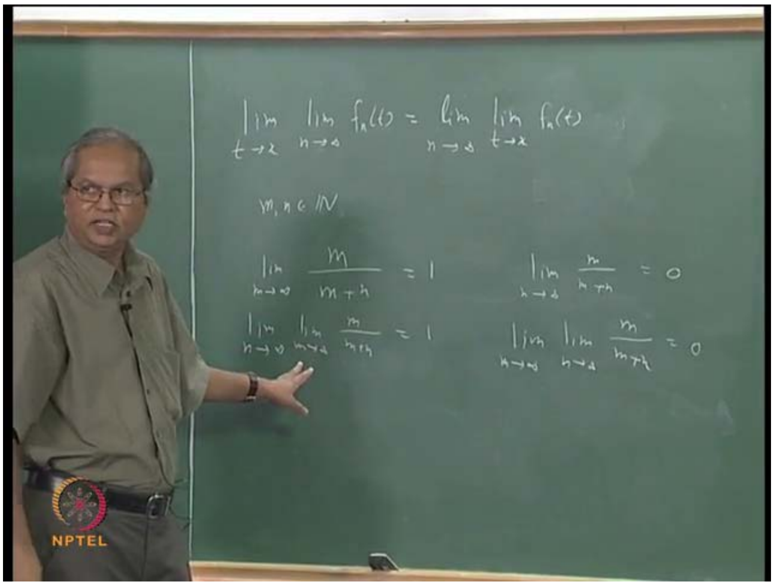

Let me, let me rewrite that here what is the left hand side limit t tends to x then limit n tends to infinity f n of t this should be same as limit n tends to infinity limit t tends to x f n of t. So, to say that f is continuous at x where f is a point wise limit of f n its same as saying whether you can interchange this two limits for the limits remember the quantity is same f n of t everywhere and limits whether you take first let t tend to x. Then let n tend to infinity or whether you first let n tend to infinity and then let t tend to x whether this two processes can be interchanged.

So, asking whether f is continuous at x is basically asking whether this two limiting process can be interchanged and practically every question which I have said so far can be can be brought into this form. Whether it is a question of differentiability, whether it is a question of integrability because each of these questions sees whether it is derivative or integral or all these are some limits.

So, each of this question basically can be transformed into questions like this whether certain limiting processes can be interchanged and from your experience you should know that in general the answers is no. Unless you put some additional restrictions, we cannot in general interchange the way in which you take the limit again let us, let us take a well known example suppose you have something like this, suppose you take something like this m by m plus n. Where both m n both are natural numbers, this is a function of both m and n, so you can first and you can consider the limit of this as m goes to infinity and n goes to infinity in two ways.

First let m tend to infinity and then whatever you get, let n tend to infinity and the other way let us see what happens. Suppose you first take limit of this as m tends to infinity then what will be that n is fixed, n is fixed and they are letting m to infinity. So, the limit will be 1, so the limit will be 1 and then after this if you let n tend to infinity, so suppose I say limit n tends to infinity and limit m tends to infinity m by m plus n this will be same as 1. Now, look at the other way suppose I take limit n tends to infinity of m divided by m plus n, m is fixed and n is going to infinity, so what will be that limit that is 0, that is 0 and that is independent of whatever is m, for every m this limit is limit is 0.

So, if you, now let m tend to infinity you are going to get 0, so limit m tends to infinity, limit n tends to infinity of m by m plus n that is 0, so the same quality m by m plus n if you take the limits in two different orders you will get you may get different limits. Of course, in some cases the limits may be the same, but you need some extra conditions for that, so that is that is what we are going to what are the extra conditions. Means we will talk of some different kind of conversions which is supposed to be stronger than this convergence and if under that convergence this properties are preserved.

That is if each f n is continuous, then f is continuous etcetera and that this is one way, that is one way of dealing which is that you strengthen the concept of convergence, there is also other way of dealing with this question. You can weaken the concept of integrability, but that is something we will not follow, that is something which that is something you will do in your next course on measure and integration. Where you will think of different types of integrals and where even point wise convergence also will be fine for looking at the integrals.

Now, one more thing that is when we come to the integrals it is not just the question of integrability that we are interested in we would also want to know that we suppose we take a series. Whether such series can be differentiated point wise term by term that is if sigma f n converges is equal to g can we say that sigma f n prime is equal to g prime or can we say that sigma integral of f n is equal to integral of g etcetera.

Even that kind of questions and in those questions which are very important in this convergence, that we are considered. Now, before proceeding further let me just give one more example to illustrate the things are not very good when with respect to point wise convergence even when we talk of integrability.

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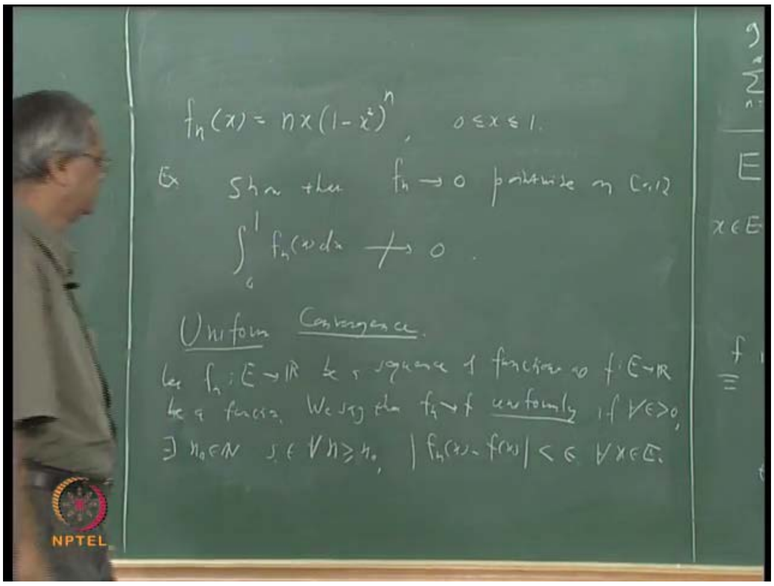

I will not give all details let us, I will give some of this things to you as an exercise, take f n x let us let us say n times x multiplied by 1 minus x square rays to n for 0 less not equal to x less not equal to 1 I will not go to all the details. Show the following I will just give as an exercise, show that f n tends to 0 point wise on 0 to 1 and remember this is a continuous function since continuous function on 0 to 1.

Hence, integrable continuous, and hence Riemann integrable, so integral 0 to one f n x d x exist for this Riemann integral exist. But, this does not go to the, integral of this function will be 0, so this does not converge to 0, this does not converge to 0 try to show that on your own we will not spend. This is simple you have to just calculate this integral that will turn on to some function of n and you show that that sequence does not go to 0.

So, the new convergence that we are going to discuss is what is called uniform convergence, uniform convergence, what is the meaning of this let us first give the definition and then will try to explain what is here. So, suppose let us say let f n from E to R be a sequence of functions and f is also function and f from E to R is also a function b of function we say that f n converges to f uniformly, that is what we want to define. We say that f n converges to f uniformly f n again as like any convergence or any limit this definition is in terms of epsilons etcetera.

So, if for every epsilon bigger than 0 we are talking about sequences, so if every epsilon bigger than 0 there exist some n 0, there exist some n 0 in n such that for every n bigger than not equal to n 0 something should happen. For every n bigger not equal to n 0 something should happen, such that there exist n 0, such that for all n bigger nor equal to n 0, what should happen is that mod f n x minus f x this is less than epsilon, mod f n x minus f x less than epsilon. This should happen for every x in E such that for all n neither bigger nor equal to n 0 mod f n x minus f x less than epsilon for every x in E.

Now, suppose you had taken some x in E and you had set for every epsilon bigger than 0 there exist n 0 such that whenever n is bigger than nor equal to n 0 mod f n x minus f x is less than epsilon. Then that would be in that f n x converges to f x, that would be f n x converges to f x and since this is happening for every x in E that means that f n x converges to f x for every x in E. So, if f n or in other words it is same as saying that if f n converges to f uniformly then f n converges to f point wise also.

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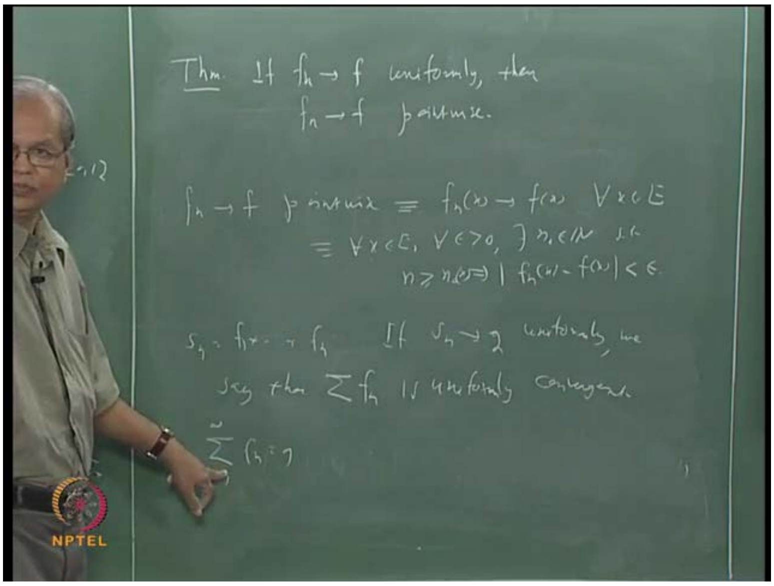

So, that is, that is an elementary observation, but usually it is given as a theorem, so if f n converges to f uniformly, then f n converges to f point wise. That means uniform convergence is stronger, uniform convergence is stronger than point wise convergence. Then there is an obvious question what about the converse that is can we say that if f n f, f n converges to f point wise then it converges uniformly if that may true, obviously then it will mean that both the notions of convergence coincide that is not a fact.

So, the converse is false, now how does one show that converse is false there is only one way that is you have to construct sequence of functions which converges point wise, but not uniformly that is the only way. We shall do it, we shall do it in the course of time because that will require some more work, so I will not go into that now, so this is about the, now what exactly is the difference see when you say f n converges to f point wise.

Let us, let us just look at this and compare the two definition what is the meaning of this f n converges to f point wise, this means or this is equivalent to saying that f n x converges to f x for every x in E, f n x converges to f x for every x. That is same as saying or this is equivalent to saying that every x in E and for every epsilon bigger than 0 there exist n 0 in n such that n bigger nor equal to n 0 implies mod f n x minus f x is less than epsilon, mod f n x minus f x. That is the meaning of saying that f n x converges to f x, now what is the difference between this and this see in this definition when we say we start with some x in E and we look at the convergence only at that point will look at that converges only at that point.

Though it happens at each point, this n 0 may depend on this x because given x in E for this epsilon we say that some n 0 exist, but this n 0 may in general depend on this x, so if you want you can write it as n 0 of x. But, here we are saying that that this n 0 that should work for every a x in E that, n 0 should work for every x in E that is the main difference that is given epsilon you should be able to find n 0, such that whatever we are saying happens for every x in E. Now, what is the meaning of saying that in a point wise convergence given x you will have some n 0 which depends on that x. If you take different x you will get different sets of difference values of x 0, suppose it is possible to take the maximum of all those n 0 then that maximum will work for all x.

But, this may or may not be possible if it is possible then that sequence converges uniformly, if it is possible then that sequence converges uniformly otherwise if it is not possible means what it means there is no maximum, there is no maximum. That is if you take n 0 x for different values of x its maximum is infinity, so that is and that is, that is what can happen in case of point wise convergence. Now, this is about the sequences in a similar thing, similarly we can talk about the series because series after all nothing but sequence of partial sums. You take the sequence of partial sums if that converges to a function g uniformly we will that a series converges uniformly.

So, let us say, now you consider say s n is equal to f 1 plus f 2 plus f n means that if s n converges to g uniformly we say that sigma f n is uniformly convergent, uniformly convergent sigma f n is uniformly convergent and we will say that again we may say the same symbol sigma f n is equal to g n going from one to infinity. But, this time we will say that g is the uniform limit that is g is the sum of sigma f n, but not point wise, but uniformly.

So, in case of series also we can make a difference whether says point wise convergent or uniform convergent and accordingly we can say that g is a point wise sum or uniform sum. So, now again you look at this difference uniform convergence we want to interpret it in a slightly different manner want we is that for every n bigger nor equal to 0 what should happen is mod f n minus f x is less than epsilon for every x in E. So, suppose I take this set of mod f n x minus f x for every x in E, I will just take this set.

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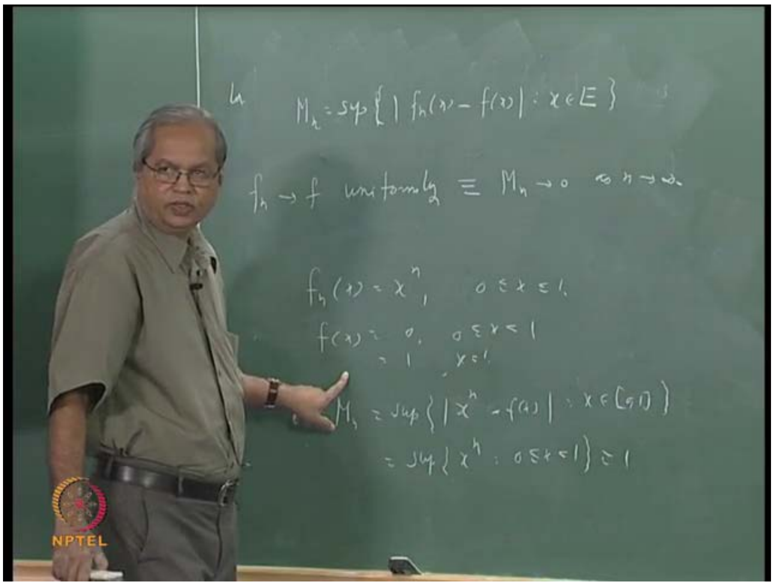

Mod f n x minus f x for x in E this is the set of positive or non negative real numbers, this is the set of non negative real numbers. Each of this is less than epsilon each of because this is less than epsilon for every x in that means this is bounded set, is a bounded set. So, I can look at this supremum or least upper bound, so suppose I call that let us, let us say m suffix n let m suffix n be the supremum of this supremum of this set, so mod f n x minus x, x belongs to E.

Then what does this say this m n is less than epsilon m n is less than epsilon for n bigger nor equal to n 0, or suppose you regret this definition what it mean for every epsilon there exist n 0 such that where ever n bigger equal to n 0 this M n is less than epsilon. Now, M n is a sequence of real numbers, so what is the meaning of saying this for every epsilon there exist n 0 such that wherever it is same as say that M n converges to 0, M n converges to 0. So, saying that f n converges to f uniformly is equivalent to saying that this sequence of real numbers M n converges to 0 and that is a very useful criteria in deciding whether certain sequences converges uniformly or not.

So, let us just record it, so f n tends to f uniformly this is equivalent to saying that M n tends to 0 as n tends to infinity and this is something which will be useful in many practical problems. That is suppose you are given sequence f n and suppose you are asked to check whether that sequence converges uniformly or not, then the best thing to do is try to find out M n if not M n some value or something. We just bounded by M n etcetera some estimate of M n and then try to conclude from that whether the sequence converges uniformly or not.

In most of the cases you will be able to find M n because see in may practical problems we come across this f n x f x those all will be continuous differentiable functions. In some interval you can always find the maximum value of that by using the methods of calculus and calculate M n so that is always possible. Let us look at the example which we have taken earlier, let us take this example f n x is equal to x to the power n for 0 less not equal to less not equal to 1 and what was f x in that case f x was 0 for 0 less not equal to x less than 1 and equal to 1 for x is equal to 1. So, what was what will be M n m n is supremum of suppose I take same definition mod f n x minus f x for x in 0 to 1.

Since f n x is defined by the same formula for every x, I will just replace by just x to the power n, here x to the power n minus f x and what is this f x, f x is 0 for 0 less nor x less than 1 and it is 1 for x equal to x equal to 1 x equal to 1 x equal to 1 what happens this is also 1. So, it is 0, so I can say that this supremum is nothing but supremum of this x to the power n for 0 less not equal to x less than 1, this is, this is same as supremum of x to the power n for 0 less not less to x less than 1.

What is this supremum 1, is it, is it obvious to everybody that if you take any n, if you take any n and look at f to the power of n and find its maximum supremum between 0 to 1 then this is nothing but 1. So, M n is equal to 1 for all n, so does this happen then m n, so it clear that this sequence f n does not converge to f uniformly, so f n we already seen that f n converges to f point wise. But, f n does not converge to f uniformly and that is what answers this question, here we ask whether the converse of this is true and, now give an example of a sequence which converges point wise, but does not converge uniformly.

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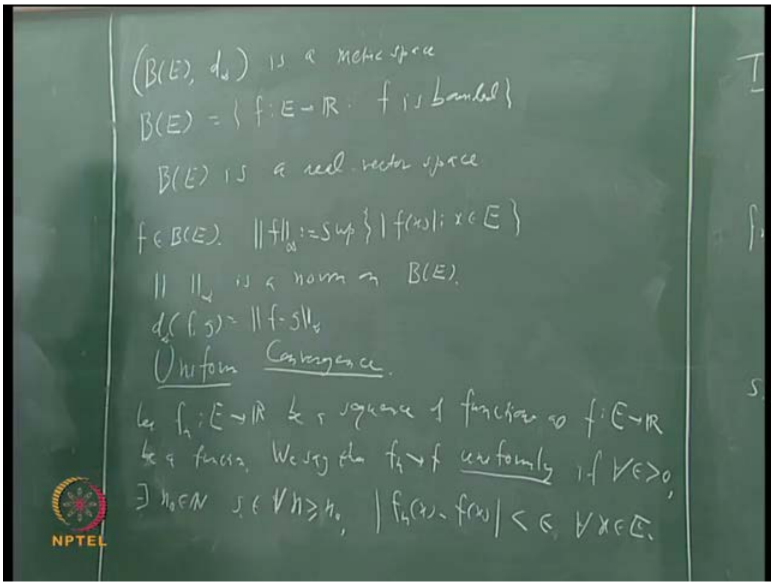

Now, there is one more way of looking at this whole uniform convergence and that is something which will be of interest to you because you have learned so many properties of metric spaces. So, what we can do is that that we can take this set E and let me define this set B of E, what is B of E let us say that it is the set of all bounded functions from E to E B for bounded.

So, B of E is this set of all f from E to R and f is bounded, now there are several properties of this say B of E its set of functions just set of functions defined on the same set E and all that we are assuming is that E of this function is bounded. Now, let us, let us use some use this set from the point of view of some properties of linear algebra is this vector space. In fact it is a well known example of vector because all that you need to check is that if f and g are bounded f plus g is bounded alpha times f is bounded etcetera.

So, B of E is a since all functions are going to R this is a real vector space B of E is a real vector space, one more thing suppose you take this sum f in this B of E can look at this number supremum of mod f x for x in E.

We assume that f is bounded f is in B of E means f is bounded, so this is a bounded set and hence this supremum exist as a finite real number infact on negative number. We will denote this as follows this is this is read as norm defined as earlier norm suffix infinity is defined as supremum of mod f x where x in E then this is a norm on this vector space. What is the meaning of this, there are a several properties of a norm which we discussed in one of the earlier classes namely that this is always bigger nor equal to 0 and this is equal to 0 if and all if f is 0 f is 0 means what it is a constant function 0.

Then norm of alpha times f is same as mod alpha times norm f and norm of f plus g is less not equal to norm f plus norm g, all this properties are very easy to prove in this case. So, we will simply I will just simply record this norm is norm on b of E, then let me again remind you that we have shown that if something is a norm. Then that norm endues a metric, that norm endues a metric on this side and what is that metric it is distance between f and g is defined by norm f minus g suffix infinity, norm f minus g suffix infinity.

This is also d suffix infinity, also d suffix infinity just to just to emphasis this is a norm given by supremum, this is a norm given by supremum, so this b suffix B of E along with this metric is a metric space. Now, come back to the definition uniform convergence and look at this last part what does it say that f n x minus f x less than epsilon for every x in E and this M n is nothing but the supremum of that.

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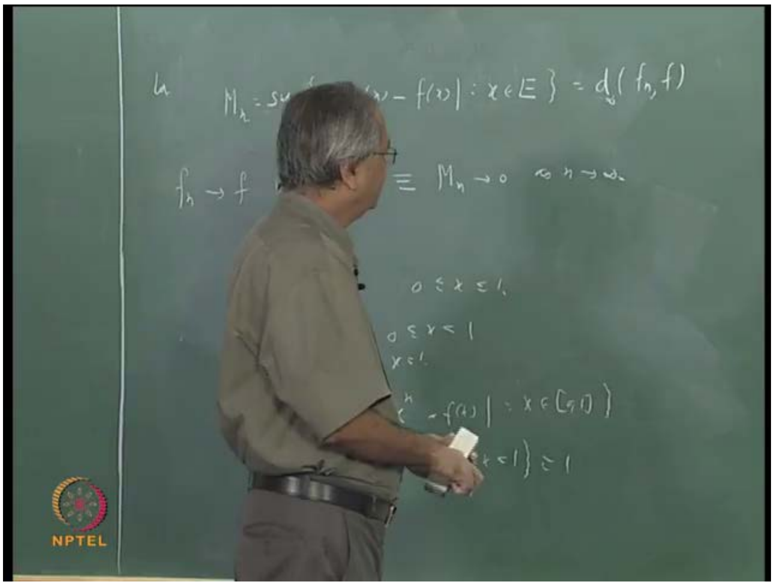

So, this M n is nothing but d f n f, d or in our case suffix infinity d f n distance between f n and f, so what does it mean suppose something was a metric space with metric d and suppose you take any sequence in that metric space. Suppose you want to say what is mean by saying that that sequence converges then what would have said the given any epsilon there exist n 0 such that whenever n is bigger than nor equal to n 0 distance between f n and f is less than epsilon.

That is what exactly is happening, here that is what exactly is happening with respect to this distance, with respect to this distance. In other words uniform convergence is nothing but the convergence in this metric uniform convergence is nothing but the convergence in this metric endues by this supremum. Once this fact becomes clear you will try to understand many other results in a different way, so let us take one more thing which is again a very famous theorem about the uniform convergence known as the Cauchy's criteria for the convergence.

Now, before going to that in another view of this we will also see, for example in this metric space when will when will you say that set is bounded in any metric space we say that the set is bounded if its diameter finite, if its diameter finite. So, suppose you take certain sequence f n we will say that f n is bounded, f n is bounded if you, if you take distance between f n and f m then that is bounded by some finite number. So, such a sequence is called uniformly bounded, such a sequence is called uniformly bounded, uniformly bounded means what.

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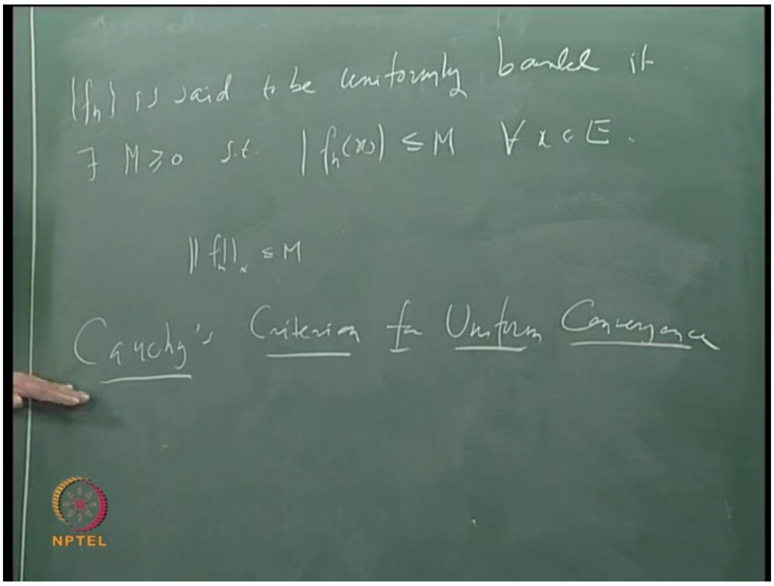

Let us, let me, let me recall the classical definition we will say that f n is said to be uniformly bounded, uniformly bounded if there exist some number M bigger nor equal to 0 such that mod of f n x less not equal to M for every x in E. In fact this is something we can talk about not only just a sequence, but any set of functions, any family of functions we will say that it is uniformly bounded if this mod f n x is less not equal to M for every x in E.

Now, coming back to this terminology is it obvious to that this is same as saying that this norm of f is saying less not equal to M norm of f n suffix infinity is less than nor equal to M because norm is nothing but supremum of this mod f n x. Now, if norm is less not equal to m for every element, here if you take any two elements is it obvious that the distance between them less not equal to 2 M, suppose you take f and g norm of f minus g will be, distance between f and g will be less not equal to norm f plus norm g.

So, that will be less equal to 2 M, so saying that uniformly bounded is nothing but saying that bounded in this metric saying that a family of functions is uniformly bounded is exactly same as saying that it is bounded in this metric. So, what I was saying is that we will want to think of what is called Cauchy's criteria for uniform convergence, Cauchy's criteria for uniform convergence. Uniform convergence, since the clock indicated time is over we will not go into discussion of the exact details of this.

But, what I want you to understand is that we shall, we shall state into this theorem and what it will mean basically is to show that this metric space is complete. What will basically show is that if you use diverge metric spaces, I can simply say that this is a complete metric space, but we will, we will see it in the next class.